On a phase field problem driven by interface area and interface curvature *

Xiaofeng Ren [†‡]
Department of Mathematics and Statistics
The George Washington University
Washington, DC 20052, USA

Juncheng Wei [§]
Department of Mathematics
Chinese University of Hong Kong
Hong Kong, PRC

Abstract

A two component system driven by both interface area and interface curvature is studied with a new phase field model. We show that if the curvature impact in the system is strong enough, there exist bubble profiles. A bubble profile describes a pattern of an inner core of one component surround by an outer membrane of the other component. It is a radial solution to a fourth order nonlinear PDE. We show the existence of such profiles in all dimensions, although the profile is unstable if the dimension is greater than two.

1 Introduction

The bending energy plays a central role in the study of vesicle membranes formed by certain amphiphilic molecules [4, 5]. In the isotropic case it may be expressed as a surface integral [13, 22]

$$E_b = \int_\Gamma \{a_1 + a_2(\kappa - c_0)^2 + a_3 G\}\, ds. \tag{1.1}$$

Here Γ is a closed surface in R^3 representing a vesicle membrane, κ is the mean curvature of the surface, and G is the Gauss curvature of the surface. The constant a_1 represents the surface tension caused by the interaction effects between the vesicle material and the ambient fluid; a_2 is the bending rigidity and a_3 is the stretching rigidity, both of which are determined by the interaction properties of the amphiphilic molecules. The last constant c_0 is the spontaneous curvature describing an asymmetric effect.

In (1.1) the integral of the first term a_1 leads to the area of the surface Γ. The integral of the third term a_3G gives a topological invariant due to the Gauss-Bonnet Theorem. We may therefore ignore the third quantity. The most interesting part in (1.1) is the second term. In the case $c_0 = 0$, it is equation to a_2 times

$$\int_\Gamma \kappa^2\, ds. \tag{1.2}$$

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†Corresponding author. Phone: 1 435 797-0755; Fax: 1 435 797-1822; E-mail: ren@math.usu.edu

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The integral (1.2) is known as the Willmore functional [25].

The functional (1.1) may be studied by a diffusive interface method [9, 10]. To explain this approach, let us first recall the Allen-Cahn equation [1] that is often used to study the phase separation phenomenon in condensed matter physics. It is a second order nonlinear parabolic equation,

$$u_t(x,t) = \epsilon^2 \Delta u(x,t) - f(u(x,t)), \quad x \in D \subset R^n, \ t > 0, \tag{1.3}$$

with the Neumann boundary condition on ∂D. The parameter ϵ is positive and small. The nonlinear function f is a balanced cubic function, such as $f(u) = u(u-1/2)(u-1)$. It can be viewed as the negative gradient flow, in L^2 space, of the free energy functional

$$I_{ac}(u) = \int_D [\frac{\epsilon^2}{2}|\nabla u|^2 + F(u)]\, dx, \tag{1.4}$$

where $F(u) = \int_0^u f(u)\, du$ is the anti-derivative of f. If $f(u) = u(u-1/2)(u-1)$, then $F(u) = \frac{1}{4}u^2(1-u)^2$. A steady state $u = u(x)$ of (1.3), i.e. a critical point of (1.4), is a solution of

$$-\epsilon^2 \Delta u(x) + f(u(x)) = 0, \ \text{if } x \in D; \qquad \frac{\partial u(x)}{\partial \nu} = 0, \ \text{if } x \in \partial D \tag{1.5}$$

where ν is the outward normal vector to ∂D.

The free energy (1.4) models a two component system whose conformation solely depends on the area of the interfaces separating the two components. If $u(x)$ is close to 0, then the first component occupies x; if $u(x)$ is close to 1, then the second component occupies x. The interfaces separating the two components are the regions where $u(x)$ is somewhat greater than 0 and less than 1. Given a configuration $u(x)$ with x in an interface region, we may roughly interpret $-\epsilon^2\Delta u + f(u)$ as the mean curvature of the interface at x. The equation (1.5) then states that at an equilibrium state, the mean curvature of the interface must be everywhere equal to 0.

In such an interface area driven system, it is difficult for the two components to co-exist. Casten and Holland [3] (and Matano [15] independently) showed that when D is bounded and convex, any non-constant solution of (1.5) must be unstable. More recently in the study of polymer blends (see Tang and Freed [24]) an additional molecular weight dependent curvature term is found to contribute to the free energy. In this case one observes two immiscible homopolymers, one forming an outer membrane and the other constituting an inner core.

This morphology pattern may be explained phenomenologically through a very simple model motivated by (1.1). Let us consider the situation in two dimensions. Suppose that the two components are separated by a closed curve Γ in R^2. We propose that the free energy I_c of the system is given by

$$I_c(\Gamma) = \int_\Gamma \kappa^2\, ds + \gamma \int_\Gamma ds, \tag{1.6}$$

where s is the length element, κ is the curvature and $\gamma > 0$ is a parameter. If we assume that Γ is a circle of radius ρ, then the curvature is everywhere $\frac{1}{\rho}$ and (1.6) becomes

$$\frac{2\pi}{\rho} + 2\pi\gamma\rho, \tag{1.7}$$

A stable configuration is obtained by minimizing (1.7) with respect to ρ. One finds that

$$\rho = \frac{1}{\sqrt{\gamma}}. \tag{1.8}$$

In this paper we study a more sophisticated phase field version of (1.6). As in the Allen-Cahn approach we let u be the phase field variable of a two component system. Again $u(x) \approx 0$ means that x is taken by one component; $u(x) \approx 1$ means that x is taken by the other component. The free energy of the system is now

$$I(u) = \frac{1}{2} \int_D |\Delta u - f(u)|^2 \, dx + \gamma \int_D [\frac{1}{2} |\nabla u|^2 + F(u)] \, dx. \tag{1.9}$$

Here $\Delta u - f(u)$ plays the role of curvature and $\frac{1}{2}|\nabla u|^2 + F(u)$ plays the role of length element. The constant $1/2$ in front of the first integral is put there merely for simplicity.

We will study (1.9) in the general case of n dimensions, i.e. $D \subset R^n$ with n being a positive integer. Although in (1.6) we have assumed that γ is positive, here we allow γ to be negative if $n \geq 3$. In this paper we are only interested in the situation where (1.9) is sufficiently different from (1.4), so we assume that $|\gamma|$ is small.

The functional (1.9) is a phase field version of (1.6) in n dimensions. As (1.6) is generalized to a problem in R^n, Γ becomes an $n-1$ dimensional hyper-surface, κ the mean curvature of the surface and ds the surface element. In the $n = 2$ and $n \geq 4$ cases we are guided by the simple problem (1.6). However when $n = 1$ or $n = 3$, (1.6) does not yield much useful information. We must rely on (1.9) only and carry out some careful analysis.

The Euler-Lagrange equation of (1.9) is a fourth order partial differential equation

$$\Delta(\Delta u - f(u)) - f'(u)(\Delta u - f(u)) - \gamma(\Delta u - f(u)) = 0. \tag{1.10}$$

If D has a boundary, then we have the Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial(\Delta u - f(u))}{\partial \nu} = 0 \text{ on } \partial D. \tag{1.11}$$

If we introduce a new variable $v = \Delta u - f(u)$, then (1.10) may be written as a system

$$\Delta u - f(u) - v = 0, \quad \Delta v - f'(u)v - \gamma v = 0 \text{ in } D. \tag{1.12}$$

If D has a boundary, then u and v both should satisfy the Neumann boundary condition there.

In this paper we study the outer membrane/inner core pattern mentioned earlier using (1.9). More specifically we seek radially symmetric solutions of (1.10). The domain D is the entire space R^n. We require that the solutions $u = u(|x|) = u(r)$ satisfy the conditions

$$u(0) > 0, \quad \lim_{r \to \infty} u(r) = 0. \tag{1.13}$$

We often call such a solution a bubble profile. Recall that γ is either positive or negative, but $|\gamma|$ is sufficiently small. This means that the curvature term in the free energy (1.9) is significant and the problem is very different from the Allen-Cahn problem (1.4). Note that the Allen-Cahn problem does not have a bubble profile solution.

Our main results are the existence of bubble profiles in 1-dimension and 2-dimensions if γ is positive and sufficiently small, and the existence of bubble profiles in $n = 3$ and $n \geq 4$ dimensions if γ is negative and sufficiently close to 0.

These results are proved by the so-called localized energy method which is a combination of the Liapunov-Schmidt reduction argument and variational techniques. Let ρ be the location of the interface of bubble profile u in the sense that $u(\rho) = 1/2$. Near ρ, u has a rather particular shape. This shape is mostly described by a function H given in (2.1). When r is much less than ρ, $u(r)$ is close to 1; when r is much larger than ρ, $u(r)$ is close to 0. Much of our paper is devoted to locating ρ. We will see that as $\gamma \to 0$, $\rho \to \infty$. The construction is divided into two steps: in the first step, we fix ρ large and solve a nonlinear problem with an orthogonal condition. In the next step, we locate ρ by finding a critical point for a reduced energy function involving ρ only. For the localized energy method used in other problems, see [2, 6, 7, 11, 12, 21, 20].

There is a well know relationship between I_{ac} and $\int_\Gamma ds$, i.e. I_c without the curvature part: $\int_\Gamma ds$ is the Gamma-limit of I_{ac} to as $\epsilon \to 0$. See De Giorgi [8], Modica and Mortola [17], Modica [16], Kohn and Sternberg [14], etc, for this theory. We do not know if a Gamma-convergence type theory between I and I_c is available. The curvature part of I_c, i.e. $\int_\Gamma \kappa^2\, ds$, the Willmore functional [25] (also see Simon [23]). There are some partial results regarding the convergence of

$$\int_D [-\epsilon^2 \Delta u + f(u)]^2\, dx \tag{1.14}$$

to the Willmore functional (see Moser [18]).

