

# Soliton-stripe patterns of a functional with an attractive-repulsive-attractive interaction

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## Abstract

We study critical points of a Ginzburg-Landau type functional with an attractive-repulsive-attractive nonlocal interaction. Using an appropriate scaling and the  $\Gamma$ -convergence method we reduce the problem to a finite dimensional one. In contrast to a similar problem with just an attractive-repulsive interaction, we obtain a richer set of solutions. The soliton-stripe patterns appear as skewed local minimizers of the free energy, and disappear or become symmetric as the number of interfaces reaches a certain threshold. We also show how other critical points can be constructed using results of the diblock copolymer problem.

## 1 Introduction

We study the free energy functional

$$I_\epsilon[u] = \int_0^1 \left( \frac{\epsilon^2}{2} |u'|^2 + W(u) \right) dx + \epsilon \int_0^1 \left( \beta u - \frac{\alpha}{2} u^2 + \frac{u}{2} (\gamma G + \alpha)[u] \right) dx, \quad (1.1)$$

defined for  $u \in W^{1,2}(0,1)$ , where  $\alpha, \beta, \gamma$  are parameters with  $\gamma > 0$ ,  $W$  is a double-well function with at least quadratic growth rate, such that  $W(0) = W(1) = 0$ ,  $W''(0) = W''(1) > 0$ , e.g.,  $W(u) = \frac{1}{4}[(u - \frac{1}{2})^2 - \frac{1}{4}]^2$ ,  $(\gamma G + \alpha)[u](x) \equiv \int_0^1 (\gamma G(x,y) + \alpha) u(y) dy$  and  $G$  is Green's function of the  $-\Delta$  operator, with the Neumann boundary conditions.

The Euler-Lagrange equation for (1.1) is

$$\begin{cases} -\epsilon^2 \Delta u + \epsilon[(\gamma G + \alpha)[u] + \beta - \alpha u] + f(u) = 0, \\ u'(0) = u'(1) = 0, \end{cases} \quad (1.2)$$

where  $f = W'$ . (1.1) is the familiar Ginzburg-Landau free energy, with the addition of the nonlocal term  $\epsilon \int_0^1 (\beta u - \frac{\alpha}{2} u^2 + \frac{\gamma}{2} (\gamma G + \alpha)[u]) dx$ . In what follows, we construct periodic local minimizers of (1.1) for small  $\epsilon > 0$ , using the  $\Gamma$ -convergence method. These minimizers appear as lamellar patterns characterized by sharp domain walls (“solitons”) delineating microdomains (“stripes”) in which the phase field  $u$  takes values close to 0 or 1 (see [19] for a similar phenomenon).

The motivation for studying (1.1) is twofold.

First, if  $\alpha = \beta = 0$  then (1.1) is another way of writing the functional

$$I_\epsilon^c[u] = \int_0^1 \left( \frac{\epsilon^2}{2} |u'|^2 + W(u) + \frac{\epsilon\gamma}{2} |(-\Delta)^{-1/2}(u - m)|^2 \right) dx. \quad (1.3)$$

When defined in  $X_m \equiv \{u \in W^{1,2}(0,1) : \int_0^1 u = m\}$ , (1.3) models the diblock copolymer system (see [9, 14, 5] for derivation). Periodic local minimizers were constructed in [15]. Later, the authors extended the analysis to lamellar patterns in higher dimensional cubes [17, 18] (see also [16] for the discussion of the global minimizer of (1.3) with another scaling). As we show below, the difference in the spaces in which the functional is defined (we do not impose a mass constraint  $\int_0^1 u = m$  on (1.1)), together with the addition of the terms which are multiples of  $\alpha$  and  $\beta$ , has a significant effect on the minimization of (1.1).

The second motivation is that in (1.1)  $\int_0^1 (-\frac{\alpha}{2} u^2 + \frac{\gamma}{2} (\gamma G + \alpha)[u]) dx$  can be written as  $-\frac{1}{4} \int_0^1 \int_0^1 (\gamma G(x,y) + \alpha)(u(x) - u(y))^2 dx dy$ , therefore (1.1) is similar to the nonlocal van der Waals functional

$$I_\epsilon^n[u] = \frac{1}{4} \int_0^1 \int_0^1 J_\epsilon(x,y)(u(x) - u(y))^2 dx dy + \int_0^1 W(u(x)) dx \quad (1.4)$$

(defined on  $L^2(0,1)$ ). (1.4) (which, together with a mass constraint, was proposed in [20]) can be derived as a mean-field limit of an Ising spin system [1]. In [3],  $J_\epsilon$  was scaled by taking  $J_\epsilon(x,y) = \frac{1}{\epsilon} J^s(\frac{x-y}{\epsilon}) - \epsilon J^l(x,y)$ , with  $J^s \geq 0$ , and  $W = W_0 + \epsilon W_1$ , with  $W_0$  balanced.  $\frac{1}{4} \int_0^1 \int_0^1 \frac{1}{\epsilon} J^s(\frac{x-y}{\epsilon})(u(x) - u(y))^2 dx dy$  has the qualitative properties of  $\int_0^1 \frac{\epsilon^2}{2} |u'(x)|^2 dx$ . E.g., without the nonlocal term in (1.1) and without  $\frac{1}{4} \int_0^1 \int_0^1 J^l(x,y)(u(x) - u(y))^2 dx dy$  in (1.4), neither  $I_\epsilon$  nor  $I_\epsilon^n$  admit non-constant local minimizers [4]. To avoid some mathematical difficulties one can therefore consider  $I_\epsilon$  as an “approximation” of  $I_\epsilon^n$ . In [3], the authors studied (1.4) with  $J^l$  being the Green’s function of  $-v'' + v = u$ ,  $v'(0) = v'(1) = 0$ . Such a  $J^l$  is positive, which makes the nonlocal interaction in (1.4) locally attractive and long range repulsive (see [3] for a more detailed explanation). In contrast,  $G$  in this paper is the solution of  $-G_{xx}(x,y) = \delta_y(x) - 1$ ,

