

Chiral symmetry breaking and the soliton-stripe pattern in Langmuir monolayers and smectic films

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Abstract

Chiral symmetry breaking in Langmuir monolayers and smectic films may be modelled by a system of two coupled fields: a scalar field that represents the chirality of the molecules and a vector field that represents the tilt direction of the molecules. In a particular parameter range, we prove the existence of a soliton-stripe pattern using the Γ -limit theory in perturbative variational calculus. This pattern, modelled by one-dimensional local minimizers of the free energy of the system, consists of stripes of molecules with distinct chirality in the film delineated by sharp domain walls.

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1. Introduction

There is a close relation between molecular chirality and pattern formation in liquid crystals. In bulk three-dimensional systems, chiral molecules can form a cholesteric phase with a helical pattern of twist in the molecular director, and a smectic- C^* phase, in which the director rotates from layer to layer [7].

In two-dimensional systems, chiral molecules can form a striped pattern of parallel defect walls. More interestingly, in some Langmuir monolayers and freely suspended smectic films of nonchiral molecules, similar striped patterns can occur. In these systems, chiral symmetry is spontaneously broken, leading to a chiral phase composed of nonchiral molecules. We study this phenomenon in this paper using a model proposed by Selinger *et al* [27].

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There are several mechanisms that can cause chiral symmetry breaking in two dimensions. We summarize them here. The reader should consult [27] for more detail and references. In a Langmuir monolayer, if it is in a tilted hexatic phase, the tilt direction can be locked at an angle between 0° and 30° from one of the local bond directions. This relation between tilt order and bond-orientational order breaks chiral symmetry. The chiral order parameter would be $u = \sin(6(\phi - \theta))$, where ϕ is the tilt azimuth and θ the bond orientation. Second, even if the monolayer is not in a tilted hexatic phase, the molecules might pack on the two-dimensional surface in two inequivalent ways that are mirror images of each other. The chiral order parameter would be the difference in the densities of the two packings. Third, if the monolayer is composed of a racemic mixture of two opposite enantiomers, the racemic mixture can separate to form chiral domains. In that case the chiral order parameter would be the differences in densities of the two enantiomers. In a freely suspended smectic thin film where the top and bottom surfaces are equivalent, \mathbf{n} is equivalent to $-\mathbf{n}$. A tilted hexatic film of cylindrical molecules always has an inversion symmetry. Hence, $\sin(6(\phi - \theta))$ is not a chiral order parameter. But chiral symmetry can still be broken by the other two mechanisms described above.

Like the Landau–de Gennes [6] model for three-dimensional smectics A, the model in [27] also has two order parameters, although its mathematical structure is simpler. The first is a scalar field, u , that characterizes the chirality of the molecules. $u > 0$ indicates one chiral state and $u < 0$ indicates the other chiral state. The second is a two-dimensional vector field, \mathbf{c} , of unit length, which characterizes the director of the molecules. It is actually the normalized projection of the three-dimensional molecular director, \mathbf{n} , onto the plane of the film. We neglect variations in the magnitude of the tilt. Suppose the film occupies $\Omega \subset \mathbf{R}^2$, so for $r \in \Omega$, $u = u(r) \in \mathbf{R}$ and $\mathbf{c} = \mathbf{c}(r) \in S^1 \subset \mathbf{R}^2$, where S^1 is the unit circle in \mathbf{R}^2 . The free energy of the system is

$$\mathcal{F}(u, \mathbf{c}) = \int_{\Omega} \left(\frac{\kappa}{2} |\nabla u|^2 - \frac{t}{2} u^2 + \frac{s}{4} u^4 + \frac{K_1}{2} (\nabla \cdot \mathbf{c})^2 + \frac{K_3}{2} (\nabla \times \mathbf{c})^2 - \lambda u \nabla \times \mathbf{c} \right) dr. \quad (1.1)$$

In (1.1), $\kappa, t, s, K_1, K_3, \lambda$ are all positive constants.