Our paper is organized as follows. In Sections 2 and 3 we show the existence of the two dimensional bubble profile when γ is positive and small. In section 4 we show the existence of the bubble profile in the $n \geq 4$ case; the $n = 1$ case is studied in Section 5 and the $n = 3$ case in Section 5. Certain steps in these proofs are similar so we omit some details in the latter cases. We include a section to discuss the stability of the solutions. A technical estimate is proved in the appendix.

The function F in (1.9) is assumed to be smooth. $F(0) = F(1) = 0$ and $F(u) > 0$ if $u \neq 0$ and $u \neq 1$. We also assume that $F''(0) = F''(1)$. As $|u| \to \infty$, $F(u) \to \infty$.

We often encounter quantities that depend on ρ, the radius of a bubble. For instance we have a family of approximate solutions w that depend on both r, the radial coordinate, and ρ. We often write $w(r; \rho)$ to emphasize w's dependence on r and ρ. However when the dependence on ρ is less important in a piece of argument, we will write $w(r)$ instead.

We use $L^2_r(R^n)$ and $H^4_r(R^n)$ to denote the subspaces of radial functions in $L^2(R^n)$ and $H^4(R^n)$ respectively. The inner product in $L^2(R^n)$ is denoted by $\langle , \rangle$. We write $\phi \perp \psi$ if $\langle \phi, \psi \rangle = 0$. We use $\| \cdot \|_\infty$ to denote the L^∞-norm and $\| \cdot \|_2$ the L^2-norm of a function.

2 $n = 2$

In two dimensions, the phase field problem (1.9) is consistent with the simple model (1.6).

Theorem 2.1 *When γ is positive and sufficiently small, there exists a bubble profile. The radius of the bubble is* $\frac{1}{\sqrt{2\gamma}} + o(\gamma^{-1/2})$.

We let $H = H(y)$ be the solution of the following ODE on $(-\infty, \infty)$.

$$H'' - f(H) = 0, \quad y \in (-\infty, \infty), \quad \lim_{y \to -\infty} H(y) = 1, \quad \lim_{y \to \infty} H(y) = 0, \quad H(0) = 1/2. \tag{2.1}$$

We define a positive constant τ by

$$\tau = \int_R (H'(y))^2\, dy. \tag{2.2}$$

Because H has a first integral $\frac{1}{2}(H')^2 - F(H) = 0$ by (2.1), we also have

$$\tau = \int_0^1 \sqrt{2F(s)}\, ds. \tag{2.3}$$

The derivative of H, $H'(y)$, decays to 0 exponentially fast as $|y| \to \infty$. More precisely we have $a > 0$ and $k < 0$ such that

$$H'(y) = ke^{-a|y|} + O(e^{-2a|y|}). \tag{2.4}$$

Necessarily

$$a^2 = f'(0) = f'(1). \tag{2.5}$$

In the special case $f(H) = H(H - 1/2)(H - 1)$

$$H(y) = \frac{1}{2}[\tanh(-\frac{x}{2\sqrt{2}}) + 1], \text{ and } a = \frac{1}{\sqrt{2}}, \quad k = -\frac{1}{\sqrt{2}}. \tag{2.6}$$

For each

$$\rho \in (\frac{1}{2\sqrt{2\gamma}}, \frac{2}{\sqrt{2\gamma}}) \tag{2.7}$$

we construct an approximate solution w to (1.10) of the form

$$w(r;\rho) = H(r - \rho) + \beta(r;\rho) \tag{2.8}$$

where

$$\beta(r;\rho) = c_{1,\rho}e^{-ar} + c_{2,\rho}re^{-ar}. \tag{2.9}$$

The constants $c_{1,\rho}$ and $c_{2,\rho}$ are so chosen that $w'(0) = w'''(0) = 0$. More explicitly

$$c_{1,\rho} = \frac{H'(-\rho)}{a} + \frac{1}{2a}(H'(-\rho) - \frac{H'''(-\rho)}{a^2}), \tag{2.10}$$

$$c_{2,\rho} = \frac{1}{2}(H'(-\rho) - \frac{H'''(-\rho)}{a^2}). \tag{2.11}$$

Note that

$$c_{1,\rho} = \frac{H'(-\rho)}{a} + O(e^{-2a\rho}), \quad c_{2,\rho} = O(e^{-2a\rho}). \tag{2.12}$$

In this paper a weighted L^∞-norm is of particular importance. Define

$$\|\zeta\|_* = \sup_{r\geq 0} |\zeta(r)|e^{\mu|r-\rho|} \tag{2.13}$$

where μ is a small positive number. This μ is independent of γ and ρ. How small μ should be will become clear later.

We denote the left side of (1.10) by $S(u)$, i.e.

$$S(u) = (\Delta - f'(u) - \gamma)[\Delta u - f(u)] \tag{2.14}$$

where $\Delta - f'(u) - \gamma$ is viewed as an operator. The equation (1.10) becomes $S(u) = 0$.

Lemma 2.2 $\|S(w)\|_* = O(\gamma)$.

Proof. We start with an estimate of $\Delta w - f(w)$. Calculations show that

$$\Delta w - f(w) = \beta'' + \frac{H' + \beta'}{r} - f(w) + f(H).$$

We consider two cases of r: $r \in (0, \theta\rho)$ and $r \in (\theta\rho, \infty)$ where $\theta \in (0,1)$. In the first case

$$\beta'' - f(w) + f(H) = O(e^{-a\rho}),$$

and

$$\begin{aligned} \frac{H' + \beta'}{r} &= \frac{H'(r-\rho) - H'(-\rho)}{r} + \frac{\beta'(r) - \beta'(0)}{r} \\ &= O(e^{-(1-\theta)a\rho}) + O(e^{-a\rho}) \end{aligned}$$

by the mean value theorem, the decay rates of $H''(y)$ on $y \in (-\infty, -(1-\theta)\rho)$ and the fact that $\|\beta\|_\infty = O(e^{-a\rho})$. Hence

$$\Delta w - f(w) = O(e^{-(1-\theta)a\rho}). \tag{2.15}$$

Consequently

$$(\Delta w(r) - f(w(r)))e^{\mu|r-\rho|} = O(e^{(-(1-\theta)a+\mu)\rho}). \tag{2.16}$$

In the second case

$$\begin{aligned} (\beta'' - f(w) + f(H))e^{\mu|r-\rho|} &= O(e^{-a\rho}(r+1)e^{-ar}e^{\mu|r-\rho|}) \\ &= O(e^{-a\rho}(r+1)e^{-ar}e^{\mu(r+\rho)}) \;=\; O(e^{-(a-\mu)\rho}) \end{aligned}$$

and

$$\frac{H'(r-\rho) + \beta'(r)}{r} e^{\mu|r-\rho|} = O(\frac{1}{\rho}).$$

Therefore we deduce that

$$\|\Delta w - f(w)\|_* = O(\frac{1}{\rho}) = O(\gamma^{0.5}). \tag{2.17}$$

If we write $S(w)$ as $\Delta z - f'(w)z - \gamma z$ with $z = \Delta w - f(w)$, then (2.17) implies that

$$\|\gamma z\|_* = O(\gamma^{1.5}). \tag{2.18}$$

The term Δz is further broken into z'' and $\frac{z'}{r}$ for which

$$z'' = \beta^{(4)} - (f(w) - f(H))'' + (\frac{H' + \beta'}{r})''$$

and

$$\frac{z'}{r} = \frac{1}{r}(\frac{H' + \beta'}{r})' + \frac{1}{r}(\beta''' - (f(H+\beta) - f(H))').$$

We again consider two cases of r. If $r \in (0, \theta\rho)$, then

$$z''(r)e^{\mu|r-\rho|} = (\frac{H' + \beta'}{r})'' e^{\mu|r-\rho|} + O(e^{-(a-\mu)\rho}).$$

Since $\beta''' - (f(H+\beta) - f(H))'$ is 0 at $r = 0$, according to the mean value theorem,

$$\frac{1}{r}(\beta''' - (f(H+\beta) - f(H))') = O(e^{-a\rho}).$$

Therefore we have, on $(0, \theta\rho)$,

$$[\Delta z - (\frac{H' + \beta'}{r})'' - \frac{1}{r}(\frac{H' + \beta'}{r})']e^{\mu|r-\rho|} = O(e^{-(a-\mu)\rho}).$$

Note that

$$(\frac{H' + \beta'}{r})'' + \frac{1}{r}(\frac{H' + \beta'}{r})' = \frac{H''' + \beta'''}{r} + \frac{H' + \beta' - rH'' - r\beta''}{r^3}$$

Together with (2.16) we find

$$[\Delta z - f'(w)z - \frac{H''' + \beta'''}{r} - \frac{H' + \beta' - rH'' - r\beta''}{r^3}]e^{\mu|r-\rho|} = O(e^{(-(1-\theta)a+\mu)\rho}). \tag{2.19}$$

Since $H''' + \beta'''$ is 0 at $r = 0$, the mean value theorem implies

$$\frac{H''' + \beta'''}{r}e^{\mu|r-\rho|} = O(e^{(-(1-\theta)a+\mu)\rho}).$$

Using Taylor expansions shows that

$$|\frac{H' + \beta' - rH'' - r\beta''}{r^3}| \le C \sup_{r\in(0,\theta\rho)}(|H^{(4)}(r-\rho)| + |\beta^{(4)}(r)|),$$

which implies

$$\frac{H' + \beta' - rH'' - r\beta''}{r^3}e^{\mu|r-\rho|} = O(e^{(-(1-\theta)a+\mu)\rho}).$$