$G_x(0, y) = G_x(1, y) = 0$ ,  $\int_0^1 G(x, y)dx = 0 \forall y \in [0, 1]$ , whose exact formula is

$$G(x, y) = \begin{cases} \frac{x^2}{2} + \frac{(1-y)^2}{2} - \frac{1}{6}, & x < y, \\ \frac{(1-x)^2}{2} + \frac{y^2}{2} - \frac{1}{6}, & x > y. \end{cases} \quad (1.5)$$

We see that  $G$  changes sign, therefore for small  $\alpha$  the nonlocal interaction in (1.1) is attractive-repulsive-attractive.

Physically,  $u$  represents a general phase-field variable. The configuration of a binary material is reflected in  $u$ , and it is natural to associate the preferred states with local minimizers of  $I_\epsilon$ . The double well function  $W$  induces segregation of the mixture into states which are zeros of  $W$ , here 0 and 1. The term  $\frac{\epsilon^2}{2}|u'|^2$  prohibits the interfacial area from being too large. The two terms taken together give the Ginzburg-Landau functional, whose only local minima are 0 and 1 (see [7, 2] for a description of evolution of the gradient flow  $u_t = u_{xx} - f(u)$ , which results in exponentially slow motion). The addition of a long-range term in (1.1) introduces a competing, oscillation inducing effect.

The scaling in (1.1) is chosen so that  $\epsilon^{-1}I_\epsilon$   $\Gamma$ -converge to  $I_0$ , where

$$I_0[u] = \begin{cases} \tau \|Du\|(0, 1) \\ + \int_0^1 (\beta u - \frac{\alpha}{2}u^2 + \frac{\gamma}{2}(\gamma G + \alpha)[u])dx & u \in BV((0, 1), \{0, 1\}), \\ \infty & \text{otherwise.} \end{cases} \quad (1.6)$$

Here,  $\tau = \sqrt{2} \int_0^1 \sqrt{W(s)}ds$  is the surface tension and  $BV((0, 1), \{0, 1\}) = \{u \in BV(0, 1) : u(x) = 0 \text{ or } 1 \text{ a.e. } x \in (0, 1)\}$ .  $\|Du\|(0, 1)$  is equal to the number of jumps that  $u$  has. The main idea behind the construction of local minimizers of  $I_\epsilon$  is that if  $I_\epsilon$  satisfies an additional uniform coercivity property, then isolated minima of the  $\Gamma$ -limit persist under small perturbation (this was first proved and used in [8]). Minimizing (1.6) turns out to be a finite dimensional problem. Such an approach provides an elegant and fast way for constructing solutions with interfaces. Our results can probably be recovered by the more complicated method of matched asymptotic expansions, i.e., by constructing inner and outer solutions, then using the Implicit Function Theorem in an appropriate way [6, 11] (see also [10], where the author developed a more general technique to construct such solutions in systems of local equations).

Since we are comparing the minimization of  $I_\epsilon$  with those of  $I_\epsilon^c$  and  $I_\epsilon^n$ , we first mention the previous results [15, 3]. Both  $\epsilon^{-1}I_\epsilon^c$  and  $\epsilon^{-1}I_\epsilon^n$   $\Gamma$ -converge to functionals  $I_0^c$  and  $I_0^n$ , which are very similar to  $I_0$  (however, for  $I_0^n$   $\tau$  is defined in a different way). Let us briefly discuss the local minima of  $I_0^c$  and  $I_0^n$ , where in  $I_0^n$  we take  $J^l$  to be the aforementioned Green's function. The structure of  $BV((0, 1), \{0, 1\})$  is rather simple. For each integer  $N \geq 0$ , we have a subset

$$A_N = \{u \in BV((0, 1), \{0, 1\}) : \|Du\|(0, 1) = N\}, \quad (1.7)$$

the set of functions with  $N$  jumps. Let  $\xi_1, \dots, \xi_N$  denote the points of jump discontinuities of  $u \in A_N$ .  $A_N$  can be further divided into  $A_N^1$  and  $A_N^0$ :

$$A_N^1 = \{u \in A_N : u(x) = 1, x \in (0, \xi_1)\},$$

$$A_N^0 = \{u \in A_N : u(x) = 0, x \in (0, \xi_1)\},$$

so that we have a mutually disjoint decomposition

$$BV((0, 1), \{0, 1\}) = \cup_{N=0}^{\infty} (A_N^1 \cup A_N^0).$$

It turns out that it suffices to minimize  $I_0^c (I_0^n)$  in  $A_N^1 (A_N^0)$ . For any fixed  $N \geq 1$ ,  $I_0^c (I_0^n)$  admits a unique critical point in  $A_N^1 (A_N^0)$ , which is then an isolated local minimum of  $I_0^c (I_0^n)$ .

We follow a similar approach in the study of  $I_\epsilon$ . However, we find that  $I_0$  admits one, two or three critical points in  $A_N^1 (A_N^0)$ , of which one can be unstable. This result is significantly different from those for  $I_0^c$  or  $I_0^n$ . Not only is the possible number of interfaces  $N$  of a local minimizer dependent on the parameters  $\alpha$ ,  $\beta$  and  $\gamma$ , but also the critical points of  $I_0$  in  $A_N^1 (A_N^0)$  are not unique anymore. Moreover,  $I_0$  admits skewed local minimizers, which is perhaps a counterintuitive result.

