Asymptotic Behavior of Energy Solutions to a Two Dimensional Semilinear Problem with Mixed Boundary Condition

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1 Introduction

This work is concerned with the asymptotic behavior of the energy solutions of the mixed boundary value problem

\[\begin{cases}
\Delta u + u^p = 0 \text{ in } \Omega \\
u = 0 \text{ on } \Gamma_0 \\
\frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma_1
\end{cases}\]  

(1.1)

where

- $\Omega$ is a $C^{0,1}$ and bounded domain in $\mathbb{R}^2$,
- $\partial\Omega$ consists of two pieces $\Gamma_0$ and $\Gamma_1$, where the 1-dimensional Hausdorff measure of $\Gamma_0$ is greater than 0,
- $\Gamma_0$ is smooth and $\Gamma_1$ is piecewise smooth,
- $\Gamma_0$ and $\Gamma_1$ are relatively closed in $\partial\Omega$,
- $\nu$ is the unit outer normal of $\Omega$,
- $p$ is a large parameter.
In this work, we shall only consider the least energy solutions, although the method can be used to study other solutions with the same decay rate of energies. Let

\[ A_p = \{ v \in W^{1,2}(\Omega) : v = 0 \text{ on } \Gamma_0, \| v \|_{L^p(\Omega)}^{p+1} = 1 \} \]

be the admissible set. Define the energy

\[ J_p(v) := \int_{\Omega} |\nabla v|^2 \, dx \]

on the admissible set \( A_p \). Standard argument shows that for any \( p > 1 \) \( J_p \) is bounded from below and the infimum is obtained by a function \( u_p' \) in \( A_p \). By the inhomogeneity of (1.1) we know that a positive multiple of \( u_p' \) solves (1.1). Throughout the rest of this paper we denote such least energy solutions by \( u_p \).

Our goal here is to understand the asymptotic behavior of \( u_p \) as \( p \), serving as a parameter, approaches \( \infty \). It is known in [10] that for the pure Dirichlet problem, i.e. \( \Gamma_1 = \emptyset \), the solutions \( u_p \) develop single or double bounded peaks in the interior of \( \Omega \) as \( p \to \infty \). In the current mixed problem, we shall see peaks on the Neumann boundary \( \Gamma_1 \) and show that \( u_p \) can develop no more than either one interior peak or two boundary peaks on \( \Gamma_1 \). We start to investigate \( c_p \) where

\[ c_p := \inf \{ [\int_{\Omega} |u|^2 \, dx]^{1/2} : u \in A_p \}, \quad (1.2) \]

According to the construction of least energy solution \( u_p \),

\[ c_p^2 = \frac{\int_{\Omega} |\nabla u_p|^2 \, dx}{\int_{\Omega} u_p^{p+1} \, dx^{2/(p+1)}}, \quad (1.3) \]

and \( c_p^{-1} \) is the optimal constant of the Sobolev embedding

\[ V(\Gamma_1, \Omega) \hookrightarrow L^{p+1}(\Omega) \]

where \( V(\Gamma_1, \Omega) = \{ v \in W^{1,2}(\Omega) : v = 0 \text{ on } \Gamma_0 \} \) is a Hilbert space equipped with the inner product

\[ < u, v > = \int_{\Omega} < \nabla u, \nabla v > \, dx. \]

We shall see that \( c_p \) possesses nice decay property as \( p \to \infty \). Next we extend some \( L^1 \) estimates of H. Brezis and F. Merle for \( \Delta \) with Dirichlet boundary condition in \( R^2 \) to mixed boundary condition. After these preparations we shall prove

**Theorem 1.1** There exist \( C_1, C_2 \), independent of \( p \), such that

\[ 0 < C_1 < \| u_p \|_{L^\infty} < C_2 < \infty \]
for large $p$. Indeed

\[
1 \leq \liminf_{p \to \infty} \|u_p\|_{L^\infty(\Omega)} \leq \limsup_{p \to \infty} \|u_p\|_{L^\infty(\Omega)} \leq \exp \left( 1 + \frac{\alpha_0}{2} \right)
\]

where $\alpha_0$, defined later in (4.4) section 4, is a constant depending on the pair $(\Gamma_1, \Omega)$ only.

To state our second result, we need a few definitions. Let

\[
v_p = \frac{u_p}{\int_\Omega u_p^p}.
\]

For a sequence $\{u_{p_n}\}$ of $\{u_p\}$ with $p_n \to \infty$ as $n \to \infty$, we define the blow-up set $S$ to be the subset of $\Omega$ such that $x \in S$ if there exist a subsequence, still denoted by $\{p_n\}$, and a sequence $x_n$ in $\Omega$ with

\[
v_{p_n}(x_n) \to \infty \text{ and } x_n \to x.
\]

Define

\[
S_I = S \cap \Omega,
S_C = S \cap (\Gamma_0 \cap \Gamma_1),
S_D = S \cap (\Gamma_0 \setminus (\Gamma_0 \cap \Gamma_1)),
S_N = S \cap (\Gamma_1 \setminus (\Gamma_0 \cap \Gamma_1)).
\]

So every blow-up point must fall in one and only one of the above 4 classes. We shall see later that $S$ contains the set of peaks of sequence $\{u_{p_n}\}$. By a peak $P \in \overline{\Omega}$ we mean that $\{u_{p_n}\}$ doesn’t vanish in $L^\infty$ norm in any small neighborhood of $P$. Theorem 1.1 in particular implies that the set of peaks of $\{u_p\}$ is not empty. In this paper we are mainly concerned with $S_I$ and $S_N$. We will use $\#S_I$ ($\#S_N$) to denote the cardinality of $S_I$ ($S_I$ respectively). Our second result says

**Theorem 1.2** For a domain $\Omega$ with the properties stated in the beginning of this article, we have

1. $S_D = \emptyset$, $\#(S_I \cup S_C \cup S_N) \geq 1$;

2. $\#S_I + \frac{1}{2} \#S_N \leq 1$

if $\Gamma_1$ is smooth;

3. $S_I = \emptyset$, and $\#S_N = 1$

if $\Gamma_1$ has convex corners; furthermore in this case if $x_0$ is the point in $S_N$, $x_0$ must be a corner point with the least angle among all the corners on $\Gamma_1$. 

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Here by a convex corner, we mean a corner having angle less than $\pi$.