Therefore, for $r \in (0, \theta\rho)$,

$$(\Delta z - f'(w)z)e^{\mu|r-\rho|} = O(e^{(-(1-\theta)a+\mu)\rho}).$$

When $r > \theta\rho$, similar argument shows that

$$[\Delta z - f'(w)z - \frac{H''' + \beta'''}{r} - \frac{H' + \beta' - rH'' - r\beta''}{r^3} - f'(w)\frac{H' + \beta'}{r}]e^{\mu|r-\rho|} = O(e^{-(a-\mu)\rho}).$$

In this case

$$\frac{H' + \beta' - rH'' - r\beta''}{r^3}e^{\mu|r-\rho|} = O(\frac{1}{\rho^2}),$$

and

$$[\frac{H''' + \beta'''}{r} - f'(w)\frac{H' + \beta'}{r}]e^{\mu|r-\rho|} = \frac{(f'(H) - f'(w))H' + \beta''' - f'(w)\beta'}{r}e^{\mu|r-\rho|} = O(\frac{e^{-(a-\mu)\rho}}{\rho}).$$

Therefore for $r > \theta\rho$,

$$(\Delta z - f'(w)z)e^{\mu|r-\rho|} = O(\frac{1}{\rho^2}).$$

Combining the two cases of r we find

$$\|\Delta z - f'(w)z\|_* = O(\frac{1}{\rho^2}) = O(\gamma),$$

which implies, by (2.18), that $\|S(w)\|_* = O(\gamma)$. $\square$

Lemma 2.3 $I(w) = 2\pi\tau(\frac{1}{2\rho} + \gamma\rho) + O(\gamma^{1.5})$ *where* τ *is given in (2.2).*

Proof. It is easy to see that

$$\begin{aligned}
\int_{R^2} |\Delta w - f(w)|^2\,dx &= 2\pi \int_0^1 [\beta'' + \frac{H' + \beta'}{r} - f(w) + f(H)]^2 r\,dr \\
&= 2\pi \int_{-\rho}^{\infty} \frac{1}{\rho}(H'(y))^2\,dy + O(\frac{1}{\rho^3}) \\
&= \frac{2\pi\tau}{\rho} + O(\gamma^{1.5}); \\
\int_{R^2} [\frac{1}{2}|\nabla w|^2 + F(w)]\,dx &= 2\pi \int_{-\infty}^{\infty} [\frac{1}{2}(H'(y))^2 + F(y)]\,rdr + O(e^{-2a\rho}) \\
&= 2\pi\tau\rho + O(e^{-2a\rho}).
\end{aligned}$$

The lemma follows. □

In the next section we will show that there is a particular ρ, called ρ_γ, and a small function $\phi(\cdot;\rho_\gamma)$ such that $S(w(\cdot;\rho_\gamma) + \phi(\cdot;\rho_\gamma)) = 0$.

3 Reduction to one dimension

Around w the linearized operator of S is L_ρ given by

$$L_\rho\phi = (\Delta - f'(w))^2\phi - (\Delta w - f(w))f''(w)\phi - \gamma(\Delta - f'(w))\phi. \tag{3.1}$$

Here $(\Delta - f'(w))^2$ and $\Delta - f'(w)$ are linear operators.

We define an approximate kernel

$$h(r;\rho) = H'(r-\rho) + b_{1,\rho}e^{-ar} + b_{2,\rho}re^{-ar} \tag{3.2}$$

where $b_{1,\rho} = O(e^{-a\rho})$ and $b_{2,\rho} = O(e^{-a\rho})$ are constants so chosen that $h'(0) = h'''(0) = 0$.

Let π_ρ be the projection operator to the subspace perpendicular to h:

$$\pi_\rho g = g - \frac{< g, h >}{\|h\|_2^2}h. \tag{3.3}$$

We view

$$M = \{w(\cdot;\rho) : \frac{1}{2\sqrt{2\gamma}} < \rho < \frac{2}{\sqrt{2\gamma}}\} \tag{3.4}$$

as a one-dimensional submanifold in $H_r^4(R^2)$. At each $w(\cdot;\rho)$ we define an approximate normal subspace

$$F_\rho = \{\phi \in H_r^4(R^2) : \phi \perp h(\cdot;\rho)\}. \tag{3.5}$$

In each F_ρ we look for a $\phi(\cdot;\rho)$ so that

$$\pi_\rho S(w(\cdot;\rho) + \phi(\cdot;\rho)) = 0. \tag{3.6}$$

We write the last equation as

$$\pi_\rho(S(w) + L_\rho\phi + N_\rho\phi) = 0$$

where the higher order, nonlinear operator N_ρ is given by

$$\begin{aligned}
N_\rho\phi &= -(\Delta - f'(w) - \gamma)(f(w+\phi) - f(w) - f'(w)\phi) \\
&\quad -(f'(w+\phi) - f'(w))(\Delta\phi - (f(w+\phi) - f(w))) \\
&\quad -(\Delta w - f(w))(f'(w+\phi) - f'(w) - f''(w)\phi). \qquad (3.7)
\end{aligned}$$

We would like to turn the last equation to the following fixed point form

$$\phi = -(\pi_\rho L_\rho)^{-1}(\pi_\rho S(w) + \pi_\rho N_\rho(\phi)). \qquad (3.8)$$

To this end we need to specify the function space in which the fixed point argument is made and also establish the fact that $\pi_\rho L_\rho$ is invertible.

First we note that L_ρ can be defined as an operator from $H^4_r(R^2)$ to $L^2_r(R^2)$.

Lemma 3.1 *The operator $\pi_\rho L_\rho$ from $\{\phi \in H^4_r(R^2) : \phi \perp h(\cdot;\rho)\}$ to $\{\eta \in L^2_r(R^2) : \eta \perp h(\cdot;\rho)\}$ satisfies the Fredholm Alternative. In particular the operator is onto if it is one-to-one.*

Proof. Let $q = f'(0) - f'(w)$ and $z = \Delta w - f(w)$, we note that

$$\begin{aligned}
L_\rho\phi &= (\Delta - f'(w))^2\phi - zf''(w)\phi - \gamma(\Delta - f'(w))\phi \\
&= (\Delta - f'(0))^2\phi - \gamma(\Delta - f'(0))\phi \\
&\quad +(\Delta - f'(0))q\phi + q(\Delta - f'(w))\phi - \gamma q\phi - zf''(w)\phi \\
&:= Q\phi + P_\rho\phi
\end{aligned}$$

and

$$\pi_\rho L_\rho\phi = Q\phi + P_\rho\phi - \frac{\langle L_\rho\phi, h\rangle}{\|h\|_2^2}h \qquad (3.9)$$

with

$$Q\phi = (\Delta - f'(0))^2\phi - \gamma(\Delta - f'(0))\phi \qquad (3.10)$$

$$P_\rho\phi = (\Delta - f'(0))q\phi + q(\Delta - f'(w))\phi - \gamma q\phi - zf''(w)\phi. \qquad (3.11)$$

Note that Q is a well-behaved operator, an isometry indeed, from $H^4_r(R^2)$ to $L^2_r(R^2)$. Let us denote $\{\phi \in L^2_r(R^2) : \phi \perp h\}$ by $\{h\}^\perp$. If we are given an equation $\pi_\rho L_\rho\phi = \eta$ with $\eta \in L^2_r(R^2) \cap \{h\}^\perp$ and with ϕ expected in $H^4_r(R^2) \cap \{h\}^\perp$, we apply the operator $\pi_\rho Q^{-1}$ to both sides to find

$$\phi + \pi_\rho[Q^{-1}P_\rho\phi - \frac{\langle L_\rho\phi, h\rangle}{\|h\|_2^2}Q^{-1}h] = \pi_\rho Q^{-1}\eta. \qquad (3.12)$$

However we must show that the operator $\pi_\rho Q^{-1}$ used to make this transformation is one-to-one and onto from $L^2_r(R^2) \cap \{h\}^\perp$ to $H^4_r(R^2) \cap \{h\}^\perp$. To show that the operator is one-to-one, we let $\pi_\rho Q^{-1}g = 0$ for some $g \in L^2_r(R^2) \cap \{h\}^\perp$. There exists $c \in R$ such that $Q^{-1}g = ch$, i.e. $cQh = g$. Multiply by h and integrate to find

$$0 = c\int_{R^2} hQh = c\int_{R^2}[|(\Delta - f'(0))h|^2 + \gamma(|\nabla h|^2 + f'(0)h^2)].$$

Hence $c = 0$, and consequently $g = 0$. To show that $\pi_\rho Q^{-1}$ is onto, we must be able to solve $\pi_\rho Q^{-1} g = \xi$ for any $\xi \in H^4_r(R^2) \cap \{h\}^\perp$, i.e. we look for $c \in R$ and $g \in L^2_r(R^2) \cap \{h\}^\perp$ such that

$$Q^{-1}g + ch = \xi, \quad \text{i.e.} \quad g + cQh = Q\xi.$$

Multiply the last equation by h and integrate to find

$$c = \frac{\int_{R^2} hQ\xi \, dx}{\int_{R^2} hQh \, dx}.$$

We then find g by setting $g = Q\xi - cQh$. This g is necessarily perpendicular to h.

For the Fredholm Alternative to hold in (3.12), the operator from $H^4_r(R^2)$ to itself on the left of (3.12) side should be of the form Identity + Compact. To see the compactness we note that both q and z in P_ρ decay to 0 at ∞. Hence by the Sobolev imbedding theory $Q^{-1}P_\rho$ is a compact operator. Also a rank one operator like $\phi \to \frac{\langle L_\rho \phi, h\rangle}{\|h\|_2^2} h$ is compact. □

More importantly we prove the following estimate for $\pi_\rho L_\rho$ under the weighted L^∞-norm.

Lemma 3.2 *There exists $C > 0$ independent of γ and ρ such that if $\pi_\rho L_\rho \phi = g$, $\phi \perp h$ and $g \in C(R)$, then*

$$\|\phi\|_* + \|\Delta\phi\|_* \le C\|g\|_*.$$

Before we prove this lemma, we need a technical estimate. This estimate was used by Ni and Wei [19]. They stated a version on a bounded ball. We include a proof for our entire space situation in the appendix.

