

Soliton-stripe Patterns of a Functional with an Attractive–repulsive–attractive Interaction*

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We study critical points of a Ginzburg–Landau type functional with an attractive–repulsive–attractive nonlocal interaction. Using an appropriate scaling and Γ -convergence method we reduce the problem to a finite dimensional one. In contrast to a similar problem with just an attractive–repulsive interaction, we obtain a richer set of solutions. The soliton-stripe patterns appear as skewed local minimizers of the free energy, and disappear or become symmetric as the number of interfaces reaches a certain threshold. We also show how other critical points can be constructed using results of the diblock copolymer problem.

KEY WORDS: Attractive–repulsive–attractive; soliton–stripe pattern; Γ -convergence.

1. INTRODUCTION

We study the free functional

$$I_\epsilon[u] = \int_0^1 \left(\frac{\epsilon^2}{2} |u'|^2 + W(u) \right) dx + \epsilon \int_0^1 \left(\beta u - \frac{\alpha}{2} u^2 + \frac{u}{2} (\gamma G + \alpha)[u] \right) dx \quad (1.1)$$

defined for $u \in W^{1,2}(0, 1)$, where α, β, γ are parameters with $\gamma > 0$, W is a double-well function with at least quadratic growth rate, such that

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$W(0) = W(1) = 0, W''(0) = W''(1) > 0$, e.g., $W(u) = (1/4)[(u - (1/2))^2 - (1/4)]^2$, $(\gamma G + \alpha)[u](x) \equiv \int_0^1 (\gamma G(x, y) + \alpha)u(y)dy$ and G is Green's function of the $-\Delta$ operator, with the Neumann boundary conditions.

The Euler–Lagrange equation for (1.1) is

$$\begin{aligned} -\epsilon^2 \Delta u + \epsilon[(\gamma G + \alpha)[u] + \beta - \alpha u] + f(u) &= 0, \\ u'(0) = u'(1) &= 0, \end{aligned} \tag{1.2}$$

where $f = W'$. (1.1) is the familiar Ginzburg–Landau free energy, with the addition of the nonlocal term $\epsilon \int_0^1 (\beta u - (\alpha/2)u^2 + (u/2)(\gamma G + \alpha)[u]) dx$. In what follows, we construct periodic local minimizers of (1.1) for small $\epsilon > 0$, using the Γ -convergence method. These minimizers appear as lamellar patterns characterized by sharp domain walls (“solitons”) delineating microdomains (“stripes”) in which the phase field u takes values close to 0 or 1 (see [19] for a similar phenomenon).

The motivation for studying (1.1) is twofold.

First, if $\alpha = \beta = 0$ then (1.1) is another way of writing the functional

$$I_\epsilon^c[u] = \int_0^1 \left(\frac{\epsilon^2}{2} |u'|^2 + W(u) + \frac{\epsilon\gamma}{2} |(-\Delta)^{-1/2}(u - m)|^2 \right) dx. \tag{1.3}$$

When defined in $X_m \equiv \{u \in W^{1,2}(0, 1) : \int_0^1 u = m\}$ (1.3) models the diblock copolymer system (see [5, 9, 14] for derivation). Periodic local minimizers were constructed in [15]. Later, the authors extended the analysis to lamellar patterns in higher dimensional cubes [17, 18] (see also [16] for the discussion of the global minimizer of (1.3) with another scaling). As we show below, the difference in the spaces in which the functional is defined (we do not impose a mass constraint $\int_0^1 u = m$ on (1.1)), together with the addition of the terms which are multiples of α and β , has a significant effect on the minimization of (1.1).

The second motivation is that in (1.1) $\int_0^1 (-\alpha/2)u^2 + (u/2)(\gamma G + \alpha)[u] dx$ can be written as $-(1/4) \int_0^1 \int_0^1 (\gamma G(x, y) + \alpha)(u(x) - u(y))^2 dx dy$, therefore (1.1) is similar to the nonlocal van der Waals functional

$$I_\epsilon^n[u] = \frac{1}{4} \int_0^1 \int_0^1 J_\epsilon(x, y)(u(x) - u(y))^2 dx dy + \int_0^1 W(u(x)) dx \tag{1.4}$$

(defined on $L^2(0, 1)$). (1.4) (which, together with a mass constraint, was proposed in [20]) can be derived as a mean-field limit of an Ising spin system [1]. In [3], J_ϵ was scaled by taking $J_\epsilon(x, y) = (1/\epsilon)J^s((x - y)/\epsilon) - \epsilon J^l(x, y)$, with $J^s \geq 0$, and $W = W_0 + \epsilon W_1$, with W_0 balanced. $(1/4) \int_0^1 \int_0^1 (1/\epsilon)J^s((x - y)/\epsilon)(u(x) - u(y))^2 dx dy$ has the qualitative properties of $\int_0^1 (\epsilon^2/2)|u'(x)|^2 dx$. For example, without the nonlocal term in