We shall also see that under extra condition of $\Omega$, $\Gamma_0$ and $\Gamma_1$, $u_p$ can develop only one peak on the Neumann boundary $\Gamma_1$. We would like to point out that like what we did in [11], most of our results can be extended to higher dimensions with $\Delta$ replaced by $\Delta_N$, the $N$-Laplacian operator ($\Delta_N u = \text{div}((|\nabla u|^{N-2}\nabla u)$), in (1.1) if $\Omega$ is a domain in $R^N$. However, we don’t know anything about $S_C$ if $\Gamma_0 \cap \Gamma_1$ is non-empty.

Our paper is organized as follows. In section 2, we give some background materials for the mixed boundary value problem. Then in section 3, we prove the decay rate of $c_p$. We prove theorem 1.1 in section 4. In section 5, we present some $L^1$ estimates. Section 6 is devoted to the proof of theorem 1.2. Finally we consider some special domains and some examples in section 7.

## 2 Preliminaries

Let $\Omega$ be a domain in $R^2$ with conditions stated in the beginning of this article. Let $\Gamma_0$ and $\Gamma_1$ be two parts of the boundary of $\Omega$ with $\Gamma_0$ having positive one dimensional Hausdorff measure. We recall that the isoperimetric constant of $\Omega$ relative to $\Gamma_1$, $Q(\Gamma_1, \Omega)$, is defined to be

$$Q(\Gamma_1, \Omega) = \sup \frac{|E|}{P_\Omega(E)}$$

(2.1)

where the supremum is taken over all measurable sets of $\Omega$ such that $\partial E \cap \Gamma_0$ has 1-dimensional Hausdorff measure 0, and $P_\Omega(E)$ denotes the De Giorgi perimeter of $E$ relative to $\Omega$, i.e.

$$P_\Omega(E) = \sup \{ \int_E \text{div}\psi dx : \psi \in [C_0^\infty(\Omega)]^2, |\psi| \leq 1 \}.$$  

(2.2)

Some properties of $P_\Omega(E)$ are stated in [8] and [6]. We also refer to [14] and [3] for more information about De Giorgi perimeter and isoperimetric inequalities. In particular we notice that

$$Q(\Gamma_1, \Omega) \geq (2\pi^{1/2})^{-1}$$

where the second is the absolute isoperimetric constant; and if $H^3(\Gamma_1) > 0$,

$$Q(\Gamma_1, \Omega) \geq (2\pi/2)^{-1/2}.$$  

From here we deduce that if $H^1(\Gamma_1) > 0$ and $Q(\Gamma_1, \Omega) < \infty$, there exists $\alpha \in [0, \pi]$ such that $Q(\Gamma_1, \Omega) = (\sqrt{2\alpha})^{-1}$ where $\alpha$ is the angle of the unitary sector $\Sigma(\alpha, 1) = \{x = (r, \theta) \in R^2 : 0 \leq r \leq 1, \ \theta \in [0, \alpha]\}$. We denote by $E_\alpha$ the class of all pairs $(\Gamma_1, \Omega)$ of the type considered above such that

$$Q(\Gamma_1, \Omega) = (\sqrt{2\alpha})^{-1}.$$  

(2.3)
By virtue of an isoperimetric inequality described in [6], any pair of a convex sector and its non-circular boundary \((\Gamma_1, \Sigma(\alpha, 1))\) belongs to \(\mathcal{E}_\alpha\) once we denote by \(\Gamma_0\) the circular part of \(\Sigma(\alpha, 1)\). Therefore
\[
Q(\Gamma_1, \Sigma(\alpha, 1)) = (\sqrt{2\alpha})^{-1}
\]
if \(\Sigma(\alpha, 1)\) is a convex sector. By the way, if \((\Gamma_1, \Omega) \in \mathcal{E}_\alpha\) and \(\beta\) is the smallest angle among all convex corners on \(\Gamma_1\),
\[
\beta \geq \alpha. \tag{2.4}
\]

Recall \(V(\Gamma_1, \Omega)\) the Hilbert space defined in section 1. Assuming \((\Gamma_1, \Omega) \in \mathcal{E}_\alpha\) for some \(\alpha \in [0, \pi]\), we have the following two dimensional Moser type embedding while the proof of this result in any dimension can be found in [6]. Also see [7].

**Proposition 2.1** There exists a universal constant \(C\) such that
\[
\int_\Omega \exp[\frac{2\alpha|u|^2}{\|\nabla u\|_{L^2(\Omega)}^2}] \leq C|\Omega|
\]
for any \(u \in V(\Gamma_1, \Omega)\) with \((\Gamma_1, \Omega) \in \mathcal{E}_\alpha\).

We also need some results concerning the relative isoperimetric constants near the boundary \(\Gamma_1\). Let us fix our notation first. For each smooth point \(x \in \Gamma_1\), we can associate a smooth flattening map \(\Phi_x\) in a neighborhood of \(x\) that maps the neighborhood of \(x\) to a neighborhood of \((0,0)\) in
\[
\{y \in \mathbb{R}^2 : y = (y_1, y_2), y_2 > 0\}
\]
and maps \(\Gamma\) near \(x\) to
\[
\{y \in \mathbb{R}^2 : y = (y_1, y_2), y_2 = 0\}
\]
near \((0,0)\). For a corner point \(x\) on \(\Gamma_1\) we associate a similar map \(\Phi_x\) in a neighborhood of \(x\) that maps the neighborhood of \(x\) to a neighborhood of \((0,0)\) in
\[
\{y \in \mathbb{R}^2 : y = (\rho \cos \theta, \rho \sin \theta), 0 \leq \theta \leq \beta\}
\]
where \(\beta\) is the angle of corner at \(x\) and that maps the boundary near \(x\) to the boundary near \((0,0)\). We further require that \(D\Phi_x = I\) at \(x\) and \(\Phi_x\) varies smoothly with respect to \(x\). From now on throughout the rest of this paper, for any \(x\) on \(\Gamma_1\), by a ball \(B_r(x_0)\), we mean \(\Phi_x^{-1}(B_r(0,0))\). Clearly it is well-defined if \(r\) is small. We can now state the following result concerning the asymptotic behavior of the relative isoperimetric constants and the quantities \(\alpha\) defined in (2.3) of \((\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0))\).
Proposition 2.2  
1. Let \( x_0 \in \Gamma_2 \) such that \( \Gamma_2 \) is smooth near \( x_0 \). Then as \( r \to 0 \),
\[
Q(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0)) \to \frac{1}{\sqrt{2\pi}},
\]
i.e.
\[
\alpha(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0)) \to \pi
\]
where \( \alpha(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0)) \) is the angle of the unit sector whose relative isoperimetric constant is the same as the one of \( (\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0)) \).