Lemma 3.3 *Let $\phi \in C^2[0,\infty)$ satisfy $\phi'(0) = 0$, $\lim_{r\to\infty} \phi(r) = 0$, and*

$$|\phi'' + \frac{\phi'}{r} - f'(0)\phi| \le c_0 e^{-\mu|r-\rho|}.$$

Then

$$|\phi(r)| \le e[|\phi(\rho)| + \frac{2ec_0}{f'(0)}]e^{-\mu|r-\rho|}.$$

Proof of Lemma 3.2. Let $\pi_\rho L_\rho \phi = g$ with $\phi \perp h$. Then there exists $d_{1,\rho} \in R$ such that

$$L_\rho \phi = g + d_{1,\rho} h.$$

If the lemma does not hold, then we may assume that $\|g\|_* = o(1)$ and $\|\phi\|_* + \|\Delta\phi\|_* = 1$. Let $\psi = \Delta\phi - f'(w)\phi$. Clearly $\|\psi\|_* = O(1)$ and ψ satisfies

$$\Delta\psi - f'(w)\psi - zf''(w)\phi - \gamma\psi = d_{1,\rho}h + g$$

where $z = \Delta w - f(w)$.

If we multiply the last equation by h and integrate over R^2, then integration by parts shows that

$$o(\rho) = d_{1,\rho}(\rho \int_R h^2 \, dr + O(1)).$$

This implies that $d_{1,\rho} = o(1)$. It follows that

$$\|\Delta\psi - f'(w)\psi\|_* \leq \|zf''(w)\|_\infty \|\phi\|_* + \gamma\|\psi\|_* + d_{1,\rho}\|h\|_* + \|g\|_* = o(1).$$

Now we prove that $\|\psi\|_* = o(1)$. Assume this is not true. Then we consider $\tilde{\psi} = \frac{\psi}{\|\psi\|_*}$, which satisfies $\|\tilde{\psi}\|_* = 1$ and

$$\|\Delta\tilde{\psi} - f'(w)\tilde{\psi}\|_* = o(1) \tag{3.13}$$

following the last estimate. Simple elliptic regularity argument shows that $\tilde{\psi}(\cdot - \rho)$ converges in $C^2_{loc}(R)$ to a function $\tilde{\Psi}$ as $\gamma \to 0$. It follows that $\|\tilde{\Psi}\|_* \leq 1$ and $\tilde{\Psi}$ satisfies $\tilde{\Psi}'' - f'(H)\tilde{\Psi} = 0$. Therefore $\tilde{\Psi} = d_2 H'$ for some $d_2 \in R$. This implies that

$$\langle \tilde{\psi}, h\rangle = d_2(2\pi\rho\tau) + o(\rho). \tag{3.14}$$

On the other hand we multiply

$$\frac{1}{\|\psi\|_*}(\Delta\phi - f'(w)\phi) = \tilde{\psi}$$

by h and integrate over R^2 to find that

$$O(1) = \langle \tilde{\psi}, h\rangle.$$

Combined with (3.14) we deduce that $d_2 = 0$ and $\tilde{\psi}(\cdot - \rho) \to 0$ in $C^2_{loc}(R)$. We now return to (3.13) and find

$$\|\Delta\tilde{\psi} - f'(0)\tilde{\psi}\|_* \leq \|\Delta\tilde{\psi} - f'(w)\tilde{\psi}\|_* + \|(f'(w) - f'(0))\tilde{\psi}\|_* = o(1). \tag{3.15}$$

Since $\tilde{\psi}(\rho) = o(1)$, Lemma 3.3 implies that $\|\tilde{\psi}\|_* = o(1)$. A contradiction to $\|\tilde{\psi}\|_* = 1$.

Finally we consider ϕ in the equation

$$\Delta\phi - f'(w)\phi = \psi$$

with $\|\psi\|_* = o(1)$. Again elliptic regularity argument shows that $\phi(\cdot - \rho) \to d_3 H'$ in $C^2_{loc}(R)$ for some $d_3 \in R$. Our assumption $\phi \perp h$ implies that $d_3 = 0$. Hence $\phi(\cdot - \rho) \to 0$ in $C^2_{loc}(R)$. As before

$$\|\Delta\phi - f'(0)\phi\|_* \leq \|\psi\| + \|(f'(w) - f'(0))\phi\|_* = o(1).$$

Lemma 3.3 again implies that $\|\phi\|_* = o(1)$. Moreover

$$\|\Delta\phi\|_* \leq \|\Delta\phi - f'(w)\phi\|_* + \|f'(w)\phi\|_* \leq \|\psi\|_* + \|f'(w)\|_\infty \|\phi\|_* = o(1).$$

We have now reached a contradiction to $\|\phi\|_* + \|\Delta\phi\|_* = 1$. □

A consequence of Lemma 3.2 is the following.

Lemma 3.4 *$\pi_\rho L_\rho$: $H^4_r(R^2) \cap \{h\}^\perp \to L^2_r(R^2) \cap \{h\}^\perp$ is a one-to-one and onto map. Moreover $(\pi_\rho L_\rho)^{-1}$ is also an operator from $\{g \in C[0,\infty) : \|g\|_* < \infty,\ g \perp h\}$ to $\{\phi \in C^2[0,\infty) :\ \phi'(0) = 0,\ \|\phi\|_* + \|\Delta\phi\|_* < \infty,\ \phi \perp h\}$, whose norm is bounded by a constant independent of γ and ρ.*

Proof. Lemma 3.2 shows that $\pi_\rho L_\rho$ is one-to-one. Hence it is onto by the Fredholm Alternative, Lemma 3.1. Since every continuous function with finite $\|\cdot\|_*$-norm is in $L^2_r(R^2)$, one can apply $(\pi_\rho L_\rho)^{-1}$ to such a function. Lemma 3.2 yields a bound of $(\pi_\rho L_\rho)^{-1}$ in this setting. □

Now we define the proper space on which the fixed point argument is done. Let

$$Z_\rho = \{\phi \in C^2[0,\infty) : \ \phi'(0) = 0, \ \|\phi\|_* < \infty, \ \|\Delta\phi\|_* < \infty, \ \phi \perp h(\cdot;\rho)\}. \tag{3.16}$$

In Z_ρ we define a norm

$$\|\phi\|_Z = \|\phi\|_* + \|\Delta\phi\|_*. \tag{3.17}$$

We write the right side of (3.8) as $T_\rho\phi$. Based on Lemmas 3.2 and 3.4 we know that T_ρ is well-defined on Z_ρ. We show that T_ρ is a contraction map with a fixed point.

Lemma 3.5 *There exists $\phi(\cdot;\rho)$ so that $\|\phi(\cdot;\rho)\|_Z = O(\gamma)$ and $\pi_\rho S(w+\phi) = 0$.*

Proof. Define a closed ball B_ρ in Z_ρ:

$$B_\rho = \{\phi \in Z_\rho : \|\phi\|_Z \le d_1\gamma\}. \tag{3.18}$$

Here d_1 is a positive constant independent of γ and ρ to be fixed soon. For each $\phi \in B_\rho$, we have, by Lemmas 2.2 and 3.2,

$$\|T_\rho\phi\|_Z \le C(\|S(w)\|_* + \|N_\rho\phi\|_*) \le C(\gamma + C_1\|\phi\|_Z^2) \le C(\gamma + C_1 d_1^2\gamma^2).$$

The last quantity is less than $d_1\gamma$ when γ is small, if we choose d_1 to be sufficiently large. This shows that T_ρ maps B_ρ into itself. Next we take ϕ_1 and ϕ_2 from B_ρ and consider

$$\|T_\rho\phi_1 - T_\rho\phi_2\|_Z \le C\|N_\rho\phi_1 - N_\rho\phi_2\|_* \le C\|\phi_1-\phi_2\|_Z(\|\phi_1\|_Z + \|\phi_2\|_Z) \le Cd_1\gamma\|\phi_1-\phi_2\|_Z.$$

Hence T_ρ is a contraction map when γ is small. This yields a unique solution ϕ of $\pi_\rho S(w+\phi) = 0$ in B_ρ. □

To find a particular $\rho = \rho_\gamma$ so that $S(w(\cdot;\rho_\gamma) + \phi(\cdot;\rho_\gamma)) = 0$, we consider the free energy of $w(\cdot;\rho) + \phi(\cdot;\rho)$: $I(w(\cdot;\rho) + \phi(\cdot;\rho))$.