We make the single-Frank-constant approximation $K_1 = K_3 = \tilde{K}$. When \mathbf{c} can be expressed in the polar coordinates globally, i.e. $\mathbf{c} = (\cos v, \sin v)$, we simplify (1.1) to

$$\mathcal{F}(u, v) = \int_{\Omega} \left(\frac{\kappa}{2} |\nabla u|^2 - \frac{t}{2} u^2 + \frac{s}{4} u^4 + \frac{\tilde{K}}{2} |\nabla v|^2 - \lambda u \left(\frac{\partial \sin v}{\partial r_1} - \frac{\partial \cos v}{\partial r_2} \right) \right) dr. \quad (1.2)$$

We assume that Ω is a square: $\Omega = (0, L) \times (0, L)$. To separate the domain effect, we scale Ω to $D = (0, 1) \times (0, 1)$ by introducing $x = (x_1, x_2) \in D$, so that $r = Lx$. We scale u , \mathbf{c} and v to

$$\psi(x) = \frac{\sqrt{s}}{2\sqrt{t}} u(r), \quad \phi(x) = v(r), \quad (1.3)$$

so that

$$\mathcal{F}(u, v) = \frac{4L^2 t^2}{s} F(\psi, \phi) - \frac{L^2 t^2}{4s}, \quad (1.4)$$

where $F(\psi, \phi)$, the rescaled dimensionless free energy, is

$$F(\psi, \phi) = \int_D \left(\frac{\epsilon^2}{2} |\nabla \psi|^2 + W(\psi) + \frac{\epsilon \gamma}{2} |\nabla \phi|^2 - \epsilon \gamma \beta \psi \left(\frac{\partial \sin \phi}{\partial x_1} - \frac{\partial \cos \phi}{\partial x_2} \right) \right) dx. \quad (1.5)$$

The new parameters in (1.5) are related to the old parameters in (1.1) via

$$\epsilon^2 = \frac{\kappa}{L^2 t}, \quad \epsilon \gamma = \frac{\tilde{K} s}{8L^2 t^2}, \quad \epsilon \gamma \beta = \frac{\lambda \sqrt{s}}{2L \sqrt{t^3}}. \quad (1.6)$$

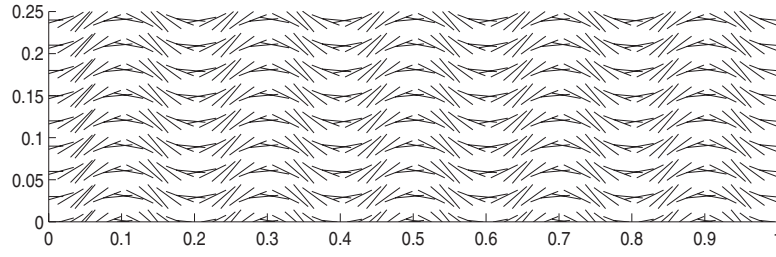


Figure 1. A soliton-stripe pattern of a chiral liquid crystal film. The director field is drawn on the two-dimensional film. The slope increases on lamellar stripes of one chiral type and decreases on stripes of the other type.

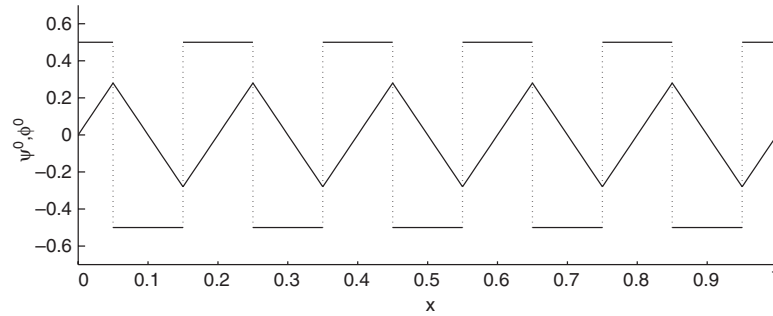


Figure 2. The discontinuous step function is ψ^0 , and the zigzag function is ϕ^0 .