The paper is organized as follows. In Section 2 we make the above heuristic discussion rigorous and present the computations in detail. In Section 3, we show how other critical points of  $I_\epsilon$  can be constructed using results for  $I_\epsilon^c$ .

## 2 Periodic local minimizers

We first review the  $\Gamma$ -convergence method.

**Proposition 2.1**  $\epsilon^{-1}I_\epsilon$   $\Gamma$ -converges to  $I_0$  as  $\epsilon \rightarrow 0$  in the following sense.

1. For every  $\{u_\epsilon\} \subset W^{1,2}(0, 1)$  with  $\lim_{\epsilon \rightarrow 0} u_\epsilon = u$  in  $L^2(0, 1)$ ,

$$\liminf_{\epsilon \rightarrow 0} \epsilon^{-1}I_\epsilon(u_\epsilon) \geq I_0(u);$$

2. For every  $u \in W^{1,2}(0, 1) \cap BV((0, 1), \{0, 1\})$ , there exists a family  $\{u_\epsilon\} \subset L^2(0, 1)$  such that  $\lim_{\epsilon \rightarrow 0} u_\epsilon = u$  in  $L^2(0, 1)$ , and

$$\limsup_{\epsilon \rightarrow 0} \epsilon^{-1}I_\epsilon(u_\epsilon) \leq I_0(u).$$

**Proof.** See [12, 13] for a proof for the Ginzburg-Landau functional. The conclusion follows since  $\Gamma$ -convergence is stable under continuous perturbations.  $\square$

**Proposition 2.2** Let  $\epsilon_n$  be a sequence of positive numbers converging to 0, and  $\{u_n\}$  a sequence in  $W^{1,2}(0, 1)$ . If  $\epsilon_n^{-1}I_{\epsilon_n}(u_n)$  is bounded above in  $n$ , then  $\{u_n\}$  is relatively compact in  $L^2(0, 1)$  and its cluster points belong to  $BV((0, 1), \{0, 1\})$ .

**Proof.** See [12, 13] or [15].  $\square$

Using Propositions 2.1 and 2.2 one can show [8, 15]

**Proposition 2.3** Let  $\delta > 0$  and  $u_0 \in L^2(0,1)$  be such that  $I_0(u_0) < I_0(u)$  for all  $B_\delta(u_0)$  with  $u \neq u_0$ . Then for small  $\epsilon$  there exists  $u_\epsilon \in B_{\delta/2}(u_0)$  with  $I_\epsilon(u_\epsilon) \leq I_\epsilon(u)$  for all  $u \in B_{\delta/2}(u_0)$ . In addition  $\lim_{\epsilon \rightarrow 0} \|u_\epsilon - u_0\|_2 = 0$ .

The next proposition allows us to focus our attention on minimizing  $I_0$  in  $A_N^1 (A_N^0)$ .

**Proposition 2.4** If  $u \in A_N^1 (A_N^0)$  is a strict local minimum of  $I_0$  restricted on  $A_N^1 (A_N^0)$ , then  $u$  is a strict local minimum of  $I_0$  defined on  $L^2(0,1)$ .

**Proof.** See [3] (Thm. 2.6, p. 146).  $\square$

Without loss of generality, we can assume  $u \in A_N^1$ . We identify  $u$  with its jump discontinuities  $\xi_1, \dots, \xi_N$ . Since  $\|Du\|(0,1) = N$  and  $N$  is fixed,  $I_0[u]$  can be expressed as a function of  $\xi_i$ 's. Let  $F(\xi_1, \dots, \xi_N) \equiv I_0[u]$ . Propositions 2.1-2.4 reduce the construction of local minimizers of  $I_\epsilon$  to the finite dimensional problem of finding strict local minima of  $F$ .

**Lemma 2.5** The critical points of  $F$  in  $A_N^1$ ,  $N \geq 1$ , are given by  $\xi_1 = \frac{\bar{u}}{N}$ ,  $\xi_2 = \frac{2-\bar{u}}{N}$ ,  $\xi_3 = \frac{2+\bar{u}}{N}$ ,  $\xi_4 = \frac{4-\bar{u}}{N}$ ,  $\dots$ . If  $u$  is determined by  $\xi_i$ 's then  $\bar{u} = \int_0^1 u$  is a solution of  $\frac{3N^2}{\gamma}[\beta - \alpha(\bar{u} - \frac{1}{2})] = \bar{u}(\bar{u} - 1)(1 - 2\bar{u})$ . Thus there are none, one, two or three solutions.