(1.1) and without $\frac{1}{4} \int_0^1 \int_0^1 J^l(x, y)(u(x) - u(y))^2 dx dy$ in (1.4), neither I_ϵ nor I_ϵ^n admit nonconstant local minimizers [4]. To avoid some mathematical difficulties one can therefore consider I_ϵ as an “approximation” of I_ϵ^n . In [3], the authors studied (1.4) with J^l being the Green’s function of $-v'' + v = u, v'(0) = v'(1) = 0$. Such a J^l is positive, which makes the non-local interaction in (1.4) locally attractive and long range repulsive (see [3] for a more detailed explanation). In contrast, G in this paper is the solution of $-G_{xx}(x, y) = \delta_y(x) - 1, G_x(0, y) = G_x(1, y) = 0, \int_0^1 G(x, y) dx = 0 \forall y \in [0, 1]$, whose exact formula is

$$G(x, y) = \begin{cases} \frac{x^2}{2} + \frac{(1-y)^2}{2} - \frac{1}{6}, & x < y, \\ \frac{(1-x)^2}{2} + \frac{y^2}{2} - \frac{1}{6}, & x > y. \end{cases} \tag{1.5}$$

We see that G changes sign, therefore for small α the nonlocal interaction in (1.1) is attractive–repulsive–attractive.

Physically, u represents a general phase-field variable. The configuration of a binary material is reflected in u , and it is natural to associate the preferred states with local minimizers of I_ϵ . The double well function W induces segregation of the mixture into states which are zeros of W , here 0 and 1. The term $(\epsilon^2/2)|u'|^2$ prohibits the interfacial area from being too large. The two terms taken together give the Ginzburg–Landau functional, whose only local minima are 0 and 1 (see [2, 7] for a description of evolution of the gradient flow $u_t = u_{xx} - f(u)$, which results in exponentially slow motion). The addition of a long-range term in (1.1) introduces a competing, oscillation inducing effect.

The scaling in (1.1) is chosen so that $\epsilon^{-1}I_\epsilon \Gamma$ -converges to I_0 , where

$$I_0[u] = \begin{cases} \tau \|Du\|(0, 1), \\ + \int_0^1 (\beta u - \frac{\alpha}{2} u^2 + \frac{\gamma}{2} (\gamma G + \alpha)[u]) dx, & u \in BV((0, 1), \{0, 1\}), \\ \infty, & \text{otherwise.} \end{cases} \tag{1.6}$$

Here, $\tau = \sqrt{2} \int_0^1 \sqrt{W(s)} ds$ is the surface tension and $BV((0, 1), \{0, 1\}) = \{u \in BV(0, 1) : u(x) = 0 \text{ or } 1 \text{ a.e. } x \in (0, 1)\}$. $\|Du\|(0, 1)$, is equal to the number of jumps that u has. The main idea behind the construction of local minimizers of I_ϵ is that if I_ϵ satisfies an additional uniform coercivity property, then isolated minima of the Γ -limit persist under small perturbation (this was first proved and used in [8]). Minimizing (1.6) turns out to be a finite dimensional problem. Such an approach provides an elegant and fast way for constructing solutions with interfaces. Our results can probably be recovered by the more complicated method of matched asymptotic expansions, i.e., by constructing inner and outer solutions, then using

the Implicit Function Theorem in an appropriate way [6,11] (see also [10], where the author developed a more general technique to construct such solutions in systems of local equations).

Since we are comparing the minimization of I_ϵ with those of I_ϵ^c and I_ϵ^n , we first mention the previous results [3,15]. Both $\epsilon^{-1}I_\epsilon^c$ and $\epsilon^{-1}I_\epsilon^n$ Γ -converge to functionals I_0^c and I_0^n , which are very similar to I_0 (however, for I_0^n , τ is defined in a different way). Let us briefly discuss the local minima of I_0^c and I_0^n , where in I_0^n we take J^l to be the aforementioned Green's function. The structure of $BV((0, 1), \{0, 1\})$ is rather simple. For each integer $N \geq 0$, we have a subset

$$A_N = \{u \in BV((0, 1), \{0, 1\}) : ||Du||((0, 1)) = N\}, \tag{1.7}$$

the set of function with N jumps. Let ξ_1, \dots, ξ_N denote the points of jump discontinuities of $u \in A_N$. A_N can be further divided into A_N^1 and A_N^0 :

$$\begin{aligned} A_N^1 &= \{u \in A_N : u(x) = 1, x \in (0, \xi_1)\}, \\ A_N^0 &= \{u \in A_N : u(x) = 0, x \in (0, \xi_1)\}, \end{aligned}$$

so that we have a mutually disjoint decomposition

$$BV((0, 1), \{0, 1\}) = \cup_{N=0}^\infty (A_N^1 \cup A_N^0).$$

It turns out that it suffices to minimize $I_0^c(I_0^n)$ in $A_N^1(A_N^0)$. For any fixed $N \geq 1$, $I_0^c(I_0^n)$ admits a unique critical point in $A_N^1(A_N^0)$, which is then an isolated local minimum of $I_0^c(I_0^n)$.