The function W in (1.5) is

$$W(\psi) = \left(\psi^2 - \frac{1}{4}\right)^2. \tag{1.7}$$

The soliton-stripe pattern [27] is a lamellar pattern of ψ and ϕ , which only vary in one direction. In the context of a chiral liquid crystal the chiral field, ψ , takes values close to $-\frac{1}{2}$ in one lamellar stripe. Next to it is another lamellar stripe where ψ is close to $\frac{1}{2}$. The film is covered by a large number of lamellar stripes of alternating types. The stripes are of finite width and are delineated by sharp domain walls (solitons)⁵. The director field, ϕ , also varies along these stripes. When ψ is close to $\frac{1}{2}$, ϕ increases continuously in $(-\pi/2, \pi/2)$, and when ψ is close to $-\frac{1}{2}$, ϕ decreases continuously in $(-\pi/2, \pi/2)$. Figure 1 shows the director field, ϕ , on a film, and figure 2 shows how ψ and ϕ vary (in their asymptotic limit) along the direction perpendicular to the stripes.

In this paper, we will find soliton-stripe solutions in the parameter range

$$0 < \epsilon \ll 1, \quad \gamma \sim 1, \quad \beta \sim 1. \tag{1.8}$$

In terms of the original parameters, we assume

$$\frac{1}{L} \sqrt{\frac{\kappa}{t}} \ll 1, \quad \frac{\tilde{K}s}{L\sqrt{\kappa t^3}} \sim 1, \quad \frac{L\lambda}{\tilde{K}} \sqrt{\frac{t}{s}} \sim 1. \tag{1.9}$$

One may think of (1.9) as a condition under which a separation of two scales occurs. The two scales are the thickness of the interfaces (solitons) and the width of the stripes. According to (1.9), if the parameters t, s, κ, \tilde{K} and λ satisfy

$$\frac{\kappa t}{\tilde{K}s} \ll 1, \quad \frac{\lambda}{t} \sqrt{\frac{s}{k}} \sim 1, \tag{1.10}$$

⁵ There is another lamellar pattern, the sinusoidal pattern, where domain walls are not sharp [27].

we may choose

$$L \sim \frac{\tilde{K}s}{\sqrt{\kappa t^3}}. \quad (1.11)$$

Then the sample size is of the same order as the stripe width, while the interface thickness is of smaller order.

Since a soliton-stripe pattern varies along one direction, we assume that (ψ, ϕ) depends on $x_1 \in (0, 1)$ only, which we denote by x throughout the rest of this paper. To eliminate unnecessary boundary effects, we identify the boundary points 0 and 1 to impose the periodic boundary condition. The interval $(0, 1)$ is now \mathbf{R}/\mathbf{Z} . And (1.5) becomes

$$F_\epsilon(\psi, \phi) = \int_0^1 \left(\frac{\epsilon^2}{2} (\psi')^2 + W(\psi) + \frac{\epsilon\gamma}{2} (\phi')^2 + \epsilon\gamma\beta\psi' \sin\phi \right) dx, \quad (1.12)$$

defined in the admissible set $W^{1,2}(\mathbf{R}/\mathbf{Z}) \times W^{1,2}(\mathbf{R}/\mathbf{Z})$. Because \mathbf{R}/\mathbf{Z} acts on \mathbf{R}/\mathbf{Z} as a translation group, we will often encounter degeneracy caused by this action. We will frequently use phrases like ‘up to translation’ or ‘modulo translation’. Also note that if we let $c(x) = \mathbf{c}(r)$, then in one dimension here c is a map from $\mathbf{R}/\mathbf{Z} \equiv S^1$ to S^1 . The assumed existence of a globally continuous ϕ so that $c = (\cos\phi, \sin\phi)$ is equivalent to the condition that the winding number of c is zero.

In view of (1.8), we hold γ and β fixed and treat ϵ as a small parameter, so (1.12) is a singularly perturbed variational problem. The Euler–Lagrange equations of (1.12) are

$$-\epsilon^2\psi'' + W'(\psi) - \epsilon\gamma\beta(\sin\phi)' = 0, \quad (1.13)$$

$$-\phi'' + \beta\psi' \cos\phi = 0 \quad (1.14)$$

with the periodic boundary condition. The main result of this paper is the following existence theorem.