**Proof.** We first determine the critical points of  $F$ , then investigate their stability.

$$\begin{aligned} \frac{\partial F}{\partial \xi_1} &= \frac{\partial}{\partial \xi_1} \left[ (\beta - \frac{\alpha}{2})(\xi_1 + \xi_3 - \xi_2 + \dots) + \frac{\alpha}{2}\bar{u}^2 + \frac{\gamma}{2} \left( \int_0^{\xi_1} G[u](x)dx \right. \right. \\ &\quad \left. \left. + \int_{\xi_2}^{\xi_3} G[u](x)dx + \dots \right) \right] = \beta - \frac{\alpha}{2} + \alpha\bar{u} + \gamma G[u](\xi_1) \end{aligned}$$

since

$$\frac{\partial G[u]}{\partial \xi_1} = \frac{\partial}{\partial \xi_1} \left[ \int_0^{\xi_1} G(x,y)dy + \int_{\xi_2}^{\xi_3} G(x,y)dy + \dots \right] = G(x, \xi_1).$$

Thus we deduce that

$$\frac{\partial F}{\partial \xi_i} = (-1)^{i+1}(\beta + \alpha(\bar{u} - \frac{1}{2}) + \gamma G[u](\xi_i)), \quad i = 1, \dots, N.$$

Recalling (1.5), the critical points  $(\xi_1, \dots, \xi_N)$  are therefore determined from the system

$$\begin{cases} -v'' = u - \bar{u}, \quad \bar{v} = 0, \quad v'(0) = v'(1) = 0, \\ \beta + \alpha(\bar{u} - \frac{1}{2}) + \gamma v(\xi_i) = 0, \quad i = 1, \dots, N. \end{cases} \quad (2.1)$$

On  $(\xi_1, \xi_2)$ ,  $v$  solves  $-v'' = u - \bar{u}$ ,  $v(\xi_1) = v(\xi_2)$ , thus  $v$  is symmetric about  $(\xi_1 + \xi_2)/2$ , and hence  $v'(\xi_1) = -v'(\xi_2)$ . On  $(0, \xi_1)$  and  $(\xi_2, \xi_3)$ ,  $v$  solves  $-v'' = u$ ,  $v'(0) = 0$ ,  $v(\xi_2) = v(\xi_3)$ .  $v(\xi_1) = v(\xi_2)$  and  $v'(\xi_1) = -v'(\xi_2)$  thus imply that

$$2\xi_1 = \xi_3 - \xi_2. \quad (2.2)$$

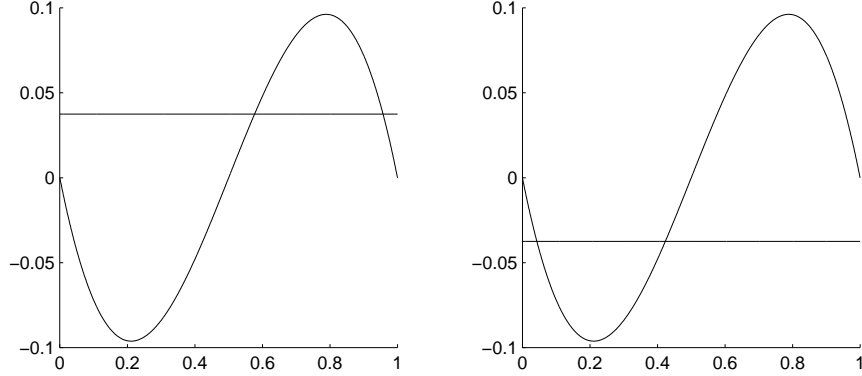


Figure 1: The two functions of  $\bar{u}$  on each graph represent  $\frac{3N^2\beta}{\gamma}$  and  $\bar{u}(\bar{u}-1)(1-2\bar{u})$ . (1) Case  $\beta > 0$  and  $\alpha = 0$ . (2) Case  $\beta < 0$  and  $\alpha = 0$ .

In a similar way we find that

$$\xi_2 - \xi_1 = \xi_4 - \xi_3. \quad (2.3)$$

Since  $\bar{u}$  can be represented as  $\bar{u} = \xi_1 + (\xi_3 - \xi_2) + \dots$  we get  $\xi_1 = \frac{\bar{u}}{N}$ . Also,  $1 = \xi_1 + (\xi_2 - \xi_1) + \dots + (1 - \xi_N) = N\xi_1 + \frac{N}{2}(\xi_2 - \xi_1)$  which gives  $\xi_2 = \frac{2-\bar{u}}{N}$ . With  $\xi_1$  and  $\xi_2$  determined, the rest of  $\xi_i$ 's are easily found using (2.2) and (2.3).

To find the dependence of  $\bar{u}$  on  $N$ , we use the equality  $\bar{v} = 0$ . To this end, first we determine from (2.1) that

$$v(x) = \begin{cases} \frac{\bar{u}-1}{2}x^2 - \frac{\bar{u}^2(\bar{u}-1)}{2N^2} - \frac{\beta + \alpha(\bar{u} - \frac{1}{2})}{\gamma}, & x \in (0, \frac{\bar{u}}{N}), \\ \frac{\bar{u}}{2}\left(x - \frac{1}{N}\right)^2 - \frac{\bar{u}(\bar{u}-1)^2}{2N^2} - \frac{\beta + \alpha(\bar{u} - \frac{1}{2})}{\gamma}, & x \in (\frac{\bar{u}}{N}, \frac{1}{N}). \end{cases}$$

After elementary calculations, the equality  $\int_0^{1/N} v(x)dx = 0$  can then be simplified to

$$\frac{3N^2}{\gamma}[\beta + \alpha(\bar{u} - \frac{1}{2})] = \bar{u}(\bar{u}-1)(1-2\bar{u}). \quad (2.4)$$

It is easily seen that for any given  $N \geq 1$  there are none, one, two or three solutions to (2.4) (Figures 1 and 2).  $\square$

We now determine the stability of the solutions of (2.4). To avoid tedious calculations, we only discuss the cases  $\alpha = 0$  or  $\beta = 0$ .

**Lemma 2.6** (a) If  $\alpha = \beta = 0$ , there exists one saddle critical point of  $F$  corresponding to  $\bar{u} = \frac{1}{2}$ .