We follow a similar approach in the study of I_ϵ . However, we find that I_0 admits one, two or three critical points in $A_N^1(A_N^0)$, of which one can be unstable. This result is significantly different from those for I_0^c or I_0^n . Not only is the possible number of interfaces N of a local minimizer dependent on the parameters α, β and γ , but also the critical points of I_0 in $A_N^1(A_N^0)$ are not unique anymore. Moreover, I_0 admits skewed local minimizers, which is perhaps a counterintuitive result.

This paper is organized as follows. In Section 2 we make the above heuristic discussion rigorous and present the computations in detail. In Section 3, we show how other critical points of I_ϵ can be constructed using results for I_ϵ^c .

2. PERIODIC LOCAL MINIMIZERS

We first review the Γ -convergence method.

Proposition 2.1. $\epsilon^{-1}I_\epsilon$ Γ -converges to I_0 as $\epsilon \rightarrow 0$ in the following sense:

1. For every $\{u_\epsilon\} \subset W^{1,2}(0, 1)$ with $\lim_{\epsilon \rightarrow 0} u_\epsilon = u$ in $L^2(0, 1)$,

$$\liminf_{\epsilon \rightarrow 0} \epsilon^{-1} I_\epsilon(u_\epsilon) \geq I_0(u);$$

2. For every $u \in W^{1,2}(0, 1) \cap BV((0, 1), \{0, 1\})$, there exists a family $\{u_\epsilon\} \subset L^2(0, 1)$ such that $\lim_{\epsilon \rightarrow 0} u_\epsilon = u$ in $L^2(0, 1)$, and

$$\limsup_{\epsilon \rightarrow 0} \epsilon^{-1} I_\epsilon u_\epsilon \leq I_0(u).$$

Proof. See [12,13] for a proof for the Ginzburg–Landau functional. The conclusion follows since Γ -convergence is stable under continuous perturbations. □

Proposition 2.2. Let ϵ_n be a sequence of positive numbers converging to 0, and $\{u_n\}$ a sequence in $W^{1,2}(0, 1)$. If $\epsilon_n^{-1} I_{\epsilon_n}(u_n)$ is bounded above in n , then $\{u_n\}$ is relatively compact in $L^2(0, 1)$ and its cluster points belong to $BV((0, 1), \{0, 1\})$.

Proof. See [12,13] or [15]. □

Using Propositions 2.1 and 2.2 one can show [8,15] the following proposition.

Proposition 2.3. Let $\delta > 0$ and $u_0 \in L^2(0, 1)$ be such that $I_0(u_0) < I_0(u)$ for all $B_\delta(u_0)$ with $u \neq u_0$. Then for small ϵ there exists $u_\epsilon \in B_{\delta/2}(u_0)$ with $I_\epsilon(u_\epsilon) \leq I_\epsilon(u)$ for all $u \in B_{\delta/2}(u_0)$. In addition $\lim_{\epsilon \rightarrow 0} \|u_\epsilon - u_0\|_2 = 0$.

Proposition 2.4 allows us to focus our attention on minimizing I_0 in $A_N^1(A_N^0)$.

Proposition 2.4. If $u \in A_N^1(A_N^0)$ is a strict local minimum of I_0 restricted on $A_N^1(A_N^0)$, then u is a strict local minimum of I_0 defined on $L^2(0, 1)$.

Proof. See [3, Theorem 2.6, p. 146]. □

Without loss of generality, we can assume $u \in A_N^1$. We identify u with its jump discontinuities ξ_1, \dots, ξ_N . Since $\|Du\|(0, 1) = N$ and N is fixed, $I_0[u]$ can be expressed as a function of ξ_i 's. Let $F(\xi_1, \dots, \xi_N) \equiv I_0[u]$. Propositions 2.1–2.4 reduce the construction of local minimizers of I_ϵ to the finite-dimensional problem of finding strict local minima of F .

Lemma 2.5. The critical points of F in $A_N^1, N \geq 1$, are given by $\xi_1 = (\bar{u}/N), \xi_2 = (2 - \bar{u})/N, \xi_3 = (2 + \bar{u})/N, \xi_4 = (4 - \bar{u})/N, \dots$ If u is determined by ξ_1 's then $\bar{u} = \int_0^1 u$ is a solution of $(3N^2/\gamma)[\beta - \alpha(\bar{u} - (1/2))] = \bar{u}(\bar{u} - 1)(1 - 2\bar{u})$. Thus there are none, one, two or three solutions.