Lemma 3.6 1. *$I(w(\cdot;\rho) + \phi(\cdot;\rho)) = 2\pi\tau(\frac{1}{2\rho} + \gamma\rho) + O(\gamma^{1.5})$.*

2. *There exists $\rho_\gamma = \frac{1}{\sqrt{2\gamma}} + o(\gamma^{-1/2})$ such that $S(w(\cdot;\rho_\gamma) + \phi(\cdot;\rho_\gamma)) = 0$.*

Proof. Expanding I shows that

$$\begin{aligned}
I(w+\phi) &= I(w) + \int_{R^2} S(w)\phi\,dx + \frac{1}{2}\int_{R^2}\phi L_\rho\phi\,dx + O(\int_{R^2}[|\phi|^3 + |\Delta\phi|^3]\,dx) \\
&= I(w) + \int_{R^2} S(w)\phi\,dx + \frac{1}{2}\int_{R^2}\phi L_\rho\phi\,dx + O(\|\phi\|_Z^3\int_0^\infty e^{-3\mu|r-\rho|}r\,dr) \\
&= I(w) + \int_{R^2} S(w)\phi\,dx + \frac{1}{2}\int_{R^2}\phi L_\rho\phi\,dx + O(\|\phi\|_Z^3\rho) \\
&= I(w) + \int_{R^2} S(w)\phi\,dx + \frac{1}{2}\int_{R^2}\phi L_\rho\phi\,dx + O(\gamma^{2.5})
\end{aligned} \tag{3.19}$$

by Lemma 3.5. From the equation $\pi_\rho S(w+\phi)=0$ we find, since $\phi \perp h$,

$$\begin{aligned} 0 &= \int_{R^2} S(w+\phi)\phi\,dx \;=\; \int_{R^2} S(w)\phi\,dx + \int_{R^2}\phi L_\rho\phi\,dx + \int_{R^2}\phi N_\rho \phi\,dx \\ &= \int_{R^2} S(w)\phi\,dx + \int_{R^2}\phi L_\rho\phi\,dx + O(\|\phi\|_*\|N_\rho\phi\|_*\int_0^\infty e^{-2\mu|r-\rho|}r\,dr) \\ &= \int_{R^2} S(w)\phi\,dx + \int_{R^2}\phi L_\rho\phi\,dx + O(\gamma^{2.5}). \end{aligned}$$

We can rewrite (3.19) as

$$I(w+\phi) = I(w) + \frac{1}{2}\int_{R^2} S(w)\phi\,dx + O(\gamma^{2.5}). \tag{3.20}$$

Using Lemmas 2.2 and 3.5 we deduce

$$\int_{R^2} S(w)\phi\,dx = O(\|S(w)\|_*\|\phi\|_*\int_{R^2} e^{-2\mu|r-\rho|}\,rdr) = O(\|S(w)\|_*\|\phi\|_*\rho) = O(\gamma^{1.5}).$$

We have now turned (3.20) to

$$I(w+\phi) = I(w) + O(\gamma^{1.5}). \tag{3.21}$$

The first part of the lemma then follows from Lemma 2.3.

If we minimize $I(w(\cdot;\rho)+\phi(\cdot;\rho))$ with respect to ρ, then based on the first part of this lemma we know that $I(w(\cdot;\rho)+\phi(\cdot;\rho))$ is minimized at some ρ_γ and

$$\rho_\gamma = \frac{1}{\sqrt{2\gamma}} + o(\gamma^{-1/2}).$$

We now show that if $\rho=\rho_\gamma$, $S(w(\cdot;\rho_\gamma)+\phi(\cdot;\rho))=0$. Since $\pi_\rho S(w+\phi)=0$, there exists c_ρ such that $S(w+\phi)=c_\rho h$. We differentiate $I(w+\phi)$ with respect to ρ to find

$$\frac{dI(w(\cdot;\rho)+\phi(\cdot;\rho))}{d\rho} = \int_{R^2} S(w+\phi)(\frac{\partial w}{\partial\rho}+\frac{\partial\phi}{\partial\rho})\,dx = c_\rho[\int_{R^2} h\frac{\partial w}{\partial \rho}\,dx + \int_{R^2} h\frac{\partial\phi}{\partial\rho}\,dx].$$

Here

$$\int_{R^2} h\frac{\partial w}{\partial\rho}\,dx = 2\pi\int_0^\infty h\frac{\partial(H(r-\rho)+\beta)}{\partial\rho}\,rdr = 2\pi(\tau\rho+o(\rho));$$

$\phi\perp h$ implies that

$$\int_{R^2} h\frac{\partial\phi}{\partial\rho}\,dx = -\int_{R^2}\frac{\partial h}{\partial\rho}\phi\,dx.$$

Then

$$|\int_{R^2} h\frac{\partial\phi}{\partial\rho}\,dx| \le \|\phi\|_* 2\pi\int_0^\infty |\frac{\partial h}{\partial\rho}|e^{-\mu|r-\rho|}\,rdr = \|\phi\|_* O(\rho) = o(\rho).$$

Therefore

$$\frac{dI(w+\phi)}{d\rho} = c_\rho(2\pi\tau\rho + o(\rho)).$$

If ρ is equal to ρ_γ, then

$$0 = c_{\rho_\gamma}(2\pi\tau\rho_\gamma + o(\rho_\gamma)),$$

i.e. $c_{\rho_\gamma}=0$ and $S(w(\cdot;\rho_\gamma)+\phi(\cdot;\rho_\gamma))=0$. □

The last lemma completes the proof of Theorem 2.1.

4 $n \geq 4$

If we consider (1.6) for $n \geq 4$, then with Γ being a $n-1$ dimensional sphere

$$I_c(\Gamma) = \omega_{n-1}\rho^{n-3} + \omega_{n-1}\gamma\rho^{n-1} \tag{4.1}$$

where ω_{n-1} is the area of $n-1$ dimensional unit sphere. It is clear that if $\gamma > 0$, the right side is increasing in ρ. Only if $\gamma < 0$, there exists a critical point, but this critical point is a maximum. In the phase field model, we have the similar phenomenon.

Theorem 4.1 *When γ is negative and sufficiently close to 0, there exists a bubble profile. The radius of the bubble is* $\sqrt{\frac{(n-1)(n-3)}{-2\gamma}} + o(\gamma^{-1/2})$.

Proof. The proof of this theorem is almost identical to the proof of Theorem 2.1. The main difference occurs in the last step:

$$I(w+\phi) = \omega_{n-1}\tau\left(\frac{(n-1)^2\rho^{n-3}}{2} + \gamma\rho^{n-1}\right) + o(\gamma^{(3-n)/2}) \tag{4.2}$$

where ω_{n-1} is the area of the $n-1$ dimensional unit sphere. If $\gamma < 0$, the above quantity has a *maximum* at

$$\rho_\gamma = \sqrt{\frac{(n-1)(n-3)}{-2\gamma}} + o(\gamma^{-1/2}).$$

The detail of the proof is left to the reader. □

5 $n = 1$

When $n = 1$, the phase field problem is far more complex than (1.6). A zero dimensional sphere is just the union of two points in R, and ρ is half the distance between the two points. This sphere has no curvature. Hence

$$I_c(\Gamma) = 2\gamma, \tag{5.1}$$

a constant independent of ρ. No conclusion can be drawn (5.1). But for the phase field problem, we have the following result.

Theorem 5.1 *When γ is positive and sufficiently small, there exists a bubble profile. The radius of the bubble is* $\frac{1}{2a}\log\frac{1}{\gamma} + o(\log\frac{1}{\gamma})$.

Let

$$\rho \in \left(\frac{1}{4a}\log\frac{1}{\gamma}, \frac{1}{a}\log\frac{1}{\gamma}\right). \tag{5.2}$$

For each ρ satisfying (5.2) we define an approximate solution w. Compared to the $n = 2$ and $n \geq 4$ cases, the construction of w is more complex. We let

$$\alpha(x;\rho) = c_{0,\rho}e^{-ax}, \text{ where } c_{0,\rho} = \frac{H'(-\rho)}{a}, \text{ so that } H'(-\rho) + \alpha'(0) = 0. \tag{5.3}$$

Now we define a function $g(y;\rho)$ on $(-\infty,\infty)$ which is the solution of

$$g'' - f'(H)g + \alpha(y+\rho)(f'(0) - f'(H)) = d_\rho H', \quad g(0) = -\alpha(\rho). \tag{5.4}$$

In (5.4) the constant d_ρ is chosen so that

$$d_\rho \int_R (H')^2\, dy = \int_R \alpha(y+\rho)(f'(0) - f'(H(y)))H'(y)\, dy. \tag{5.5}$$

This ensures that (5.4) is solvable. The condition $g(0) = -\alpha(\rho)$ gives a unique solution. We calculate the right side of (5.5):

$$\begin{aligned} \int_R \alpha(y+\rho)(f'(0) - f'(H(y)))H'(y)\, dy &= \alpha(\rho)\int_R e^{-ay}(f'(0)H' - H''')\, dy \\ = \alpha(\rho)[-e^{-ay}H''(y) - ae^{-ay}H'(y)]|_{y=-\infty}^{y=\infty} &= \alpha(\rho)\lim_{y\to-\infty}[e^{-ay}H''(y) + ae^{-ay}H'(y)] \\ = 2ak\alpha(\rho) &= 2k^2e^{-2a\rho} + o(e^{-2a\rho}). \end{aligned}$$

Therefore

$$d_\rho = \frac{2k^2e^{-2a\rho}}{\tau} + o(e^{-2a\rho}). \tag{5.6}$$

We include g in the construction of w. One last term is β which is given as

$$\beta(x;\rho) = c_{1,\rho}e^{-ax} + c_{2,\rho}xe^{-ax} + \frac{f''(0)c_{0,\rho}^2}{6a^2}e^{-2ax}. \tag{5.7}$$

It is a solution of

$$(D^2 - f'(0))^2\beta = (D^2 - f'(0))\frac{f''(0)\alpha^2}{2}. \tag{5.8}$$

The constants $c_{1,\rho}$ and $c_{2,\rho}$ are chosen so that

$$\beta'(0) = -H'(-\rho) - \alpha'(0) - g'(-\rho) = -g'(-\rho), \quad \beta'''(0) = -H'''(-\rho) - \alpha'''(0) - g'''(-\rho). \tag{5.9}$$

Here

$$c_{1,\rho} = O(e^{-2a\rho}), \quad c_{2,\rho} = O(e^{-2a\rho}). \tag{5.10}$$

Now we set

$$w(x;\rho) = H(x-\rho) + \alpha(x;\rho) + g(x-\rho;\rho) + \beta(x;\rho). \tag{5.11}$$

Our choice of β ensures that $w'(0) = w'''(0) = 0$. Note that this β is different from the one (2.9) used in the $n = 2$ case.