(b) If  $\alpha = 0$  and  $\beta \neq 0$ , there exist two critical points of  $F$  if  $|\frac{3\beta N^2}{\gamma}| \in (0, \frac{\sqrt{3}}{12})$ , and no critical points if  $|\frac{3\beta N^2}{\gamma}| > \frac{\sqrt{3}}{12}$ . A critical point corresponding

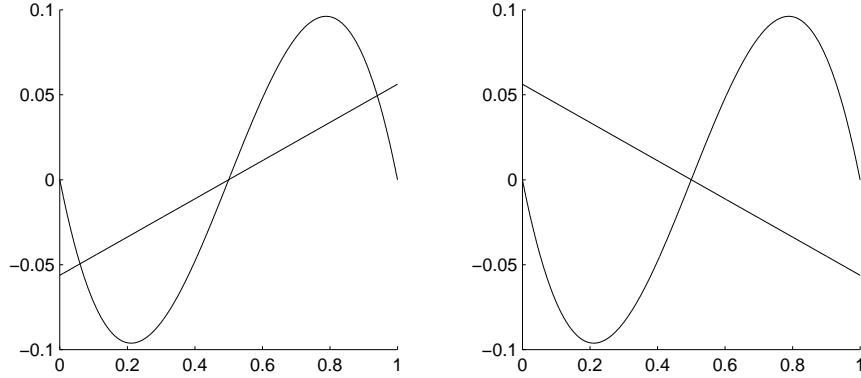


Figure 2: The two functions of  $\bar{u}$  on each graph represent  $\frac{3N^2\alpha}{\gamma}(\bar{u} - \frac{1}{2})$  and  $\bar{u}(\bar{u} - 1)(1 - 2\bar{u})$ . (1) Case  $\alpha > 0$  and  $\beta = 0$ . (2) Case  $\alpha < 0$  and  $\beta = 0$ .

to  $\bar{u} \in (0, \frac{1}{2} - \frac{\sqrt{3}}{6}) \cup (\frac{1}{2} + \frac{\sqrt{3}}{6}, 1)$  is a strict local minimum. A critical point corresponding to  $\bar{u} \in (\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} + \frac{\sqrt{3}}{6})$  is a saddle.

(c) If  $\beta = 0$  and  $\alpha \neq 0$ , there exists one saddle critical point  $\bar{u} = \frac{1}{2}$  if  $\alpha < 0$ . If  $\alpha > 0$ , there exist three critical points if  $\frac{3N^2\alpha}{\gamma} < \frac{1}{2}$  and one critical point if  $\frac{3N^2\alpha}{\gamma} > \frac{1}{2}$ , which is a strict local minimum. In the case of three critical points, the ones corresponding to  $\bar{u} \in (0, \frac{1}{2} - \frac{\sqrt{3}}{6}) \cup (\frac{1}{2} + \frac{\sqrt{3}}{6}, 1)$  are strict local minima, the one corresponding to  $\bar{u} = \frac{1}{2}$  is a saddle.

**Proof.** The existence part for case (a) is obvious. For case (b), note that  $\bar{u}(\bar{u} - 1)(1 - 2\bar{u})$  has local extrema at  $\frac{1}{2} \pm \frac{\sqrt{3}}{6}$ , with values  $\mp \frac{\sqrt{3}}{12}$ . Existence then is obtained with the help of Figure 1. For case (c), note that the derivative of the right side of (2.4) with respect to  $\bar{u}$  at  $\frac{1}{2}$  is  $\frac{1}{2}$ , and see Figure 2.

To determine the stability of the critical points, we need to find the eigenvalues of the hessian of  $F$  at  $(\xi_1, \dots, \xi_N)$ . The  $(i, j)$  entry of this matrix  $H$  is

$$H_{ij} \equiv \frac{\partial^2 F}{\partial \xi_i \partial \xi_j} = (-1)^{i+j} \alpha + \gamma \begin{cases} G(\xi_i, \xi_i) - \frac{\bar{u}(1-\bar{u})}{N}, & i = j, \\ (-1)^{i+j} G(\xi_i, \xi_j), & i \neq j. \end{cases}$$

To find the eigenvalues of  $H$  we first discuss the matrix  $[G(\xi_i, \xi_j)]$ . According to [17, Section 5 and Appendix B], it has  $N - 1$  eigenvalues whose eigenvectors are perpendicular to  $(1, 1, \dots, 1)^T$ . These eigenvalues are

$$\frac{1}{A + B - q_j}, \quad (2.5)$$

where

$$A = \frac{N}{2\bar{u}}, \quad B = \frac{N}{2(1-\bar{u})}, \quad (2.6)$$

and  $q_j$  is

$$\pm\sqrt{A^2 + B^2 + 2AB \cos \theta} \quad (\theta = \frac{2\pi(j-1)}{N}, j = 2, 3, \dots, \frac{N+1}{2}), \text{ if } N \text{ is odd} \quad (2.7)$$

or

$$\pm\sqrt{A^2 + B^2 + 2AB \cos \theta} \quad (\theta = \frac{2\pi(j-1)}{N}, j = 2, 3, \dots, \frac{N}{2}), A-B, \text{ if } N \text{ is even.} \quad (2.8)$$

The remaining eigenvalue corresponds to the eigenvector  $(1, 1, \dots, 1)^T$ . It is

$$\begin{aligned} \sum_{i=1}^N G(\xi_i, \xi_j) &= \sum_{i=1}^j \left( \frac{\xi_i^2}{2} + \frac{(1-\xi_j)^2}{2} - \frac{1}{6} \right) + \sum_{i=j+1}^N \left( \frac{(1-\xi_i)^2}{2} + \frac{\xi_j^2}{2} - \frac{1}{6} \right) \\ &= \sum_{i=1}^N \frac{\xi_i^2}{2} - \sum_{i=j+1}^N \xi_i + \frac{N\xi_j^2 - 2j\xi_j}{2} + \frac{N}{3}. \end{aligned} \quad (2.9)$$