Proof. We first determine the critical points of F , then investigate their stability.

$$\frac{\partial F}{\partial \xi_1} = \frac{\partial}{\partial \xi_1} \left[\left(\beta - \frac{\alpha}{2} \right) (\xi_1 + \xi_3 - \xi_2 + \dots) + \frac{\alpha}{2} \bar{u}^2 + \frac{\gamma}{2} \left(\int_0^{\xi_1} G[u](x) dx + \int_{\xi_2}^{\xi_3} G[u](x) dx + \dots \right) \right] = \beta - \frac{\alpha}{2} + \alpha \bar{u} + \gamma G[u](\xi_1)$$

since

$$\frac{\partial G[u]}{\partial \xi_1} = \frac{\partial}{\partial \xi_1} \left[\int_0^{\xi_1} G(x, y) dy + \int_{\xi_2}^{\xi_3} G(x, y) dy + \dots \right] = G(x, \xi_1).$$

Thus we deduce that

$$\frac{\partial F}{\partial \xi_i} = (-1)^{i+1} \left(\beta + \alpha \left(\bar{u} - \frac{1}{2} \right) + \gamma G[u](\xi_i) \right), \quad i = 1, \dots, N.$$

Recalling (1.5), the critical points (ξ_1, \dots, ξ_N) are therefore determined from the system

$$\begin{aligned} -v'' &= u - \bar{u}, & \bar{v} &= 0, & v'(0) &= v'(1) = 0, \\ \beta + \alpha \left(\bar{u} - \frac{1}{2} + \gamma v(\xi_i) \right) &= 0, & i &= 1, \dots, N. \end{aligned} \tag{2.1}$$

On (ξ_1, ξ_2) , v solves $-v'' = u - \bar{u}$, $v(\xi_1) = v(\xi_2)$, thus v is symmetric about $(\xi_1 + \xi_2)/2$, and hence $v'(\xi_1) = -v'(\xi_2)$. On $(0, \xi_1)$ and (ξ_2, ξ_3) , v solves $-v'' = u$, $v'(0) = 0$, $v(\xi_2) = v(\xi_3)$. $v(\xi_1) = v(\xi_2)$ and $v'(\xi_1) = -v'(\xi_2)$ thus imply that

$$2\xi_1 = \xi_3 - \xi_2. \tag{2.2}$$

In a similar way we find that

$$\xi_2 - \xi_1 = \xi_4 - \xi_3. \tag{2.3}$$

Since \bar{u} can be represented as $\bar{u} = \xi_1 + (\xi_3 - \xi_2) + \dots$ we get $\xi_1 = (\bar{u}/N)$. Also, $1 = \xi_1 + (\xi_2 - \xi_1) + \dots + (1 - \xi_N) = N\xi_1 + (N/2)(\xi_2 - \xi_1)$ which gives $\xi_2 = (2 - \bar{u})/N$. With ξ_1 and ξ_2 determined, the rest of ξ_i 's are easily found using (2.2) and (2.3).

To find the dependence of \bar{u} on N , we use the equality $\bar{v} = 0$. To this end, first we determine from (2.1) that

$$v(x) = \begin{cases} \frac{\bar{u} - 1}{2} x^2 - \frac{\bar{u}^2(\bar{u} - 1)}{2N^2} - \frac{\beta + \alpha(\bar{u} - \frac{1}{2})}{\gamma}, & x \in \left(0, \frac{\bar{u}}{N} \right), \\ \frac{\bar{u}}{2} \left(x - \frac{1}{N} \right)^2 - \frac{\bar{u}(\bar{u} - 1)^2}{2N^2} - \frac{\beta + \alpha(\bar{u} - \frac{1}{2})}{\gamma}, & x \in \left(\frac{\bar{u}}{N}, \frac{1}{N} \right). \end{cases}$$

After elementary calculations, the equality of $\int_0^{1/N} v(x) dx = 0$ can then be simplified to

$$\frac{3N^2}{\gamma} \left[\beta + \alpha \left(\bar{u} - \frac{1}{2} \right) \right] = \bar{u}(\bar{u} - 1)(1 - 2\bar{u}). \tag{2.4}$$

It is easily seen that for any given $N \geq 1$ there are none, one, two or three solutions to (2.4) (Figs. 1 and 2). \square

We now determine the stability of the solutions of (2.4). To avoid tedious calculations, we only discuss the cases $\alpha = 0$ or $\beta = 0$.

Lemma 2.6. (a) *If $\alpha = \beta = 0$, there exists one saddle critical point of F corresponding to $\bar{u} = (1/2)$.*

(b) *If $\alpha = 0$ and $\beta \neq 0$, there exists two critical points of F if $|(3\beta N^2)/\gamma| \in (0, (\sqrt{3}/12))$, and no critical points if $|(3\beta N^2)/\gamma| > (\sqrt{3}/12)$. A critical point corresponding to $\bar{u} \in (0, (1/2) - (\sqrt{3}/6)) \cup ((1/2) + (\sqrt{3}/6), 1)$ is a strict local minimum. A critical point corresponding to $\bar{u} \in ((1/2) - (\sqrt{3}/6), (1/2) + (\sqrt{3}/6))$ is a saddle.*

(c) *If $\beta = 0$ and $\alpha \neq 0$, there exists one saddle critical point $\bar{u} = (1/2)$ if $\alpha < 0$. If $\alpha > 0$, there exist three critical points if $(3N^2\alpha)/\gamma < (1/2)$ and one critical point if $(3N^2\alpha)/\gamma > (1/2)$, which is a strict local minimum. In the case of three critical points, the ones corresponding to $\bar{u} \in (0, (1/2) - (\sqrt{3}/6)) \cup ((1/2) + (\sqrt{3}/6), 1)$ are strict local minima, the one corresponding to $\bar{u} = (1/2)$ is a saddle.*