We again need the weighted L^∞ norm:

$$\|\zeta\|_* = \sup_{x\in R}|\zeta(x)|e^{\mu|x-\rho|}. \tag{5.12}$$

Lemma 5.2 *There exists $\delta > 0$ independent of γ and ρ such that $\|S(w)\|_* = O(\gamma^{1+\delta})$.*

Proof. We start with $w'' - f(w)$. Note that

$$\begin{aligned} w'' - f(w) &= H'' + \alpha'' + g'' + \beta'' - f(H + \alpha + g + \beta) \\ &= \alpha'' + g'' + \beta'' + f(H) - f(H + \alpha + g + \beta) \\ &= \alpha'' + g'' + \beta'' - f'(H)(\alpha + g + \beta) \\ &\quad -(f(H + \alpha + g + \beta) - f(H) - f'(H)(\alpha + g + \beta)) \\ &= d_\rho H' + \beta'' - f'(H)\beta - (f(H + \alpha + g + \beta) - f(H) - f'(H)(\alpha + g + \beta)). \end{aligned}$$

At this point we consider two cases of x: $x \in (0, \theta\rho)$ and $x \in (\theta\rho, \infty)$ where $\theta \in (0, 1)$. In the first case we write

$$w'' - f(w) = d_\rho H' + \beta'' - f'(0)\beta - \frac{f''(0)}{2}\alpha^2 + N_1(x; \rho) \tag{5.13}$$

with

$$N_1(x; \rho) = (f'(0) - f'(H))\beta - (f(H + \alpha + g + \beta) - f(H) - f'(H)(\alpha + g + \beta) - \frac{f''(0)}{2}\alpha^2). \tag{5.14}$$

Note that

$$\sup\{|N_1(x; \rho)| + |N_1'(x, \rho)| + |N_1''(x, \rho)| : \ x \in (0, \theta\rho)\} = O(e^{-3\theta a\rho}).$$

Hence

$$\sup\{(|N_1(x; \rho)| + |N_1'(x, \rho)| + |N_1''(x, \rho)|)e^{\mu|x-\rho|} : \ x \in (0, \theta\rho)\} = O(e^{-(3\theta a-\mu)\rho}). \tag{5.15}$$

From the equation that β satisfies we see that $\beta'' - f'(0)\beta - \frac{f''(0)}{2}\alpha^2 = c_{3,\rho}e^{-ax}$. Hence (5.13) becomes

$$w'' - f(w) = d_\rho H' + c_{3,\rho}e^{-ax} + N_1(r; \rho).$$

To estimate the size of $c_{3,\rho}$ note that the derivative of $w'' - f(w)$ at $x = 0$ is 0, by our construction of w. Therefore

$$0 = d_\rho H''(-\rho) - ac_{3,\rho} + O(e^{-3\theta a\rho}).$$

This shows, with (5.6), that $c_{3,\rho} = O(e^{-3\theta a\rho})$ and

$$\beta'' - f'(0)\beta - \frac{f''(0)}{2}\alpha^2 = O(e^{-3\theta a\rho}).$$

Now we can write (5.13) as

$$w'' - f(w) = d_\rho H' + M(x; \rho) \tag{5.16}$$

with M satisfying

$$\sup\{(|M(x; \rho)| + |M'(x, \rho)| + |M''(x, \rho)|)e^{\mu|x-\rho|} : \ x \in (0, \theta\rho)\} = O(e^{-(3\theta a-\mu)\rho}). \tag{5.17}$$

For the second case, $x > \theta\rho$, since $\alpha(x) = O(e^{-a\rho})e^{-ax}$ and $\beta(x) = O(e^{-2a\rho})xe^{-ax}$,

$$\begin{aligned} w'' - f(w) &= d_\rho H' + \beta'' - f'(H)\beta - (f(H + \alpha + g + \beta) - f(H) - f'(H)(\alpha + g + \beta)) \\ &= d_\rho H' + M(r; \rho) \end{aligned}$$

with

$$\sup\{(|M(x;\rho)|+|M'(x,\rho)|+|M''(x,\rho)|)e^{\mu|x-\rho|}:\ x\in(\theta\rho,\infty)\}=O(e^{-(2a+\delta_1)\rho}) \tag{5.18}$$

for some $\delta_1>0$.

Combining (5.17) and (5.18) we have

$$w''-f(w)=d_\rho H'+M$$

with

$$\|M\|_*+\|M''\|_*=O(e^{-(2a+\delta_1)\rho}) \tag{5.19}$$

by choosing θ to be sufficiently close to 1 and μ sufficiently small.

Now we estimate $S(w)$ by (5.19). In

$$S(w)=(D^2-f'(w)-\gamma)(d_\rho H'+M)=d_\rho(f'(H)-f'(w))H'-\gamma d_\rho H'+(D^2-f'(w)-\gamma)M,$$

clearly $\|\gamma d_\rho H'\|_*=O(\gamma^2)$ and $\|(D^2-f'(w)-\gamma)M\|_*=O(\gamma^{1+\delta})$ for some $\delta>0$. As for $d_\rho(f'(H)-f'(w))H'$, note that

$$\|d_\rho(f'(H)-f'(w))H'\|_*\le d_\rho\|f'(H)-f'(w)\|_\infty\|H'\|_*=d_\rho O(e^{-a\rho})=O(e^{-3a\rho}).$$

Hence $\|S(w)\|_*=O(\gamma^{1+\delta})$. □

Lemma 5.3 $I(w)=\dfrac{4k^4e^{-4a\rho}}{\tau}+\gamma(2\tau-\dfrac{2k^2e^{-2a\rho}}{a})+o(\gamma^2)$.

Proof. . Using (5.18) and (5.6) we find

$$\begin{aligned}
&\frac{1}{2}\int_R (w''-f(w))^2\,dx\\
&= d_\rho^2\int_0^\infty (H'(x-\rho))^2\,dx+o(e^{-4a\rho}) \;=\; d_\rho^2\tau+o(e^{-4a\rho}) \;=\; \frac{4k^4e^{-4a\rho}}{\tau}+o(\gamma^2);\\
&\int_R[\frac{1}{2}|w'|^2+F(w)]\,dx\\
&= \int_0^\infty[|w'|^2+2F(w)]\,dx \;=\; \int_0^\infty[|H'(x-\rho)+\alpha'(x)|^2+2F(H+\alpha)]\,dx+o(e^{-2a\rho})\\
&= 2\tau-2\int_{-\infty}^{-\rho}H'(y)^2\,dy+\int_0^\infty[2H'\alpha'+2f(H)\alpha+(\alpha')^2+f'(H)\alpha^2]\,dx+o(e^{-2a\rho})\\
&= 2\tau-2\int_{-\infty}^{-\rho}H'(y)^2\,dy+2H'(x-\rho)\alpha(x)|_{x=0}^{x=\infty}+\int_0^\infty(\alpha')^2\,dx+\int_0^\infty f'(H)\alpha^2\,dx+o(e^{-2a\rho})\\
&= 2\tau-2\int_{-\infty}^{-\rho}k^2e^{2ay}\,dy-2H'(-\rho)\alpha(0)+2a^2\int_0^\infty\alpha^2\,dx+o(e^{-2a\rho})\\
&= 2\tau-\frac{k^2e^{-2a\rho}}{a}-\frac{2k^2e^{-2a\rho}}{a}+\frac{k^2e^{-2a\rho}}{a}+o(e^{-2a\rho}) \;=\; 2\tau-\frac{2k^2e^{-2a\rho}}{a}+o(\gamma).
\end{aligned}$$

This proves the lemma. □

The rest of the proof is analogous to that of Theorem 2.1. Define h as in (3.2). For each ρ we find $\phi(\cdot;\rho) \perp h(\cdot;\rho)$ so that $\pi_\rho S(w+\phi) = 0$. The Contraction Mapping Principle used in the argument also shows, with the help of Lemma 5.2, that

$$\|\phi(\cdot;\rho)\|_* = O(\gamma^{1+\delta}), \quad \|\phi''(\cdot;\rho)\|_* = O(\gamma^{1+\delta}). \tag{5.20}$$

We then expand $I(w+\phi)$ as follows.

$$\begin{aligned} I(w+\phi) &= I(w) + \int_R S(w)\phi\,dx + \frac{1}{2}\int_R \phi L_\rho \phi\,dx + O(\gamma^{3+3\delta}) \\ &= I(w) + \frac{1}{2}\int_R S(w)\phi\,dx + O(\gamma^{3+3\delta}) \\ &= I(w) + O(\|S(w)\|_*\|\phi\|_*)\int_0^\infty e^{-2\mu|x-\rho|}\,dx + O(\gamma^{3+3\delta}) \\ &= I(w) + O(\gamma^{2+2\delta}). \end{aligned}$$

Finally we minimize $I(w(\cdot;\rho)+\phi(\cdot;\rho))$ with respect to ρ. Lemma 5.3 shows that $I(w+\phi)$ is minimized at some

$$\rho_\gamma = -\frac{1}{2a}\log\frac{\gamma\tau}{4ak^2} + o(\log\frac{1}{\gamma}). \tag{5.21}$$

This completes the proof of Theorem 5.1.

6 $n = 3$

When $n=3$, for a sphere $\kappa = \frac{1}{\rho}$ and $\int_\Gamma \kappa^2\,ds = \frac{1}{\rho^2}4\pi\rho^2 = 4\pi$. Hence

$$I_c(\Gamma) = 4\pi + 4\pi\gamma\rho^2, \tag{6.1}$$

which has no critical point for positive ρ. The phase field problem is again very different.

Let $l(s)$ be the inverse function of

$$\rho \to -\frac{2k^2 e^{-2a\rho}}{\tau\rho}. \tag{6.2}$$

Here $l : (-\infty, 0) \to (0, \infty)$. As s tends to 0, $l(s)$ grows to ∞, but more slowly than $-\frac{1}{2a}\log(-s)$ does.

Theorem 6.1 *When γ is negative and sufficiently close to 0, there exists a bubble profile. The radius of the bubble is $l(\gamma) + o(l(\gamma))$.*

We let

$$\rho \in (\frac{l(\gamma)}{2}, 2l(\gamma)). \tag{6.3}$$

We define a family of approximate solutions

$$w(r) = H(r-\rho) + \beta(r;\rho) \tag{6.4}$$

where β is the same as the one (2.9) used in the $n=2$ case.