2. Let \( x_0 \in \Gamma_2 \) such that \( x_0 \) is the vertex of a convex corner with angle \( \beta_0 \) in \( \Gamma_2 \). Then as \( r \to 0 \),
\[
Q(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0)) \to \frac{1}{\sqrt{2\beta_0}}
\]
i.e.
\[
\alpha(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0)) \to \beta_0.
\]

To prove, one just invokes the variable change formula in standard integration theory to compare the relative isoperimetric constants above with the relative isoperimetric constants of sectors computed in [6]. We leave the details of this argument to reader.

3 Some Estimates for \( c_p \)

Recall \( c_p \) defined in (1.2). We have the following refined Sobolev embedding.

Lemma 3.1 For every \( t \geq 2 \) there is \( D_t \) such that
\[
\|u\|_{L^t} \leq D_t t^{1/2} \|\nabla u\|_{L^2}
\]
for all \( u \in V(\Gamma_1, \Omega) \) with \( (\Gamma_1, \Omega) \in \mathcal{E}_\alpha \); furthermore
\[
\lim_{t \to \infty} D_t = (4\alpha e)^{-1/2}.
\]

Proof. Let \( u \in V(\Gamma_1, \Omega) \). We know
\[
\frac{1}{\Gamma(s+1)} x^s \leq e^x
\]
for all \( x \geq 0, s \geq 0 \) where \( \Gamma \) is the \( \Gamma \) function. Using proposition 2.1, we have
\[
\int_{\Omega} \exp[2\alpha(\frac{u}{\|\nabla u\|_{L^2}})^2] dx \leq C|\Omega|
\]
where $C$ doesn’t depend on any thing and $|\Omega|$ is the Lebesgue measure of $\Omega$. Therefore
\[
\frac{1}{\Gamma\left(\frac{t}{2} + 1\right)} \int_{\Omega} u' \, dx
= \frac{1}{\Gamma\left(\frac{t}{2} + 1\right)} \int_{\Omega} \left[2\alpha\left(\frac{u}{\|\nabla u\|_{L^2}}\right)^{2\alpha} \right]^{t/2} \, dx \cdot (2\alpha)^{-t/2} \|\nabla u\|_{L^2}^t \\
\leq \int_{\Omega} \exp\left(2\alpha\left(\frac{u}{\|\nabla u\|_{L^2}}\right)^2\right) \, dx \cdot (2\alpha)^{-t/2} \|\nabla u\|_{L^2}^t \\
\leq C|\Omega|(2\alpha)^{-t/2} \|\nabla u\|_{L^2}^t
\]
Hence
\[
\left(\int_{\Omega} u' \, dx\right)^{1/t} \leq \left(\Gamma\left(\frac{t}{2} + 1\right)\right)^{1/t} C^{1/t}(2\alpha)^{-1/2} |\Omega|^{1/t} \|\nabla u\|_{L^2(\Omega)}^{1/t}
\]
Notice that, according to Stirling’s formula,
\[
\left(\Gamma\left(\frac{t}{2} + 1\right)\right)^{1/t} \sim \left(\frac{4}{e}t^{1/2}\sqrt{t}\right)^{1/t} \sim \left(\frac{1}{2e}\right)^{1/2} t^{1/2}
\]
where $0 < \theta_t < \frac{1}{2e}$. Choosing $D_t$ to be
\[
\left(\Gamma\left(\frac{t}{2} + 1\right)\right)^{1/t} C^{1/t}(2\alpha)^{-1/2} |\Omega|^{1/t} t^{-1/2}
\]
we get the desired result. □

An immediate consequence is

**Corollary 3.2**

\[
\liminf_{p \to \infty} p^{1/2} c_p \geq (4\alpha e)^{1/2}.
\]

Next we prove an upper bound for $p^{1/2} c_p$.

**Lemma 3.3** For domains $\Omega$ with smooth $\Gamma_1$

\[
\limsup_{p \to \infty} p^{1/2} c_p \leq (4\pi e)^{1/2};
\]

if the domain $\Omega$ has convex corners on $\Gamma_1$,

\[
\limsup_{p \to \infty} p^{1/2} c_p \leq (4\beta e)^{1/2}
\]

where $\beta$ is the smallest angle among all convex corners on $\Gamma_1$. 

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Proof. Let us first assume that \( \Omega \) contains \( \{ (x_1, x_2) : x_2 > 0, \ x_1^2 + x_2^2 \leq L \} \) with \( \{ (x_1, x_2) : x_2 = 0, \ x_1^2 + x_2^2 \leq L \} \) being part of the Neumann boundary. We construct a Moser type test function near \((0, 0)\). Letting

\[
 m_l(x) = \frac{1}{\sqrt{\pi}} \begin{cases} 
 \frac{(\log L - \log l)^{1/2}}{\log l - \log |x|}, & 0 \leq |x| \leq l \\
 \frac{|\log L - \log l|^{1/2}}{0}, & |x| \geq L, 
\end{cases}
\] (3.1)

we have \( m_l \in V(\Gamma_1, \Omega) \), \( \| \nabla m_l \|_{L^2(\Omega)} = 1 \) and

\[
 \int_{\Omega} m_l^{p+1}(x) \, dx \\
 = \left[ \frac{1}{\sqrt{\pi}} (\log \frac{L}{l})^{1/2} \right]^{p+1} |B_l| \\
 + \left[ \frac{1}{\sqrt{2\pi}} (\log \frac{L}{l})^{-1/2} \right]^{p+1} \int_{l < |x| < L} (\log \frac{L}{|x|})^{p+1} \, dx \\
 := I_1 + I_2
\]

where

\[
 I_1 = \left[ \frac{1}{\sqrt{\pi}} (\log \frac{L}{l})^{1/2} \right]^{p+1} \frac{\pi l^2}{4} \\
 I_2 = \left[ \frac{1}{\sqrt{2\pi}} (\log \frac{L}{l})^{-1/2} \right]^{p+1} \int_{l < |x| < L} (\log \frac{L}{|x|})^{p+1} \, dx.
\]