Proof. The existence part for case (a) is obvious. For case (b), note that $\bar{u}(\bar{u} - 1)(1 - 2\bar{u})$ has local extrema at $(1/2) \pm (\sqrt{3}/6)$, with values $\mp(\sqrt{3}/12)$. Existence then is obtained with the help of Fig. 1. For case (c),

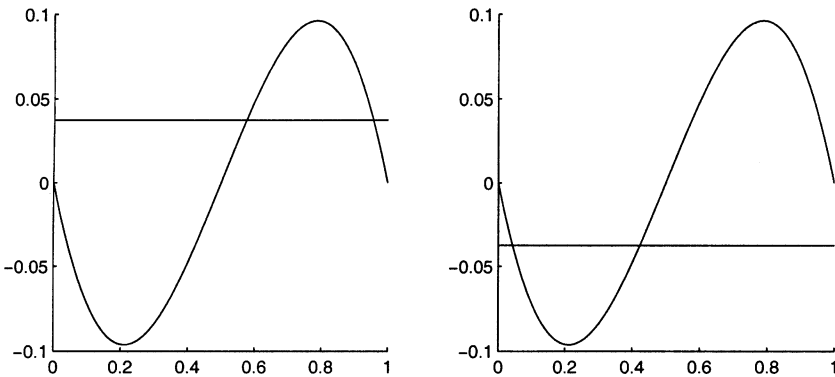


Figure 1. The two function of \bar{u} on each graph represent $(3N^2\beta)/\gamma$ and $\bar{u}(\bar{u} - 1)(1 - 2\bar{u})$. (1) Case $\beta > 0$ and $\alpha = 0$. (2) Case $\beta < 0$ and $\alpha = 0$.

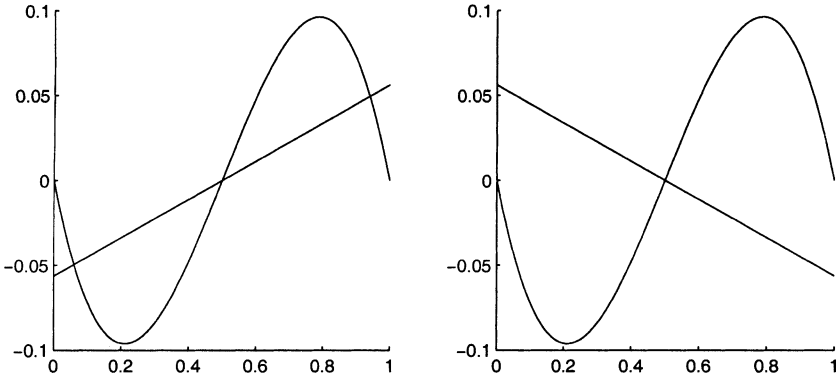


Figure 2. The two functions of \bar{u} on each graph represent $(3N^2\alpha)/\gamma(\bar{u} - (1/2))$ and $\bar{u}(\bar{u} - 1)(1 - 2\bar{u})$. (1) Case $\alpha > 0$ and $\beta = 0$. (2) Case $\alpha < 0$ and $\beta = 0$.

note that the derivative of the right-hand side of (2.4) with respect to \bar{u} at $(1/2)$ is $(1/2)$, and see Fig. 2.

To determine the stability of the critical points, we need to find the eigenvalues of the hessian of F at (ξ_1, \dots, ξ_N) . The (i, j) entry of this matrix H is

$$H_{ij} \equiv \frac{\partial^2 F}{\partial \xi_i \partial \xi_j} = (-1)^{i+j} \alpha + \gamma \begin{cases} G(\xi_i, \xi_i) - \frac{\bar{u}(1-\bar{u})}{N}, & i = j, \\ (-1)^{i+j} G(\xi_i, \xi_j), & i \neq j. \end{cases}$$

To find the eigenvalues of H we first discuss the matrix $[G(\xi_i, \xi_j)]$. According to [17, Section 5 and Appendix B], it has $N - 1$ eigenvalues whose eigenvectors are perpendicular to $(1, 1, \dots, 1)^T$. These eigenvalues are

$$\frac{1}{A + B - q_j}, \tag{2.5}$$

where

$$A = \frac{N}{2\bar{u}}, \quad B = \frac{N}{2(1-\bar{u})} \tag{2.6}$$

and q_j is

$$\pm \sqrt{A^2 + B^2 + 2AB \cos \theta} \left(\theta = \frac{2\pi(j-1)}{N}, j = 2, 3, \dots, \frac{N+1}{2} \right), \quad \text{if } N \text{ is odd} \tag{2.7}$$

or

$$\pm\sqrt{A^2 + B^2 + 2AB \cos \theta} \left(\theta = \frac{2\pi(j-1)}{N}, j = 2, 3, \dots, \frac{N}{2} \right), \quad A - B$$

if N is even. (2.8)