Lemma 6.2 *There exists $\delta > 0$ independent of γ and ρ so that $\|S(w)\|_* = O(\gamma^{(1+\delta)/2})$.*

Proof. We start with

$$\Delta w - f(w) = \beta'' + \frac{2H' + 2\beta'}{r} - f(w) + f(H).$$

As in (2.17),

$$\|\Delta w - f(w)\|_* = O(\frac{1}{\rho}) = O(\frac{1}{l(\gamma)}). \tag{6.5}$$

Let $z = \Delta w - f(w)$. In $S(w) = (\Delta - f'(w))z - \gamma z$ we first note from the above estimate that

$$\|\gamma z\|_* = O(\frac{\gamma}{l(\gamma)}).$$

It suffices to estimate $\Delta z - f'(w)z$. In the $n = 3$ case, $\Delta = \frac{1}{r}D^2 r$ for radial functions. Hence we deduce that

$$\begin{aligned} \Delta z - f'(w)z &= \frac{1}{r}(D^2 - f'(w))(rz) \\ &= \frac{1}{r}(D^2 - f'(w))[(r\beta)'' - r(f(w) - f(H)) + 2H'] \\ &= \frac{1}{r}\{(D^2 - f'(w))[(r\beta)'' - r(f(w) - f(H))] - 2(f'(w) - f'(H))H'\}. \end{aligned}$$

When $r \in (\theta\rho, \infty)$ where $\theta \in (0, 1)$ is independent of ρ and γ, we find

$$|\Delta z - f'(w)z| \le \frac{C}{\rho}[e^{-a\rho} r e^{-ar} + e^{-a\rho} r e^{-ar} e^{-a|r-\rho|}];$$

hence for small μ

$$|(\Delta z - f'(w)z)|e^{\mu|r-\rho|} = O(e^{-(a+\delta_1)\rho}), \quad r \in (\theta, \rho), \tag{6.6}$$

for some $\delta_1 > 0$ where δ_1 is independent of ρ and γ.

When $r \in (0, \theta\rho)$ we write

$$\begin{aligned} &\Delta z - f'(w)z \\ &= \frac{1}{r}\{(D^2 - f'(w))[(r\beta)'' - f'(0)r\beta + f'(0)r\beta - r(f(w) - f(H))] - 2(f'(w) - f'(H))H'\} \\ &= \frac{1}{r}\{(D^2 - f'(0))^2(r\beta) + (f'(0) - f'(w))(D^2 - f'(0))(r\beta) \\ &\quad +(D^2 - f'(w))(f'(0)r\beta - r(f(w) - f(H)) - 2(f'(w) - f'(H))H'\}. \end{aligned}$$

The largest term appears to be $(D^2 - f'(0))^2(r\beta)$. However

$$(D^2 - f'(0))^2(r\beta) = r(D^2 - f'(0))^2\beta + 4(D^2 - f'(0))\beta' = 4(D^2 - a^2)\beta' = 8a^2 c_{2,\rho} e^{-ar},$$

which is small. By choosing θ close to 1 and μ small we find $\delta_1 > 0$ such that

$$|\Delta z - f(w)z|e^{\mu|r-\rho|} = O(e^{-(a+\delta_1)\rho}), \ r \in (0, \theta\rho). \tag{6.7}$$

From (6.6) and (6.7) we deduce that

$$\|\Delta z - f'(w)z\|_* = O(e^{-(a+\delta_1)\rho}) = O(\gamma^{(1+\delta)/2})$$

for some $\delta > 0$ independent of ρ and γ, and consequently by (6.5) $\|S(w)\|_* = O(\gamma^{(1+\delta)/2})$. □

Lemma 6.3 $I(w) = 4\pi[2\tau - \frac{2k^2 e^{-2a\rho}}{a} + \gamma\tau\rho^2] + O(\gamma^{1+\delta})$ *for some* $\delta > 0$ *independent of* ρ *and* γ.

Proof. It is easy to see that

$$\int_{R^3} [\frac{1}{2}|\nabla w|^2 + F(w)]\, dx = 4\pi \int_0^\infty [\frac{1}{2}(H'(r-\rho)+\beta')^2 + F(H+\beta)]\, r^2 dr = 4\pi\tau\rho^2 + O(\rho e^{-a\rho}). \quad (6.8)$$

The estimate of the first part of $I(w)$ is a bit more involved. Note that

$$\begin{aligned}
&\frac{1}{2}\int_{R^3} |\Delta w - f(w)|^2\, dx \\
= &\ 2\pi \int_0^\infty |H'' + \beta'' + \frac{2(H'+\beta')}{r} - f(H+\beta)|^2\, r^2 dr \\
= &\ 2\pi \int_0^\infty |r\beta'' + 2\beta' + 2H' - rf'(H)\beta|^2\, dr + O(e^{-(2a+\delta_1)\rho}) \\
= &\ 2\pi \int_0^\infty |2H' - 2ac_{1,\rho}e^{-ar} + rc_{1,\rho}e^{-ar}(f'(0) - f'(H))|^2\, dr + O(e^{-(2a+\delta_1)\rho})
\end{aligned}$$

for some $\delta_1 > 0$, where δ_1 is independent of ρ and γ. Here $c_{1,\rho}$ comes from the definition (2.9) of β. We now write the last quantity as $2\pi(T_1 + T_2 + T_3 + T_4 + T_5 + T_6)$ where

$$\begin{aligned}
T_1 &= \int_0^\infty 4(H')^2\, dr = \int_{-\rho}^\infty 4(H')^2\, dy = 4\tau - \int_{-\infty}^{-\rho} 4(H')^2\, dy = 4\tau - \frac{2k^2 e^{-2a\rho}}{a} + O(e^{-3a\rho}) \\
T_2 &= \int_0^\infty 4a^2 c_{1,\rho}^2 e^{-2ar}\, dr = 2c_{1,\rho}^2 a = \frac{2(H'(-\rho))^2}{a} = \frac{2k^2 e^{-2a\rho}}{a} + O(e^{-3a\rho}) \\
T_3 &= \int_0^\infty r^2 c_{1,\rho}^2 e^{-2ar}(f'(0) - f'(H))^2\, dr = O(e^{-(2a+\delta_1)\rho}) \\
T_4 &= \int_0^\infty -8aH'c_{1,\rho}e^{-ar}\, dr = -8ac_{1,\rho}e^{-a\rho} \int_{-\rho}^\infty H'(y)e^{-ay}\, dy \\
T_5 &= \int_0^\infty 4c_{1,\rho}H're^{-ar}(f'(0) - f'(H))\, dr = 4c_{1,\rho}e^{-a\rho} \int_{-\rho}^\infty (y+\rho)H'e^{-ay}(f'(0) - f'(H))\, dy \\
T_6 &= -4ac_{1,\rho}^2 \int_0^\infty e^{-2ar} r(f'(0) - f'(H))\, dr = O(e^{-(2a+\delta_1)\rho}).
\end{aligned}$$

We focus on T_5. Note that

$$T_5 = 4c_{1,\rho}e^{-a\rho}[\int_{-\rho}^\infty H'(y)y(f'(0) - f'(H))e^{-ay}\, dy + \int_{-\rho}^\infty H'(y)\rho(f'(0) - f'(H))e^{-ay}\, dy]$$

where

$$
\begin{aligned}
&\int_{-\rho}^{\infty} H'(y)y(f'(0)-f'(H))e^{-ay}\,dy \\
&= \int_{-\rho}^{\infty}(H'f'(0)-H''')ye^{-ay}\,dy \\
&= \int_{-\rho}^{\infty}[H'f'(0)ye^{-ay}-H'(ye^{-ay})'']\,dy - H''(y)ye^{-ay}|_{-\rho}^{\infty} + H'(y)(ye^{-ay})'|_{-\rho}^{\infty} \\
&= 2a\int_{-\rho}^{\infty} H'(y)e^{-ay}\,dy - H''(-\rho)\rho e^{a\rho} - H'(-\rho)e^{a\rho} - aH'(-\rho)\rho e^{a\rho}; \\
&\int_{-\rho}^{\infty} H'(y)\rho(f'(0)-f'(H))e^{-ay}\,dy] \\
&= \rho\int_{-\rho}^{\infty}(H'f'(0)-H''')e^{-ay}\,dy \\
&= \rho[-H''(y)e^{-ay}|_{-\rho}^{\infty} - aH'(y)e^{-ay}|_{-\rho}^{\infty}] \\
&= \rho[H''(-\rho)e^{a\rho} + aH'(-\rho)e^{a\rho}].
\end{aligned}
$$

Hence

$$T_5 = 4c_{1,\rho}e^{-a\rho}[2a\int_{-\rho}^{\infty} H'(y)e^{-ay}\,dy - H'(-\rho)e^{a\rho}],$$

and consequently

$$T_1 + ... + T_6 = 4\tau - 4c_{1,\rho}H'(-\rho) + O(e^{-(2a+\delta_1)\rho}) = 4\tau - \frac{4k^2e^{-2a\rho}}{a} + O(e^{-(2a+\delta_1)\rho}).$$

Therefore

$$\frac{1}{2}\int_{R^3}|\Delta w - f(w)|^2\,dx = 4\pi(2\tau - \frac{2k^2e^{-2a\rho}}{a}) + O(e^{-(2a+\delta_1)\rho}). \tag{6.9}$$

The lemma now follows from (6.8) and (6.9). □

Note that in $I(w)$,

$$2\tau - \frac{2k^2e^{-2a\rho}}{a} + \gamma\tau\rho^2$$

is maximized at $\rho = l(\gamma)$. For each ρ satisfying (6.3) we find $\phi \perp h$ so that $\pi_\rho S(w+\phi) = 0$. Here h is again given by (3.2). As we use the fixed point argument, Lemma 6.2 implies that $\|\phi\|_* = O(\gamma^{(1+\delta)/2})$. Then we find

$$
\begin{aligned}
I(w+\phi) &= I(w) + \frac{1}{2}\int_{R^3} S(w)\phi\,dx + O(\gamma^{1+\delta}) \\
&= I(w) + O(\|S(w)\|_*\|\phi\|_*)\int_0^{\infty} e^{-2\mu|r-\rho|}\,r^2dr \\
&= I(w) + O(\|S(w)\|_*\|\phi\|_*\rho^2) \\
&= I(w) + O(\gamma^{1+\delta}l^2(\gamma)).
\end{aligned}
$$

Finally we maximize $I(w+\phi)$ with respect to ρ. Theorem 6.1 follows from Lemma 6.3.