Choosing \( l = Le^{-(p+1)/4} \), we have

\[
 \| m_l \|_{L^{p+1}} \geq I_1^{1/(p+1)} \\
 \geq \left[ \frac{1}{4\pi e} \right]^{1/2} (p + 1)^{1/2} (\pi L^2)^{1/(p+1)}.
\]

Hence

\[
 c_p \leq \left[ 4\pi e \right]^{1/2} (p + 1)^{-1/2} (\pi L^2)^{-1/(p+1)},
\]

i.e.

\[
 \limsup_{p \to \infty} p^{1/2} c_p \leq (4\pi e)^{1/2}.
\]

For a domain \( \Omega \) with smooth \( \Gamma_1 \), we can first flatten the boundary and construct the same test function with small \( L \). Sending \( L \) to 0, we still get the desired result.

If the domain \( \Omega \) has a corner on \( \Gamma_1 \), we can first transform it to a sector by a smooth map. Then we construct a similar test function on that sector. Finally we let \( L \) tend to 0. \( \square \)
Corollary 3.4 1. For domains $\Omega$ with smooth $\Gamma_1$,

$$\limsup_{p \to \infty} p \int_{\Omega} u_p^{p+1} \leq (4\pi e) \text{ and } \limsup_{p \to \infty} p \int_{\Omega} |\nabla u_p|^2 \leq 4\pi e.$$  

2. For domains $\Omega$ having convex corners on $\Gamma_1$,

$$\limsup_{p \to \infty} p \int_{\Omega} u_p^{p+1} \leq 4\beta e \text{ and } \limsup_{p \to \infty} p \int_{\Omega} |\nabla u_p|^2 \leq 4\beta e$$

where $\beta$ is the smallest angle among all convex corners on $\Gamma_1$.

Proof. From (1.3), we know

$$c_p = \frac{\|\nabla u_p\|_{L^2(\Omega)}}{\|u_p\|_{L^{p+1}(\Omega)}}.$$  

If we multiply (1.1) by $u_p$ and integrate by parts, we have

$$\int_{\Omega} |\nabla u_p|^2 = \int_{\Omega} u_p^{p+1}.$$  

Therefore

$$\int_{\Omega} u_p^{p+1} = c_p^{2(p+1)} \text{ and } \int_{\Omega} |\nabla u_p|^2 = c_p^{2(p+1)}. $$

The results follow immediately from lemma 3.3. \[ \square \]

As another consequence of lemma 3.3, we prove a crucial estimate for quantity

$$L_0 = \limsup_{p \to \infty} \frac{p \int_{\Omega} u_p^p}{e}. \quad (3.2)$$

The proof follows easily from lemma 3.3 and the Holder’s inequality.

Corollary 3.5 1. For domains $\Omega$ with smooth $\Gamma_1$

$$L_0 \leq 4\pi;$$

2. For domains $\Omega$ having convex corners on $\Gamma_1$,

$$L_0 \leq 4\beta$$

where $\beta$ is the smallest angle among all convex corners on $\Gamma_1$.  

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4 Proof of Theorem 1.1

A uniform lower bound indeed exists for any positive solutions to (1.1). Let $\lambda_1$ be the first eigenvalue of $-\Delta$ with the same boundary condition as the one in (1.1) and $\varphi$ be a corresponding positive eigenfunction. Then for any solution $u$

$$\int_{\Omega} [u \Delta \varphi - \varphi \Delta u] = \int_{\partial \Omega} [u \frac{\partial \varphi}{\partial \nu} - \varphi \frac{\partial u}{\partial \nu}] = 0. \quad (4.1)$$

Therefore

$$\int_{\Omega} (u^p - \lambda_1 u) \varphi = 0. \quad (4.2)$$

Hence

$$\|u\|_{L^\infty(\Omega)} \geq \frac{\lambda_1}{(p-1)} \to 1 \quad \text{as} \quad p \to \infty$$

which yields a uniform lower bound in $p$ for $\|u\|_{L^\infty(\Omega)}$ when $p > 1 + \epsilon$, $\epsilon > 0$.

To get an upper bound for $\{u_p\}$, we use an iteration argument. Define

$$\gamma_0 = \beta/\alpha \quad (4.3)$$

where $\beta$ is the smallest angle among all convex corners on $\Gamma_1$ and $(\Gamma_1, \Omega)$ is in class $E$.

Then $\gamma_0 \geq 1$ by (2.4). Let $\alpha_0$ be such that

$$\exp \alpha_0 = \gamma_0(1 + \alpha_0). \quad (4.4)$$

Fix $t$ and $\epsilon$ that will be chosen later. Letting $\nu = (1 + t)(p + 1)$, from lemma 3.1, we have

$$\left[ \int_{\Omega} u_p^{\nu} \right]^{1/\nu} \leq (4\alpha e)^{-1/2} E_{(1+t)(p+1)}^{1/2} \|\nabla u_p\|_{L^2(\Omega)}$$

where

$$\lim_{p \to \infty} E_{(1+t)(p+1)} = 1. \quad (4.6)$$

But from corollary 3.4 we know

$$\limsup_{p \to \infty} p \int_{\Omega} |\nabla u_p|^2 \leq 4\beta e. \quad (4.6)$$

Hence there exists $P_0$ such that for all $p > P_0$,

$$\int_{\Omega} u_p^{\nu} \leq (\gamma_0(1 + t + \epsilon))^{\nu/2}. \quad (4.5)$$

Multiplying (1.1) both side by $u_p^{2s-1}$, we get, after integrating by parts,

$$\frac{2s-1}{s^2} \int_{\Omega} |\nabla u_p|^2 = \int_{\Omega} u_p^{p-1+2s}. \quad (4.6)$$
Using lemma 3.1 again, we have
\[
\left[ \int_\Omega u_p^{s_j} \right]^{1/\nu} \leq D_{\nu s} \nu \frac{s_0^2}{2 - 1} \int_\Omega u_p^{-1+2s} \\
\leq C_1 \nu s \int_\Omega u_p^{-1+2s}
\]
where $D_{\nu s}$ is defined in lemma 3.1 and $C_0$ and $C_1$ are constant independent of $p > P_1$. Hence we have
\[
\left[ \int_\Omega u_p^{s_j} \right]^{2/\nu} \leq C_1 \nu s \int_\Omega u_p^{-1+2s}.
\]  
(4.7)