Theorem 1.1. *For each positive even integer, K , the functional F_ϵ has a local minimizer $(\psi_\epsilon, \phi_\epsilon)$ when ϵ is sufficiently small. It satisfies the Euler–Lagrange equations (1.13) and (1.14) and has the properties $\lim_{\epsilon \rightarrow 0} \|\psi_\epsilon - \psi^0\|_2 = 0$, $\lim_{\epsilon \rightarrow 0} \|\phi_\epsilon - \phi^0\|_{1,2} = 0$ modulo translation and $\lim_{\epsilon \rightarrow 0} \epsilon^{-1} F_\epsilon(\psi_\epsilon, \phi_\epsilon) = J(\psi^0)$.*

$\|\cdot\|_2$ denotes the L^2 -norm, and $\|\cdot\|_{1,2}$ denotes the $W^{1,2}$ -norm. J is defined in (2.20). That ψ_ϵ develops a soliton-stripe pattern of K domain walls as $\epsilon \rightarrow 0$ lies in the fact that the limiting profile, ψ_0 , of ψ_ϵ is a step function with K jump points of equal distance:

$$\psi^0(x) = \begin{cases} \frac{1}{2} & \text{on } \left(0, \frac{1}{2K}\right), \\ -\frac{1}{2} & \text{on } \left(\frac{1}{2K}, \frac{3}{2K}\right), \\ \frac{1}{2} & \text{on } \left(\frac{3}{2K}, \frac{5}{2K}\right), \\ -\frac{1}{2} & \text{on } \left(\frac{5}{2K}, \frac{7}{2K}\right), \\ \dots & \\ -\frac{1}{2} & \text{on } \left(\frac{2K-3}{2K}, \frac{2K-1}{2K}\right), \\ \frac{1}{2} & \text{on } \left(\frac{2K-1}{2K}, 1\right). \end{cases} \quad (1.15)$$

The corresponding ϕ^0 is

$$\phi^0(x) = \begin{cases} 2K\xi x & \text{on } \left(0, \frac{1}{2K}\right), \\ -2K\xi \left(x - \frac{1}{K}\right) & \text{on } \left(\frac{1}{2K}, \frac{3}{2K}\right), \\ 2K\xi \left(x - \frac{2}{K}\right) & \text{on } \left(\frac{3}{2K}, \frac{5}{2K}\right), \\ -2K\xi \left(x - \frac{3}{K}\right) & \text{on } \left(\frac{5}{2K}, \frac{7}{2K}\right), \\ \dots & \dots \\ -2K\xi \left(x - \frac{k-1}{K}\right) & \text{on } \left(\frac{2K-3}{2K}, \frac{2K-1}{2K}\right), \\ 2K\xi(x-1) & \text{on } \left(\frac{2K-1}{2K}, 1\right). \end{cases} \tag{1.16}$$

Here, ξ is the unique solution of

$$\cos \xi = \frac{4K}{\beta} \xi, \tag{1.17}$$

in $(0, \pi/2)$ (see figure 2). Note that ϕ^0 satisfies (1.14) when $\psi = \psi^0$.

Mathematical studies on periodic patterns with sharp domain walls have started rather recently. Many works have been done on the block copolymer problem. The literature includes Nishiura and Ohnishi [15], Ohnishi *et al* [16], Ren and Wei [18–23, 26], Choksi [3], Fife and Hilhorst [10], Henry [11] and Choksi and Ren [4]. Ren and Wei [24] deal with bending membranes, and [25] studies charged monolayers. Also see Ren and Truskinovsky [17] and Chmaj and Ren [2].

2. The Γ -limit

The Γ -limit theory of De Giorgi [8] is a rigorous singular perturbation theory in the calculus of variations. An introduction to the theory may be found in Dal Maso [5]. In this theory, there is a perturbed variational problem, which is often a standard one with a small parameter, say ϵ . The Euler–Lagrange equation is a differential equation. The limiting problem, as $\epsilon \rightarrow 0$, is usually a geometric problem whose extremals solve a free boundary problem. Certain properties of the limiting problem are carried over to the perturbed problem (corollary 2.4). In this sense, the perturbed problem is reduced to the limiting problem.