7 Discussion

The stability of the bubble profiles constructed in this paper should depend on the dimension of the space. But first it is obvious that by differentiating the equation of a bubble solution $u = u(x_1, x_2, ..., x_n)$ with respect to x_j, $j = 1, 2, ..., n$, one obtains an eigenfunction with eigenvalue 0. This 0 eigenvalue is a consequence of the translation invariance of our problem. We can only discuss stability modulo translation.

In the cases $n = 3$ and $n \geq 4$, we have obtained the solutions by maximizing $I(w + \phi)$ with respect to ρ. This means that a solution is a maximum of I when restricted in the submanifold $\{w(\cdot;\rho) + \phi(\cdot;\rho)\}$. Hence the solution must be unstable; actually it must be a saddle point.

In the cases $n = 1$ and $n = 2$, our conjecture is that the solutions are stable modulo translation. We will present a complete spectral analysis of all the bubble solutions elsewhere.

A Proof of Lemma 3.3

The proof mainly consists of comparison argument. We separate the two cases $r \in (\rho, \infty)$ and $r \in (0, \rho)$.

In the case $r \in (\rho, \infty)$, we consider an auxiliary function

$$A(y) = e^{-\mu y}, \ y = r - \rho \in (0, \infty). \tag{1.1}$$

In this appendix we work with the y-coordinate instead of the r-coordinate. Then

$$\begin{aligned} \Delta A - f'(0)A &= A'' + \frac{A'}{y+\rho} - f'(0)A \ = \ \mu^2 e^{-\mu y} - \frac{\mu e^{-\mu y}}{y+\rho} - f'(0)e^{-\mu y} \\ &= (\mu^2 - \frac{\mu}{y+\rho} - f'(0))e^{-\mu y} \ < \ -\frac{f'(0)}{2}e^{-\mu y} \end{aligned}$$

provided that μ is sufficiently small. Now we have ϕ, as a function of y as well, such that

$$|\phi'' + \frac{\phi'}{y+\rho} - f'(0)\phi| \leq c_0 e^{-\mu y}, \quad \lim_{y\to\infty} \phi(y) = 0.$$

Let

$$R(y) = [|\phi(0)| + \frac{2c_0}{f'(0)}]A(y) - \phi(y),$$

and we have

$$R(0) = |\phi(0) + \frac{2c_0}{f'(0)}| - \phi(0) \geq 0, \quad \lim_{y\to\infty} R(y) = 0,$$

and

$$R'' + \frac{R'}{y+\rho} - f'(0)R \leq -[|\phi(0)| + \frac{2c_0}{f'(0)}]\frac{f'(0)}{2}A(y) + c_0 e^{-\mu y} \leq 0. \tag{1.2}$$

We claim $R(y) \geq 0$ on $(0, \infty)$. Otherwise there exists $y_* \in (0, \infty)$ such that $R(y) \geq R(y_*)$ for all $y \in (0, \infty)$ and $R(y_*) < 0$. But at this minimum point y_*,

$$R''(y_*) \geq 0, \ R'(y_*) = 0, \ -f'(0)R(y_*) > 0.$$

Therefore

$$R''(y_*) + \frac{R'(y_*)}{y_* + \rho} - f'(0)R(y_*) > 0,$$

a contradiction to (1.2). Hence $R(y) \geq 0$, i.e.

$$\phi(y) \leq [|\phi(0)| + \frac{2c_0}{f'(0)}]e^{-\mu}.$$

One can carry out a similar argument with

$$\tilde{R}(y) = [|\phi(0)| + \frac{2c_0}{f'(0)}]A(y) + \phi(y)$$

to conclude that

$$-\phi(y) \leq [|\phi(0)| + \frac{2c_0}{f'(0)}]e^{-\mu}.$$

Hence we obtain

$$|\phi(y)| \leq [|\phi(0)| + \frac{2c_0}{f'(0)}]e^{-\mu y}. \tag{1.3}$$

In the case $r \in (0, \rho)$, we construct $A(y)$, with $y = r - \rho \in (-\rho, 0)$ exactly as in [19]. Let χ be a smooth cut-off function such that

$$\chi(t) = 1 \text{ for } |t| \leq 1,\ \chi(t) = 0 \text{ for } |t| \geq 2,\ 0 \leq \chi \leq 1. \tag{1.4}$$

Define the following auxiliary function

$$A(y) = e^{\mu y} + (e^{\mu y_0} - e^{\mu y})\chi(\mu(y + \rho)) \tag{1.5}$$

where

$$y_0 = -\rho + \frac{1}{\mu}. \tag{1.6}$$

If $y \in (-\rho, y_0)$, then $A(y) = e^{\mu y_0}$ and

$$A'' + \frac{A'}{y + \rho} - f'(0)A = -f'(0)e^{\mu y_0} \leq -f'(0)e^{\mu y}.$$

If $y \in (y_0, y_0 + \frac{1}{\mu})$, then

$$e^{\mu y_0} \leq e^{\mu y} \leq e e^{\mu y_0},\ A(y) \geq e^{\mu y_0},\ \frac{\mu}{2} \leq \frac{1}{y + \rho} \leq \mu;$$

hence

$$A'' + \frac{A'}{y + \rho} - f'(0)A \leq O(\mu^2)e^{\mu y} - f'(0)e^{\mu y_0} \leq O(\mu^2)e^{\mu y} - e^{-1}f'(0)e^{\mu y} \leq -\frac{f'(0)}{2e}e^{\mu y}$$

since μ is sufficiently small. If $y \in (y_0 + \frac{1}{\mu}, 0)$, then $A(y) = e^{\mu y}$ and $\frac{1}{y+\rho} \leq \frac{\mu}{2}$; hence

$$A'' + \frac{A'}{y + \rho} - f'(0)A \leq [\mu^2 + \frac{\mu^2}{2} - f'(0)]e^{\mu y} \leq -\frac{f'(0)}{2}e^{\mu y},$$

provided μ is small. Therefore for all $y \in (-\rho, 0)$,

$$A'' + \frac{A'}{y+\rho} - f'(0)A \leq -\frac{f'(0)}{2e}e^{\mu y}, \quad A'(-\rho) = 0. \tag{1.7}$$

If we have ϕ, as a function of y, such that

$$|\phi'' + \frac{\phi'}{y+\rho} - f'(0)\phi| \leq c_0 e^{\mu y} \text{ on } (-\rho, 0), \quad \phi'(-\rho) = 0,$$

then let

$$R(y) = [|\phi(0)| + \frac{2ec_0}{f'(0)}]A(y) - \phi(y).$$

Calculations show that

$$R'' + \frac{R'}{y+\rho} - f'(0)R \leq -[|\phi(0)| + \frac{2ec_0}{f'(0)}]\frac{f'(0)}{2e}e^{\mu y} + c_0 e^{\mu y} \leq 0, \tag{1.8}$$

and

$$R(0) \geq 0, \;\; R'(-\rho) = 0.$$

We then claim that $R(y) \geq 0$ for all $y \in (-\rho, 0)$. Otherwise there exists $y_* \in [-\rho, 0)$ such that $R(y_*) < 0$ and $R(y) \geq R(y_*)$ for all $y \in (-\rho, 0)$. However at this y_*, $R''(y_*) \geq 0$, $R'(y_*) = 0$, and $-f'(0)R(y_*) > 0$. A contradiction to (1.8). Therefore $R(y) \geq 0$, i.e.

$$\phi(y) \leq [|\phi(0)| + \frac{2ec_0}{f'(0)}]A(y) \leq e[|\phi(0)| + \frac{2ec_0}{f'(0)}]e^{\mu y}.$$

In the last inequality we have used the fact that

$$A(y) < ee^{\mu y} \text{ for all } y \in (-\rho, 0).$$

To see this note that $A(y) = e^{\mu y}$ if $y > y_0 + \frac{1}{\mu}$. If $y \in (y_0, y_0 + \frac{1}{\mu})$, $A(y) = e^{\mu y} + (e^{\mu y_0} - e^{\mu y})\chi(\mu(y + \rho)) \leq e^{\mu y}$ since $y_0 < y$ and $e^{\mu y_0} - e^{\mu y} < 0$ there. If $y \in (-\rho, y_0)$, then $y_0 - \frac{1}{\mu} = -\rho < y$, i.e. $y_0 < y + \frac{1}{\mu}$, and $A(y) = e^{\mu y_0} < e^{\mu(y+\frac{1}{\mu})} = ee^{\mu y}$. Similarly if we consider

$$\tilde{R}(y) = [|\phi(0)| + \frac{2ec_0}{f'(0)}]A(y) + \phi(y),$$

we find that

$$-\phi(y) \leq e[|\phi(0)| + \frac{2ec_0}{f'(0)}]e^{\mu y}.$$

In summary we have, on $(-\rho, 0)$,

$$|\phi(y)| \leq e[|\phi(0)| + \frac{2ec_0}{f'(0)}]e^{\mu y}.$$

The lemma is proved.

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