We now define two sequences $\{s_j\}$ and $\{M_j\}$ by
\[
\left\{
\begin{array}{l}
p - 1 + 2s_0 = \nu \\
p - 1 + 2s_{j+1} = \nu s_j \\
M_0 = \gamma_0 (1 + t + \epsilon)^{\nu s_j} \\
M_{j+1} = [C_1 \nu s_j M_j]^{\nu s_j/2}
\end{array}
\right.
\]  
(4.8)

where $C_1$ is the constant in (4.7). From (4.5) and (4.7), we have, by induction,
\[
\int_\Omega u_p^{s_j} \leq M_j.
\]  
(4.9)

Next we claim
\[
M_j \leq \exp[m(\gamma_0, t, p, \epsilon) \nu s_{j-1}]
\]  
(4.10)

where $m(\gamma_0, t, p, \epsilon)$ is a constant depending on $\gamma_0, t, p, \epsilon$ and
\[
\lim_{p \to \infty} m(\gamma_0, t, p, \epsilon) = \frac{1 + t}{2t} \log[\gamma_0 (1 + t + \epsilon)].
\]

In fact, we can write down $\{s_j\}$ explicitly.
\[
s_j = \left( \frac{\nu}{\nu - 2} \right) \left( \frac{\nu}{2} \right)^{j+1} (\nu - 1 - p - 1) + p - 1.
\]  
(4.11)

Put
\[
\sigma_j = \frac{\nu}{2} \log(C_1 \nu s_j), \quad \mu_j = \log M_j.
\]

Hence
\[
\mu_{j+1} = \frac{\nu \mu_j}{2} + \sigma_j.
\]
Therefore it is easy to see
\[\sigma_j = \nu \left( \log \sqrt{\nu \tau_0} + \log \left( \frac{\nu}{2} \right)^{j+1} \left( \nu - p - 1 \right) \right) \leq \nu \log \sqrt{\nu \tau_0 (j + 1)}.
\]

Now we define \(\{\tau_j\}\) by
\[\tau_j = \frac{\nu}{2} \nu \tau_{j+1} + \left( \nu \log \sqrt{2C_1} \nu \right) (j + 1).
\]

Clearly \(\theta_j \leq \tau_j\). Moreover we have
\[
\tau_j = \left( \frac{\nu}{2} \right)^j \mu_0 + 2\nu \log \sqrt{2C_1} \nu \nu (j + 1) \leq \mu_0 + 2\nu \log \sqrt{2C_1} \nu \left( \nu - 2 \right)^{-1} (\nu - p - 1) \nu s_{j-1}
\]
where
\[\lim_{\nu \to \infty} m(\gamma_0, t, p, \epsilon) = \frac{1 + t}{2t} \log [\gamma_0 (1 + t + \epsilon)].
\]

Therefore we get
\[\|u_p\|_{L^{\nu s_{j-1}}(\Omega)} \leq \exp [m(\gamma_0, t, p, \epsilon)].\]

Sending \(j \to \infty\), we see
\[\|u_p\|_{L^\infty(\Omega)} \leq \exp [m(\gamma_0, t, p, \epsilon)].\]

Sending \(p \to \infty\), we have
\[\limsup_{p \to \infty} \|u_p\|_{L^\infty} \leq [\gamma_0 (1 + t + \epsilon)]^{\frac{1+t}{2t}}.
\]

Sending \(\epsilon \to 0\), we deduce
\[\limsup_{p \to \infty} \|u_p\|_{L^\infty} \leq [\gamma_0 (1 + t)]^{\frac{1+t}{2t}}.
\]

If we let \(f(t) = [\gamma_0 (1 + t)]^{\frac{1+t}{2t}}\), standard calculus argument shows that \(\log f(t)\) achieves its minimum at \(\alpha_0\) where
\[\alpha_0 = \log [\gamma_0 (1 + \alpha_0)].\]
defined in (4.4). So we obtain
\[ \limsup_{p \to \infty} \|u_p\|_{L^\infty} \leq \exp \frac{1 + \alpha_0}{2}. \]
\[ \Box \]

We include a consequence of theorem 1.1 here which will be used later.

**Corollary 4.1** There exist \( C_1 \) and \( C_2 \) such that
\[ \frac{C_1}{p} \leq \int_\Omega u_p^p \leq \frac{C_2}{p} \]

**Proof.** The first inequality follows from theorem 1.1 and the first limit of corollary 2.3; the second inequality follows from the first limit of corollary 2.3 through an interpolation argument. \( \Box \)

## 5 Some Apriori Estimates

In this section we collect some less well-known estimates for \( \Delta \) on two dimensional domains.

We first state a boundary estimate lemma. The proof of the lemma is standard. One combines the moving plane method in [4] with a Kelvin transform. We refer to [2] and [4] for details. This lemma actually excludes the possibility that \( u_p \) develop a peak on \( \Gamma_0 \). See Remark 6.5.

**Lemma 5.1** Let \( u \) be a positive solution of
\[
\begin{align*}
\Delta u + f(u) &= 0 \text{ in } \Omega \subset \mathbb{R}^2 \\
\left. u \right|_{\Gamma_0} &= 0
\end{align*}
\]
where \( \Gamma_0 \) is a smooth piece of \( \partial \Omega \) and \( f \) is a smooth function. Then for every \( \Gamma \subset\subset \text{int (} \Gamma_0 \text{)} \) with respect to the relative topology of \( \partial \Omega \) there exist a neighborhood \( \omega \) of \( \Gamma \) and a constant \( C \) both depending on the geometry of \( \Omega \) and \( \Gamma \) only such that
\[ \|u\|_{L^\infty(\omega)} \leq C\|u\|_{L^1(\Omega)}. \]

Next we state an \( L^1 \) estimate of H. Brezis and F. Merle, theorem 1 [1].