Note that

$$\xi_i = \frac{i-1/2 + (-1)^i(1/2 - \bar{u})}{N}. \quad (2.10)$$

Using (2.10) we compute the first three terms in (2.9).

$$\sum_{i=1}^N \frac{\xi_i^2}{2} = \frac{N}{6} - \frac{1}{24N} + \frac{(1/2 - \bar{u})^2}{2N} + \frac{(1/2 - \bar{u})(-1)^N}{2N}, \quad (2.11)$$

$$\sum_{i=j+1}^N \xi_i = \frac{N^2 - j^2 + (1/2 - \bar{u})(-1)^{j+1}(1 - (-1)^{N-j})}{2N}, \quad (2.12)$$

$$\frac{N\xi_j^2 - 2j\xi_j}{2} = \frac{((-1)^j(1/2 - \bar{u}) - 1/2)^2 - j^2}{2N}. \quad (2.13)$$

Putting (2.11) (2.12) and (2.13) back to (2.9) we find the last eigenvalue of  $[G(\xi_i, \xi_j)]$ :

$$\frac{1}{12N} + \frac{(1/2 - \bar{u})^2}{N}. \quad (2.14)$$

Now we consider the matrix  $\tilde{H}$  whose  $(i, j)$  entry is

$$\tilde{H}_{ij} = \alpha + \gamma G(\xi_i, \xi_j) - \gamma \begin{cases} \frac{\bar{u}(1-\bar{u})}{N} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \quad (2.15)$$

Since  $H_{ij} = (-1)^{i+j} \tilde{H}_{ij}$ ,  $H$  and  $\tilde{H}$  share the same eigenvalues, because if  $\lambda$  and  $(c_j)_{j=1}^N$  are an eigenpair of  $\tilde{H}$ , then  $\lambda$  and  $((-1)^j c_j)_{j=1}^N$  are an eigenpair of  $H$ . The matrix  $\tilde{H}$  consists of three parts: The rank one matrix  $[\alpha]$ , the matrix  $[G_{ij}]$  and a scalar multiple of the identity matrix. The rank one matrix  $[a]$  has one eigenvalue  $N$  corresponding to the eigenvector  $(1, 1, \dots, 1)^T$  and

another eigenvalue 0, of multiplicity  $N - 1$ , whose eigenspace is the  $N - 1$  dimensional subspace perpendicular to  $(1, 1, \dots, 1)^T$ . Hence the eigenvalues of  $\tilde{H}$  are as follows. Corresponding to the eigenvector  $(1, 1, \dots, 1)^T$ , the eigenvalue of  $\tilde{H}$  is

$$\alpha N + \gamma \left[ \frac{1}{12N} + \frac{(1/2 - \bar{u})^2}{N} - \frac{\bar{u}(1 - \bar{u})}{N} \right].$$

Other eigenvectors of  $\tilde{H}$  are perpendicular to  $(1, 1, \dots, 1)^T$ . They are

$$\gamma \left[ \frac{1}{A + B - q_j} - \frac{\bar{u}(1 - \bar{u})}{N} \right].$$

By (2.5), (2.6), (2.7) and (2.8) we find that these  $N - 1$  eigenvalues are positive, since

$$\frac{1}{A + B - q_j} - \frac{\bar{u}(1 - \bar{u})}{N} > \frac{1}{A + B + A + B} - \frac{\bar{u}(1 - \bar{u})}{N} = 0.$$

Therefore whether  $H$  is positive definite depends on whether

$$\alpha N + \gamma \left[ \frac{1}{12N} + \frac{(\bar{u} - \frac{1}{2})^2}{N} - \frac{\bar{u}(1 - \bar{u})}{N} \right] \quad (2.16)$$

is positive. Thus a critical point of  $F$  corresponding to  $\bar{u}$  is a strict local minimum if and only if (2.16) is positive. In case (b), this is equivalent to  $\bar{u} \in (0, \frac{1}{2} - \frac{\sqrt{3}}{6}) \cup (\frac{1}{2} + \frac{\sqrt{3}}{6}, 1)$ . In case (c), this is equivalent to  $\frac{3N^2\alpha}{\gamma} > \frac{1}{2}$ .  $\square$

Let  $B_\delta(u)$  denote an  $L^2(0, 1)$  ball with radius  $\delta$  and center  $u$ . Using Proposition 2.4 and Lemma 2.6, we conclude that

**Theorem 2.7** *Let  $u_{N,0} \in A_N^1$ ,  $N \geq 1$ , be a strict local minimum of  $I_0$  corresponding to*

*$\bar{u} \in (0, \frac{1}{2} - \frac{\sqrt{3}}{6}) \cup (\frac{1}{2} + \frac{\sqrt{3}}{6}, 1)$  in the case  $\alpha = 0$ ,  $\beta \neq 0$  and  $|\frac{3\beta N^2}{\gamma}| \in (0, \frac{\sqrt{3}}{12})$ , or*

*$\bar{u} \in (0, \frac{1}{2} - \frac{\sqrt{3}}{6}) \cup (\frac{1}{2} + \frac{\sqrt{3}}{6}, 1)$  in the case  $\beta = 0$ ,  $\alpha \neq 0$  and  $\frac{3N^2\alpha}{\gamma} < \frac{1}{2}$ , or*