The remaining eigenvalue corresponds to the eigenvector $(1, 1, \dots, 1)^T$. It is

$$\begin{aligned} \sum_{i=1}^N G(\xi_i, \xi_j) &= \sum_{i=1}^j \left(\frac{\xi_i^2}{2} + \frac{(1-\xi_j)^2}{2} - \frac{1}{6} \right) + \sum_{i=j+1}^N \left(\frac{(1-\xi_i)^2}{2} + \frac{\xi_j^2}{2} - \frac{1}{6} \right) \\ &= \sum_{i=1}^N \frac{\xi_i^2}{2} - \sum_{i=j+1}^N \xi_i + \frac{N\xi_j^2 - 2j\xi_j}{2} + \frac{N}{3}. \end{aligned} \tag{2.9}$$

Note that

$$\xi_i = \frac{i - 1/2 + (-1)^i(1/2 - \bar{u})}{N}. \tag{2.10}$$

Using (2.10) we compute the first three terms in (2.9).

$$\sum_{i=1}^N \frac{\xi_i^2}{2} = \frac{N}{6} - \frac{1}{24N} + \frac{(1/2 - \bar{u})^2}{2N} + \frac{(1/2 - \bar{u})(-1)^N}{2N}, \tag{2.11}$$

$$\sum_{i=j+1}^N \xi_i = \frac{N^2 - j^2 + (1/2 - \bar{u})(-1)^{j+1}(1 - (-1)^{N-j})}{2N}, \tag{2.12}$$

$$\frac{N\xi_j^2 - 2j\xi_j}{2} = \frac{((-1)^j(1/2 - \bar{u}) - 1/2)^2 - j^2}{2N}. \tag{2.13}$$

Putting (2.11), (2.12) and (2.13) back to (2.9) we find the last eigenvalue of $[G(\xi_i, \xi_j)]$:

$$\frac{1}{12N} + \frac{(1/2 - \bar{u})^2}{N}. \tag{2.14}$$

Now we consider the matrix \tilde{H} whose (i, j) entry is

$$\tilde{H}_{ij} = \alpha + \gamma G(\xi_i, \xi_j) - \gamma \begin{cases} \frac{\bar{u}(1-\bar{u})}{N} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{2.15}$$

Since $H_{ij} = (-1)^{i+j} \tilde{H}_{ij}$, H and \tilde{H} share the same eigenvalues, because if λ and $(c_j)_{j=1}^N$ are an eigenpair of \tilde{H} , then λ and $((-1)^j c_j)_{j=1}^N$ are an eigenpair of H . The matrix \tilde{H} consists of three parts: The rank one matrix $[\alpha]$, the matrix $[G_{ij}]$ and a scalar multiple of the identity matrix. The

rank one matrix $[\alpha]$ has one eigenvalue N corresponding to the eigenvector $(1, 1, \dots, 1)^T$ and another eigenvalue 0 , of multiplicity $N - 1$ whose eigenspace is the $N - 1$ dimensional subspace perpendicular to $(1, 1, \dots, 1)^T$. Hence the eigenvalues of \tilde{H} are as follows. Corresponding to the eigenvector $(1, 1, \dots, 1)^T$, the eigenvalue of \tilde{H} is

$$\alpha N + \gamma \left[\frac{1}{12N} + \frac{(1/2 - \bar{u})^2}{N} - \frac{\bar{u}(1 - \bar{u})}{N} \right].$$

Other eigenvectors of \tilde{H} are perpendicular to $(1, 1, \dots, 1)^T$. They are

$$\gamma \left[\frac{1}{A + B - q_j} - \frac{\bar{u}(1 - \bar{u})}{N} \right].$$

By (2.5)–(2.8) we find that these $N - 1$ eigenvalues are positive, since

$$\frac{1}{A + B - q_j} - \frac{\bar{u}(1 - \bar{u})}{N} > \frac{1}{A + B + A + B} - \frac{\bar{u}(1 - \bar{u})}{N} = 0.$$

Therefore whether H is positive definite depends on whether

$$\alpha N + \gamma \left[\frac{1}{12N} + \frac{(\bar{u} - (1/2))^2}{N} - \frac{\bar{u}(1 - \bar{u})}{N} \right] \tag{2.16}$$

is positive. Thus a critical point of F corresponding to \bar{u} is a strict local minimum if and only if (2.16) is positive. In case (b), this is equivalent to $\bar{u} \in (0, (1/2) - (\sqrt{3}/6)) \cup ((1/2) + (\sqrt{3}/6), 1)$. In case (c), this is equivalent to $(3N^2\alpha)/\gamma > (1/2)$. □

Let $B_\delta(u)$ denote an $L^2(0, 1)$ ball with radius δ and center u . Using Proposition 2.4 and Lemma 2.6, we conclude the following result.