**Lemma 5.2** Let \( u \) be a solution of
\[
\begin{align*}
-\Delta u &= f \text{ in } \Omega \\
\left. u \right|_{\partial \Omega} &= 0
\end{align*}
\]
where \( \Omega \) is a smooth bounded domain in \( \mathbb{R}^2 \). We have for \( 0 < \epsilon < 4\pi \)
\[
\int_\Omega \exp\left[\frac{(4\pi - \epsilon)|u(x)|}{\|f\|_{L^1}}\right] dx \leq \frac{4\pi \text{Area}(\Omega)}{\epsilon}.
\]
Remark 5.3 In their paper, Brezis and Merle used \((\text{Diameter}(\Omega))^2\) instead of \(\text{Area}(\Omega)\) in lemma 5.2. It turns out from the following symmetrization approach that \(\text{Area}(\Omega)\) is more appropriate.

We need a similar \(L^1\) estimate as above to take care of the mix boundary condition.

Lemma 5.4 Let \(u\) be a solution of
\[
\begin{aligned}
-\Delta u &= f \text{ in } \Omega \\
|u|^\gamma &= 0 \\
\frac{\partial u}{\partial \nu}|_{\Gamma} &= 0 \\
\end{aligned}
\]
where the boundary condition is the same as the one in (1.1) and \((\Gamma_1, \Omega) \in \mathcal{E}_\alpha\). Then for every \(0 < \epsilon < 2\alpha\),
\[
\int_{\Omega} \exp\left[\frac{(2\alpha - \epsilon)|u(x)|}{\|f\|_{L^1}}\right] \, dx \leq \frac{2\alpha \text{Area}(\Omega)}{\epsilon}.
\]

Proof. Because of the maximum principle, we may assume \(f \geq 0\). Otherwise we just replace \(f\) by \(|f|\). We use the symmetrization approach here. Let \(\Sigma(\alpha, R)\) be the sector having the same area as \(\Omega\) and the same relative isoperimetric constant as \(\Omega\). Define as in [8] the \(\alpha\)-symmetrization to be the transformation that associates \(u(x)\) with \(u_\alpha := u_*(\frac{x}{2}|x|^2)\) for \(x \in \Sigma(\alpha, R)\) where \(u_*\) is the standard decreasing rearrangement. Namely
\[
u_* := \inf\{t \geq 0 : \mu(s) < t\}
\]
and
\[
\mu(t) = \text{meas}\{x \in \Omega : |u(x)| > t\}.
\]
\(u_\alpha\) has the similar properties to those of standard Schwartz symmetrization. In particular
\[
\int_{\Omega} F(u(x)) \, dx = \int_{\Sigma(\alpha, R)} F(u_\alpha(x)) \, dx.
\]
for real Borel function \(F\). Moreover, let \(u\) be a solution to the equation in lemma 5.4, and \(v\) be the solution of
\[
\begin{aligned}
-\Delta v &= f_\alpha \text{ in } \Sigma(\alpha, R) \\
|v|^\gamma &= 0 \\
\frac{\partial v}{\partial \nu}|_{\Gamma_1} &= 0 \\
\end{aligned}
\]
where
\[
\overline{\Gamma}_0 = \{x \in \partial \Sigma(\alpha, R) : |x| = R\},
\]
\[ \Gamma_1 = \{ x \in \partial \Sigma(\alpha, R) : |x| \leq R \} \]

and \( f_\alpha \) is the \( \alpha \)-symmetrization of \( f \). Standard argument shows that \( v \) is radially symmetric. From [8], we assert

\[ u_\alpha(x) \leq v(x) \quad (5.2) \]

where \( u_\alpha \) is the \( \alpha \)-symmetrization of the solution \( u \) in lemma 5.4. But since it is radially symmetric, \( v \) satisfies

\[
\begin{cases}
  v''(t) + \frac{1}{t}v'(t) + f_\alpha(t) = 0 \\
  v'(0) = 0 \\
  v(R) = 0.
\end{cases}
\]

Therefore solving the O.D.E., we have

\[
v(r) \leq \log\left(\frac{R}{r}\right) \int_0^R s f_\alpha(s) ds;
\]

\[
\int_{\Sigma(\alpha, R)} \exp\left(\frac{(2\alpha - \epsilon)v}{\|f_\alpha\|_{L^1(\Omega)}}\right) \leq \frac{2\alpha \text{Area}(\Sigma(\alpha, R))}{\epsilon} = 2\alpha \text{Area}(\Omega) \frac{\epsilon}{\epsilon}.
\]

Combining this with (5.1) and (5.2), we have the desired result. \( \square \)

6 Proof of Theorem 1.2

lemma 5.4 implies that \( \{v_p\} \) is uniformly bounded in \( L^1(\Omega) \). Therefore lemma 5.1 implies that \( \{v_p\} \) is uniformly bounded in \( L^\infty(\omega) \) where \( \omega \) is a neighborhood of any compact subset of \( \text{int}(\Gamma_0) \). Since

\[
\max_{x \in \Omega} v_n(x) \geq \frac{C}{\nu_p} \to \infty,
\]

from theorem 1.1 and corollary 4.1, we deduce \( S \neq \emptyset \). However, since \( S_D = \emptyset \), we conclude that \( \#(S_I \cup S_C \cup S_N) \geq 1 \). This proves part 1. To prove the rest of the theorem, define

\[
L_0 = \lim_{p \to \infty} \frac{\nu_p}{\epsilon}
\]

(6.1)

where

\[
\nu_p = \int_\Omega u_p^p.
\]

(6.2)

We denote any sequence \( u_{p_n} \) of \( u_p \) with \( p_n \to \infty \) by \( u_n \). Let

\[
v_n := \frac{v_{p_n}}{\nu_{p_n}}.
\]

(6.3)
\[ f_n := f_{p_n} := \frac{u_{p_n}}{\int_{\Omega} u_{p_n}^p} = \nu_{p_n} v_n. \] (6.4)

Because
\[ \int_{\Omega \cup \Gamma_1} f_n = 1, \]
we can subtract a subsequence of \( f_n \), still denoted by \( f_n \), so that there is a positive bounded measure \( \mu \) in \( M(\Omega \cup \Gamma_1) \), the set of all real bounded Borel measures on \( \Omega \cup \Gamma_1 \), such that
\[ \int_{\Omega \cup \Gamma_1} f_n \varphi \to \int_{\Omega \cup \Gamma_1} \varphi d\mu \] (6.5)
for all \( \varphi \in C^\infty_0(\Omega \cup \Gamma_1) \).