*$\bar{u} = \frac{1}{2}$  in the case  $\beta = 0$ ,  $\alpha \neq 0$  and  $\frac{3N^2\alpha}{\gamma} > \frac{1}{2}$ .*

*There exists a  $\delta > 0$  such that for small  $\epsilon > 0$  there exist local minima  $u_{N,\epsilon}$  of  $I_\epsilon$  in  $B_\delta(u_{N,0})$ , satisfying  $\lim_{\epsilon \rightarrow 0} \|u_{N,\epsilon} - u_{N,0}\|_2 = 0$ ,*

**Remark 2.8** *1. In a similar way, we can construct local minima of  $I_\epsilon$  that are close to those members in  $A_N^0$  which are strict local minima of  $I_0$ .*

*2. For  $I_\epsilon$  having a balanced double-well function, one might expect only solutions corresponding to  $\bar{u} = \frac{1}{2}$  to possibly be local minimizers. Instead we see that for  $\alpha = 0$ ,  $\beta \neq 0$  and  $|\frac{3\beta N^2}{\gamma}| \in (0, \frac{\sqrt{3}}{12})$ , or  $\beta = 0$ ,  $\alpha > 0$  and  $\frac{3N^2\alpha}{\gamma} < \frac{1}{2}$ ,  $I_0$  admits skewed strict local minima.*

Denote by  $\chi_N^i$ ,  $i = 1, 2, 3$ , the critical points of  $I_0$  in  $A_N^1$ , with  $\beta = 0$  and  $\alpha > 0$ . We conclude this section with a discussion of  $I_0(\chi_N^i)$  as a function of  $N$ . In [15, 3] it was shown that  $I_0^c(\chi_N)$  and  $I_0^n(\chi_N)$  were convex functions of  $N$ ,

where  $\chi_N$  denote the unique minimizers of  $I_0^c$  and  $I_0^n$  with  $N$  interfaces. Here the difference is that there are three branches of solutions to consider. Since

$$\begin{aligned} \int_0^1 uG[u]dx &= \int_0^1 v'^2 dx = N \left( \int_0^{\bar{u}/N} (\bar{u}-1)^2 x^2 dx + \int_{\bar{u}/N}^{1/N} \bar{u}^2 \left(x - \frac{1}{N}\right)^2 dx \right) \\ &= \frac{\bar{u}^2(\bar{u}-1)^2}{3N^2}, \end{aligned}$$

we get

$$I_0(\chi_N^i) = \tau N - \frac{\alpha\bar{u}}{2} + \frac{\alpha\bar{u}^2}{2} + \frac{\gamma\bar{u}^2(\bar{u}-1)^2}{6N^2}.$$

Let  $\chi_N^2$  be the solution corresponding to  $\bar{u} = \frac{1}{2}$ . Then since  $\bar{u}$  is constant with respect to  $N$  we see that  $I_0(\chi_N^2)$  a convex function of  $N$ . If  $\bar{u} \neq \frac{1}{2}$  then (2.4) can be written as  $\frac{3N^2\alpha}{-2\gamma} = \bar{u}(\bar{u}-1)$ . Using this equality we get for  $i = 1, 3$

$$I_0(\chi_N^i) = \tau N - \frac{3N^2\alpha^2}{8\gamma}.$$

We see that for  $i = 1, 3$ ,  $I_0(\chi_N^i)$  is a concave function of  $N$ .

### 3 Other critical points

Proposition 2.4 enabled us to conclude that if  $I_0$  has strict local minimizers, as determined in Lemma 2.6, then  $I_\epsilon$  also admits local minimizers for  $\epsilon > 0$  small enough, which converge to those of  $I_0$  as  $\epsilon \rightarrow 0$  in  $L^2$  norm. However, if  $I_0$  has unstable critical points, e.g., the saddles discussed in Lemma 2.6, we do not know if one can use any argument based on  $\Gamma$ -convergence to obtain existence of unstable critical points of  $I_\epsilon$ , converging to the saddles of  $I_0$ .

Here we make a connection with the diblock copolymer problem to construct critical points of  $I_\epsilon$  for  $\alpha = 0$ , converging to the saddles of  $I_0$ . Note that the Euler-Lagrange equation for  $I_\epsilon^c$  (1.3) defined in  $X_m$  is

$$\begin{cases} -\epsilon^2 \Delta u + \epsilon \gamma G[u] + f(u) = \lambda_\epsilon^m \\ u'(0) = u'(1) = 0. \end{cases} \quad (3.1)$$

A local minimizer  $u_\epsilon^m$  solves (3.1), thus the Lagrange multiplier  $\lambda_\epsilon^m = \int_0^1 f(u_\epsilon^m)$ . If  $-\epsilon\beta$  is equal to  $\lambda_\epsilon^m$ , then the solution of (3.1) is also a solution of (1.2). We show that this is actually the case for small  $\epsilon > 0$  by using an asymptotic expansion of  $\lambda_\epsilon^m$  in  $\epsilon > 0$  determined in [17].

We first establish the following continuity property of  $\lambda_\epsilon^m$ 's.

Let  $\chi_N^m$  be the strict local minimizer of  $I_0^c$  in  $A_N^1 \cap X_m$  [15]. It was shown in [15] that  $\chi_N^m$  is continuous in  $m$  in  $L^2(0,1)$  norm and for small enough  $\epsilon > 0$  there exist local minimizers  $u_{N,\epsilon}^m$  of  $I_\epsilon^c$  in  $X_m$ , such that  $\|u_{N,\epsilon}^m - \chi_N^m\|_2 \rightarrow 0$  as  $\epsilon \rightarrow 0$ . In [17] it was shown that  $u_{N,\epsilon}^m$  are locally unique local minimizers of  $I_\epsilon^c$  in  $X_m$ , in a small  $L^2(0,1)$  neighbourhood of  $\chi_N^m$ .