Theorem 2.7. *Let $u_{N,0} \in A_N^1, N \geq 1$, be a strict local minimum of I_0 corresponding to $\bar{u} \in (0, (1/2) - (\sqrt{3}/6)) \cup ((1/2) + (\sqrt{3}/6), 1)$ in the case $\alpha = 0, \beta \neq 0$ and $|(3\beta N^2)/\gamma| \in (0, (\sqrt{3}/12))$, or $\bar{u} \in (0, (1/2) - (\sqrt{3}/6)) \cup ((1/2) + (\sqrt{3}/6), 1)$ in the case $\beta = 0, \alpha \neq 0$ and $(3N^2\alpha)/\gamma < (1/2)$, or $\bar{u} = (1/2)$ in the case $\beta = 0, \alpha \neq 0$ and $(3N^2\alpha)/\gamma > (1/2)$.*

There exists a $\delta > 0$ such that for small $\epsilon > 0$ there exist local minima $u_{N,\epsilon}$ of I_ϵ in $B_\delta(u_{N,0})$, satisfying $\lim_{\epsilon \rightarrow 0} \|u_{N,\epsilon} - u_{N,0}\|_2 = 0$.

Remark 2.8. 1. In a similar way, we can construct local minima of I_ϵ that are close to those members in A_N^0 which are strict local minima of I_0 .

2. For I_ϵ having a balanced double-well function, one might expect only solutions corresponding to $\bar{u} = (1/2)$ to possibly be local minimizers.

Instead we see that for $\alpha=0, \beta \neq 0$ and $|(3\beta N^2)/\gamma| \in (0, (\sqrt{3}/12))$, or $\beta=0, \alpha > 0$ and $(3N^2\alpha)/\gamma < (1/2)$, I_0 admits skewed strict local minima.

Denote by $\chi_N^i, i = 1, 2, 3$, the critical points of I_0 in A_N^1 , with $\beta=0$ and $\alpha > 0$. We conclude this section with a discussion of $I_0(\chi_N^i)$ as a function of N . In [15,3] it was shown that $I_0^c(\chi_N)$ and $I_0^n(\chi_N)$ were convex functions of N , where χ_N denote the unique minimizers of I_0^c and I_0^n with N interface. Here the difference is that there are three branches of solutions to consider. Since

$$\int_0^1 uG[u]dx = \int_0^1 v^2 dx = N \left(\int_0^{\bar{u}/N} (\bar{u} - 1)^2 x^2 dx + \int_{\bar{u}/N}^{1/N} \bar{u}^2 (x - \frac{1}{N})^2 dx \right) = \frac{\bar{u}^2(\bar{u} - 1)^2}{3N^2},$$

we get

$$I_0(\chi_N^i) = \tau N - \frac{\alpha \bar{u}}{2} + \frac{\alpha \bar{u}^2}{2} + \frac{\gamma \bar{u}^2(\bar{u} - 1)^2}{6N^2}.$$

Let χ_N^2 be the solution corresponding to $\bar{u} = 1/2$. Then since \bar{u} is constant with respect to N we see that $I_0(\chi_N^2)$ a convex function of N . If $\bar{u} \neq 1/2$ then (2.4) can be written as $(3N^2\alpha)/-2\gamma = \bar{u}(\bar{u} - 1)$. Using this equality we get for $i = 1, 3$

$$I_0(\chi_N^i) = \tau N - \frac{3N^2\alpha^2}{8\gamma}.$$

We see that for $i = 1, 3, I_0(\chi_N^i)$ is a concave function of N .

3. OTHER CRITICAL POINTS

Proposition 2.4 enabled us to conclude that if I_0 has strict local minimizers, as determined in Lemma 2.6, then I_ϵ also admits local minimizers for $\epsilon > 0$ small enough, which converge to those of I_0 as $\epsilon \rightarrow 0$ in L^2 norm. However, if I_0 has unstable critical points, e.g, the saddles discussed in Lemma 2.6, we do not know if one can use any argument based on Γ -convergence to obtain existence of unstable critical points of I_ϵ , converging to the saddles of I_0 .

Here we make a connection with the diblock copolymer problem to construct critical points of I_ϵ for $\alpha = 0$, converging to he saddles of I_0 . Note that the Euler–Lagrange equation for I_ϵ^c (1.3) defined in X_m is

$$\begin{aligned} -\epsilon^2 \Delta u + \epsilon \gamma G[u] + f(u) &= \lambda_\epsilon^m, \\ u'(0) &= u'(1) = 0. \end{aligned} \tag{3.1}$$

A local minimizer u_ϵ^m solves (3.1), thus the Lagrange multiplier $\lambda_\epsilon^m = \int_0^1 f(u_\epsilon^m)$. If $-\epsilon\beta$ is equal to λ_ϵ^m , then the solution of (3.1) is also a solution of (1.2). We show that this is actually the case for small $\epsilon > 0$ by using an asymptotic expansion of λ_ϵ^m in $\epsilon > 0$ determined in [17].

We first establish the following continuity property of λ_ϵ^m 's.

Let χ_N^m be the strict local minimizer of I_0^c in $A_N^1 \cap X_m$ [15]. It was shown in [15] that χ_N^m is continuous in m in $L^2(0,1)$ norm and for small enough $\epsilon > 0$ there exist local minimizers $u_{N,\epsilon}^m$ of I_ϵ^c in X_m , such that $\|u_{N,\epsilon}^m - \chi_N^m\|_2 \rightarrow 0$ as $\epsilon \rightarrow 0$. In [17] it was shown that $u_{N,\epsilon}^m$ are locally unique local minimizers of I_ϵ^c in X_m , in a small $L^2(0,1)$ neighbourhood of χ_N^m .