Recall \( S_I \) and \( S_N \) defined in (1.6). For any \( \delta > 0 \) we call \( x_0 \in \Omega \cup (\Gamma_1 \setminus (\Gamma_1 \cap \Gamma_0)) \) a \( \delta \)-regular point if
- \( x_0 \in \Omega \) and there is \( \varphi \in C_0(\Omega) \), \( 0 \leq \varphi \leq 1 \), \( \varphi = 1 \) in a neighborhood of \( x_0 \), such that
  \[ \int_{\Omega \cup \Gamma_1} \varphi d\mu \leq \frac{4\pi}{L_0 + 2\delta} \] (6.6)
  where \( L_0 \) is the quantity defined in (3.2), or
- \( x_0 \in \Gamma_1 \setminus (\Gamma_1 \cap \Gamma_0) \) and there is \( \varphi \in C_0(\Omega \cup \Gamma_1) \), \( 0 \leq \varphi \leq 1 \), \( \varphi = 1 \) in a neighborhood of \( x_0 \), such that
  \[ \int_{\Omega \cup \Gamma_1} \varphi d\mu \leq \frac{2\alpha(x_0)}{L_0 + 2\delta} \] (6.7)
  where \( \alpha(x_0) := \lim_{r \to 0} \alpha(\Gamma_1 \cap B_r(x_0), \Omega \cap B_r(x_0)) \) considered in proposition 2.2.

We let \( \alpha(x_0) = 2\pi \) if \( x_0 \in \Omega \). We say \( x_0 \in \Omega \cup \Gamma_1 \setminus (\Gamma_0 \cap \Gamma_1) \) is \( \delta \)-irregular if \( x_0 \) is not \( \delta \)-regular. Clearly
\[ \mu(x_0) \geq \frac{2\alpha(x_0)}{L_0 + 2\delta} \]
for all \( \delta \)-irregular point \( x_0 \).

**Lemma 6.1** If \( x_0 \) is a \( \delta \)-regular point for \( \delta > 0 \), then \( \{v_n\} \) is uniformly bounded in \( L^\infty(B_{R_0}(x_0) \cup \overline{\Omega}) \) for some \( R_0 > 0 \).

**Proof.** We first consider the case where \( x_0 \in \Gamma_1 \setminus (\Gamma_0 \cap \Gamma_1) \). Let \( x_0 \) be a \( \delta \)-regular point on \( \Gamma_1 \setminus (\Gamma_0 \cap \Gamma_1) \). Then there exists \( R_0 \) such that
\[ \int_{B_{R_0}(x_0) \cap \Omega} f_n \leq \frac{2\alpha(x_0)}{L_0 + \delta} \]
for \( n \) large enough.

Split \( v_n \) into two parts, \( v_n = v_{1n} + v_{2n} \) where \( v_{1n} \) solves

\[
\begin{align*}
\Delta v_{1n} + f_n &= 0 \text{ in } B_{R_0}(x_0) \cap \Omega \\
v_{1n} &= 0 \text{ on } \partial B_{R_0}(x_0) \cap \Omega \\
\frac{\partial v_{1n}}{\partial \nu} &= 0 \text{ on } B_{R_0}(x_0) \cap \Gamma_1
\end{align*}
\]

and \( v_{2n} \) solves

\[
\begin{align*}
\Delta v_{2n} &= 0 \text{ in } B_{R_0}(x_0) \cap \Omega \\
v_{2n} &= v_n \text{ on } \partial B_{R_0}(x_0) \cap \Omega \\
\frac{\partial v_{2n}}{\partial \nu} &= 0 \text{ on } B_{R_0}(x_0) \cap \Gamma_1
\end{align*}
\]

Then \( v_{1n} \leq v_n \) and \( v_{2n} \leq v_n \) by the maximum principle. Now from the standard elliptic boundary estimate for harmonic functions with Neumann data, we have

\[
\|v_{2n}\|_{L^\infty(B_{R_0}(x_0) \cap \Omega)} \leq C\|v_{2n}\|_{L^1(B_{R_0}(x_0) \cap \Omega)} \leq C'
\]

where \( C' \) is a constant independent of \( n \) and the last inequality follows from lemma 5.4. So we need only to estimate \( v_{1n} \).

We first claim that when \( n \) is large enough

\[
f_n(x) \leq \exp(L_0 + \delta/2)v_n(x)
\]

for all \( x \in \Omega \).

Now observe

\[
\log x \leq \frac{x}{e}
\]

for \( x > 0 \). We have

\[
\begin{align*}
p_n \log \frac{u_n}{u_n^{1/p_n}} &\leq \frac{p_n}{e} \frac{u_n}{u_n^{1/p_n}} \\
&\leq \frac{L_0 + \delta/3}{v_n} \frac{u_n}{v_n^{1/p_n}} \leq \frac{t' - \delta/6}{v_n} \frac{u_n}{v_n^{1/p_n}} \leq t' \frac{u_n}{v_n}
\end{align*}
\]

for \( n \) large enough because

\[
\lim_{n \to \infty} v_n^{1/p_n} = 1
\]

which follows from corollary 4.1. Hence

\[
f_n \leq \exp[(L_0 + \delta/2)v_n]
\]

Next we claim that \( \{f_n\} \) is uniformly bounded in \( L^{1+\delta_0}(B_{R_1/2}) \) for \( \delta_0 \) sufficiently small. Because \( \{v_{2n}\} \) is uniformly bounded in \( B_{R_1/2}(x_0) \), we see from the previous claim

\[
\int_{B_{R_1/2}} f_n^{1+\delta_0} \leq \int_{B_{R_1/2}} \exp[(1 + \delta_0)(L_0 + 0.5\delta)v_n]
\]
\[
\leq C \int_{B_{R_1/2}} \exp[(1 + \delta_0)(L_0 + 0.5\delta)v_{1n}]
\leq C \int_{B_{R_1/2}} \exp\left(\frac{4\pi(1 + \delta_0)L_0 + 0.5\delta}{L_0 + 0.5\delta}v_{1n}\right) \leq C'
\]

with the aid of lemma 5.4 if we choose \(\delta_0\) sufficiently small. So we have proved
the claim 6.10.