**Lemma 3.1** For any  $N \geq 1$  and  $\epsilon > 0$ ,  $\lambda_{N,\epsilon}^m \equiv \int_0^1 f(u_{N,\epsilon}^m)$  is continuous in  $m$ .

**Proof.** Let  $i(m) = \inf I_\epsilon^c[u]$ , where the infimum is taken over a small  $L^2(0,1)$  neighbourhood of  $\chi_N^m$  intersected with  $X_m$ . Denote  $u_m = u_{N,\epsilon}^m$  and  $i(m) = I_\epsilon^c[u_m]$ . Suppose there is a sequence  $m_n \rightarrow m_0 \in (0,1)$  for which  $\|u_{m_n} - u_{m_0}\|_2 \not\rightarrow 0$ . There exists a subsequence  $m \rightarrow m_0$  and some  $u^*$ , such that  $u_m \rightarrow u^*$  strongly in  $L^2(0,1)$  and weakly in  $W^{1,2}(0,1)$ , as  $m \rightarrow m_0$ . Thus

$$\liminf_{m \rightarrow m_0} I_\epsilon^c[u_m] \geq I_\epsilon^c[u^*] \geq i(m_0).$$

On the other hand,  $i(m) \leq I_\epsilon^c[u_0 + m - m_0] = i(m_0) + o(m - m_0)$ , so we get  $I_\epsilon^c[u^*] \leq i(m_0)$ . From the local uniqueness discussed above,  $u^* = u_0$ , a contradiction. It now easily follows that  $\int_0^1 f(u_m) \rightarrow \int_0^1 f(u_{m_0})$ .  $\square$

The asymptotic expansion of  $\lambda_{N,\epsilon}^m$  is as follows.

**Proposition 3.2**

$$\lambda_{N,\epsilon}^m = \frac{\epsilon\gamma}{N^2} \int_0^1 G[\mathcal{U}^0 - m](1 - m)d\xi + O(N^2\epsilon^2)$$

where  $\mathcal{U}^0$  is 0 on  $(0, 1 - m)$  and 1 on  $(1 - m, 1)$ .

**Proof.** Since we assumed  $f'(0) = f'(1)$ , from [17, Lemma A.4] we get

$$\int_0^1 f(\mathcal{U})d\xi = \frac{\epsilon\gamma}{N^2} \int_0^1 G[\mathcal{U}^0 - m](1 - m)dx + O(N^2\epsilon^2),$$

where  $\mathcal{U} \in X_m$  is the locally unique one layer local minimizer of

$$\int_0^1 \left[ \frac{\epsilon^2 N^2}{2} |\mathcal{U}'|^2 + \frac{\epsilon\gamma}{2N^2} \left| \left( -\frac{d^2}{d\xi^2} \right)^{-1/2} (\mathcal{U} - m) \right|^2 + W(\mathcal{U}) \right] d\xi$$

that is close to  $\mathcal{U}^0$  [16]. It was shown in [17, Theorem 2.3] that  $u_{N,\epsilon}^m$  has the shape of  $N$  rescaled copies of  $\mathcal{U}$  or its reversal. Thus  $\int_0^1 f(\mathcal{U}) = \int_0^1 f(u_{N,\epsilon}^m)$  from which the Proposition follows.  $\square$

We can now obtain the existence result.

**Theorem 3.3** For any  $N \geq 1$  and  $|\frac{3\beta N^2}{\gamma}| \in (0, \frac{\sqrt{3}}{12})$ , for small  $\epsilon > 0$  there exist two solutions  $u_{N,\epsilon}^i$ ,  $i = 1, 2$ , of (1.2) with  $\alpha = 0$ , satisfying  $\lim_{\epsilon \rightarrow 0} \|u_{N,\epsilon}^i - u_{N,0}^i\|_2 = 0$ , where  $u_{N,0}^i$  are the critical points of  $I_0$  in  $A_N^0$ .

**Proof.** Since  $\int_0^1 G[\mathcal{U}^0 - m](1 - m)dx = \frac{1}{3}m(1 - m)(1 - 2m)$ , Proposition 3.2 implies that for any  $N \geq 1$  there exist solutions of

$$\begin{cases} -\epsilon^2 \Delta u + \epsilon\gamma G[u] + f(u) - \epsilon \frac{\gamma}{3N^2} m(1 - m)(1 - 2m) - O(N^2\epsilon^2) = 0, \\ u'(0) = u'(1) = 0. \end{cases}$$

From Lemma 3.1, the term  $O(N^2\epsilon^2)$  is continuous in  $m$ , therefore, recalling (2.4) and Figure 1, we see that for  $|\frac{3\beta N^2}{\gamma}| \in (0, \frac{\sqrt{3}}{12})$  and  $\epsilon > 0$  small, there exist two solutions  $u_{N,\epsilon}^i$ ,  $i = 1, 2$ , of (1.2).  $\lim_{\epsilon \rightarrow 0} \|u_{N,\epsilon}^i - u_{N,0}^i\|_2 = 0$  follows from [15].  $\square$

**Remark 3.4** *Theorem 3.3 gives us only the existence of solutions of (1.2) with  $\alpha = 0$ . It is quite likely that for a fixed  $N \geq 1$ , the skewed solution coincides with the local minimizer constructed in Theorem 2.7.*

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