Lemma 3.1. *For any $N \geq 1$ and $\epsilon > 0$, $\lambda_{N,\epsilon}^m \equiv \int_0^1 f(u_{N,\epsilon}^m)$ is continuous in m .*

Proof. Let $i(m) = \inf I_\epsilon^c[u]$, where the infimum is taken over a small $L^2(0,1)$ neighbourhood of χ_N^m intersected with X_m . Denote $u_m = u_{N,\epsilon}^m$ and $i(m) = I_\epsilon^c[u_m]$. Suppose there is a sequence $m_n \rightarrow m_0 \in (0, 1)$ for which $\|u_{m_n} - u_{m_0}\|_2 \not\rightarrow 0$. There exists a subsequence $m \rightarrow m_0$ and some u^* , such that $u_m \rightarrow u^*$ strongly in $L^2(0, 1)$ and weakly in $W^{1,2}(0, 1)$, as $m \rightarrow m_0$. Thus

$$\liminf_{m \rightarrow m_0} I_\epsilon^c[u_m] \geq I_\epsilon^c[u^*] \geq i(m_0).$$

On the other hand, $i(m) \leq I_\epsilon^c[u_0 + m - m_0] = i(m_0) + o(m - m_0)$, so we get $I_\epsilon^c[u^*] \leq i(m_0)$. From the local uniqueness discussed above, $u^* = u_0$, a contradiction. It now easily follows that $\int_0^1 f(u_m) \rightarrow \int_0^1 f(u_{m_0})$. □

The asymptotic expansion of $\lambda_{N,\epsilon}^m$ is as follows.

Proposition 3.2.

$$\lambda_{N,\epsilon}^m = \frac{\epsilon\gamma}{N^2} \int_0^1 G[\mathcal{W}^0 - m](1 - m) d\xi + O(N^2\epsilon^2),$$

where \mathcal{W}^0 is 0 on $(0, 1 - m)$ and 1 on $(1 - m, 1)$.

Proof. Since we assumed $f'(0) = f'(1)$, from [17, Lemma A.4] we get

$$\int_0^1 f(\mathcal{W}) d\xi = \frac{\epsilon\gamma}{N^2} \int_0^1 G[\mathcal{W}^0 - m](1 - m) dx + O(N^2\epsilon^2),$$

where $\mathcal{W} \in X_m$ is the locally unique one layer local minimizer of

$$\int_0^1 \left[\frac{\epsilon^2 N^2}{2} |\mathcal{W}'|^2 + \frac{\epsilon\gamma}{2N^2} \left| \left(-\frac{d^2}{d\xi^2}\right)^{-1/2} (\mathcal{W} - m) \right|^2 + W(\mathcal{W}) \right] d\xi$$

that is close to \mathcal{U}^0 [16]. It was shown in [17, Theorem 2.3] that $u_{N,\epsilon}^m$ has the shape of N rescaled copies of \mathcal{U} or reversal. Thus $\int_0^1 f(\mathcal{U}) = \int_0^1 f(u_{N,\epsilon}^m)$ from which the Proposition follows. \square

We can now obtain the existence result.

Theorem 3.3. *For any $N \geq 1$ and $|(3\beta N^2)/\gamma| \in (0, (\sqrt{3}/12))$, for small $\epsilon > 0$ there exist two solutions $u_{N,\epsilon}^i$, $i = 1, 2$, of (1.2) with $\alpha = 0$, satisfying $\lim_{\epsilon \rightarrow 0} \|u_{N,\epsilon}^i - u_{N,0}^i\|_2 = 0$, where $u_{N,0}^i$ are the critical points of I_0 in A_N^0 .*

Proof. Since $\int_0^1 G[\mathcal{U}^0 - m](1 - m)dx = (1/3)m(1 - m)(1 - 2m)$, Proposition 3.2 implies that for any $N \geq 1$ there exist solutions of

$$-\epsilon^2 \Delta u + \epsilon \gamma G[u] + f(u) - \epsilon \frac{\gamma}{3N^2} m(1 - m)(1 - 2m) - O(N^2 \epsilon^2) = 0, \\ u'(0) = u'(1) = 0.$$

From Lemma 3.1, the term $O(N^2 \epsilon^2)$ is continuous in m , therefore, recalling (2.4) and Fig. 1, we see that for $|(3\beta N^2)/\gamma| \in (0, \sqrt{3}/12)$ and $\epsilon > 0$ small, there exist two solutions $u_{N,\epsilon}^i$, $i = 1, 2$, of (1.2). $\lim_{\epsilon \rightarrow 0} \|u_{N,\epsilon}^i - u_{N,0}^i\|_2 = 0$ follows from [15]. \square

Remark 3.4. Theorem 3.3 gives us only the existence of solutions of (1.2) with $\alpha = 0$. It is quite likely that for a fixed $N \geq 1$, the skewed solution coincides with the local minimizer constructed in Theorem 2.7.

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