Now take \(B_{R_1/4}(x_0)\). We conclude from the weak Hanack inequality (Theorem 8.17, [5]),
\[
\|v_n\|_{L^\infty(B_{R_1/4}(x_0))} \leq C\|v_n\|_{L^2(B_{R_1/2}(x_0))} + \|f_n\|_{L^{1+\delta_0}(B_{R_1/2}(x_0))} \leq C.
\]

Here the boundedness of \(\{v_n\}\) in \(L^2(B_{R_1/2}(x_0))\) follows from lemma 5.4.

The case where \(x_0 \in \Omega\) is similar. We just use lemma 5.2 in place of lemma 5.4. \(\square\)

**Lemma 6.2** For any \(\delta > 0\), \(x_0 \in S_I \cup S_N\) if and only if \(x_0\) is \(\delta\)-irregular.

**Proof.** Let \(x_0\) be a \(\delta\)-irregular point. Then by lemma 6.1, \(\{v_n\}\) is bounded in \(L^\infty(B_{R_1} \cap \Omega)\) for some \(R_1\). Hence \(x_0 \notin S_I \cup S_N\). Conversely suppose \(x_0\) is a \(\delta\)-irregular point. Then we have for every \(R > 0\)
\[
\lim_{n \to \infty} \|v_n\|_{L^\infty(B_R(x_0) \cap \Omega)} = \infty.
\]

Otherwise, there would be some \(R_0 > 0\) and a subsequence, still denoted by \(\{v_n\}\), such that
\[
\|v_{1n}\|_{L^\infty(B_{R_0}(x_0) \cap \overline{\Omega})} \leq C
\]
for some \(C\) independent of \(n\). Then
\[
f_n = v_n^{p_n-1}v_n^{p_n} \leq \left(\frac{M}{p_n}\right)^{p_n-1}C^{p_n} \to 0
\]
uniformly as \(n \to \infty\) on \(B_{R_0}(x_0) \cap \overline{\Omega}\). Here \(M\) is a uniform upper bound of \(u_p\) obtained in theorem 1.1. Then
\[
\int_{B_{R_0}(x_0) \cap \overline{\Omega}} f_n \leq \epsilon_0 \leq \frac{2\alpha(x_0)}{L_0 + 2\delta}
\]
which implies that \(x_0\) is a \(\delta\)-regular point. A contradiction. \(\square\)

Back to the measure \(\mu\) defined earlier in this section. Clearly we have
\[
1 \geq \mu(\Gamma_1 \cup \Omega) \geq \sum_{x_0 \in S_I \cup S_N} \frac{2\alpha(x_0)}{L_0 + 2\delta}
\]
which in turn, if we let \(\delta \to 0\), implies
Proposition 6.3

\[ \sum_{x_0 \in S_I \cup S_N} \alpha(x_0) \leq \frac{1}{2} L_0. \]

From this proposition, with the aid of proposition 2.2 and corollary 3.5, we obtain part 2 and part 3 of theorem 1.2.

Remark 6.4 We see that every peak \( P \) in \( \Omega \) is a blow-up point of \( v_p = \frac{u_p}{\nu_p} \) because by corollary 4.1 \( \nu_p \to 0 \) as \( p \to \infty \).

7 Further Results and Examples

In this section we shall focus on some special domains \( \Omega \) where the corresponding quantities \( L_0 \) are indeed smaller than what we get in corollary 3.5. In these special cases, we can actually prove that the solutions of (1.1) possess single-peaks on the Neumann boundary of \( \Omega \). Let us first formulate a general result.

Theorem 7.1 Let \( (\Gamma_1, \Omega) \) be a pair such that \( \alpha_0 \), defined in (4.4), with respect to this pair is strictly less than 1, i.e. \( \gamma_0 < \frac{\pi}{2} \). Then for every sequence \( \{u_{n_p}\} \) of solutions on \( \Omega \) with the Neumann boundary \( \Gamma_1 \), there is a subsequence, again denoted by \( \{u_{n_p}\} \), such that the interior blow-up set \( S_I \) is empty and the \( \Gamma_1 \)-boundary blow-up set \( S_N \) contains at most one point.

Proof. If we check the proof of lemma 6.1 carefully, we can see that we can use a refined inequality

\[ \frac{\log x}{x} \leq \frac{\log y}{y} \]

if \( x \leq y \leq e \) instead of (6.11). Notice that since we assume \( \alpha_0 < 1 \),

\[ \limsup_{n \to \infty} \frac{u_{n_p}}{\nu_{n_p}^{1/p_n}} \leq \exp \frac{1 + \alpha_0}{2} < e. \]

Let

\[ L_0' = \limsup_{n \to \infty} (1 + \alpha_0) \int_{\Omega} u_{n_p}^p \]

We still have, as proposition 6.3, with the aid of corollary 3.5,

\[ \sum_{x_0 \in S_I \cup S_N} \alpha(x_0) \leq \frac{1}{2} L_0' \leq 2\beta. \]  \hspace{1cm} (7.1)

If \( S_I \neq \emptyset \), then, with the aid of proposition 2.2, \( \alpha(x_0) = 2\pi \) for some \( x_0 \in S_I \).

If \#S_N \geq 2, then, with the aid of proposition 2.2 again, \( \alpha(x_1) + \alpha(x_2) \geq 2\beta \) for two different \( x_1 \) and \( x_2 \) in \( S_N \). In any case, we reach a contradiction to (7.1). \( \Box \)
Example 7.2 Let $\Omega = \{x \in R^2 : r < |x| < R\}$, $\Gamma_1 = \{x \in R^2 : |x| = r\}$ and $\Gamma_0 = \{x \in R^2 : |x| = R\}$.

In this case the constant $\alpha$ with respect to $(\Gamma_1, \Omega)$ is equal to $\pi$ (See Example 3.3 [8]) and the constant $\beta$ is clearly $\pi$. Hence $\gamma_0 = 1 < e/2$ and the condition of theorem 7.1 is satisfied. Indeed, since the two boundaries has no intersection, passing to a subsequence if necessary, $S_N = \{x_0\}$.

Example 7.3 Let $\Omega = \Sigma(\alpha, R)$, $0 \leq \alpha \leq \pi$, and $\Gamma_1$ be the union of two sides of the sector.

In this case $\beta = \alpha$ (see [6]). Hence $\gamma_0 = 1 \leq e/2$ and the condition of theorem 7.1 is again satisfied.

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References