Different from our earlier works based on the Γ -convergence theory [18, 19, 23–25], here, we build a local version of the theory near ψ^0 and ϕ^0 .

Lemma 2.1. ϕ^0 is a stable solution of (1.14) where $\psi = \psi^0$ in the sense that the eigenvalues of the linearized operator of (1.14) at ϕ^0 are all positive.

Proof. The eigenvalue problem at ϕ^0 is

$$-q'' - \beta(\psi^0)'(\sin \phi^0)q = \lambda q. \tag{2.1}$$

Note that $(\psi^0)' = \sum_{j=1}^K (-1)^j \delta(\cdot - x_j^0)$ and $\sin(\phi^0(x_j^0)) = (-1)^{j+1} \sin \xi$. Hence (2.1) is simplified to

$$-q'' + \beta \sin \xi \left(\sum_{j=1}^K \delta(\cdot - x_j^0) \right) q = \lambda q. \tag{2.2}$$

Multiplying (2.2) by q and integrating by parts, we obtain

$$\int_0^1 (q')^2 dx + \beta \sin \xi \sum_{j=1}^K (q(x_j))^2 = \lambda \int_0^1 q^2 dx. \quad (2.3)$$

On the other hand,

$$\int_0^1 q^2 dx \leq 2(q(x_1))^2 + 2 \int_0^1 (q')^2 dx. \quad (2.4)$$

Equations (2.3) and (2.4) give a positive lower bound for λ :

$$\lambda \geq \frac{1}{2} \min\{1, \beta \sin \xi\}, \quad (2.5)$$

from which the lemma follows. \square

Lemma 2.2. *There is an L^2 neighbourhood of ψ^0 such that for every ψ in the neighbourhood there is a solution ϕ of (1.14). The (ψ, ϕ) pairs form a smooth curve under the $L^2 \times W^{1,2}$ -norm that passes through (ψ^0, ϕ^0) . Furthermore, the eigenvalues of the linearized operator of (1.14) at every ϕ are all positive.*

Proof. We define $Q: L^2(\mathbf{R}/\mathbf{Z}) \times W^{1,2}(\mathbf{R}/\mathbf{Z}) \rightarrow W^{1,2}(\mathbf{R}/\mathbf{Z})$ by

$$Q(\psi, \phi) = \phi + \left(-\frac{d^2}{dx^2} + \beta\right)^{-1} (-\beta\phi + \beta\psi' \cos \phi), \quad (2.6)$$

where $(-d^2/dx^2 + \beta)^{-1}$ is from $W^{-1,2}(\mathbf{R}/\mathbf{Z})$ to $W^{1,2}(\mathbf{R}/\mathbf{Z})$. Then (1.14) is equivalent to

$$Q(\psi, \phi) = 0. \quad (2.7)$$

Lemma 2.1 implies that the Fréchet derivative

$$D_\phi Q(\psi^0, \phi^0) : W^{1,2}\left(\frac{\mathbf{R}}{\mathbf{Z}}\right) \rightarrow W^{1,2}\left(\frac{\mathbf{R}}{\mathbf{Z}}\right)$$

is a bounded bijective linear operator. By the implicit function theorem, the solutions of (2.7) near (ψ^0, ϕ^0) form a smooth curve that passes through (ψ^0, ϕ^0) .

As in lemma 2.1, solutions from this curve continue to be stable. To see this, consider the eigenvalue problem at ϕ :

$$-q'' - \beta\psi'(\sin \phi)q = \lambda q. \quad (2.8)$$

Multiplying (2.8) by q and integrating over $(0,1)$, we obtain

$$\int_0^1 ((q')^2 - \beta\psi'(\sin \phi)q^2) dx = \lambda \int_0^1 q^2 dx. \quad (2.9)$$

The integral on the right-hand side is estimated as in (2.4). The left-hand side is written as

$$\begin{aligned} \int_0^1 ((q')^2 - \beta\psi'(\sin \phi)q^2) dx &= \int_0^1 ((q')^2 - \beta(\psi^0)'(\sin \phi^0)q^2) dx \\ &\quad + \beta \int_0^1 ((\psi^0)'(\sin \phi^0)q^2 - \psi'(\sin \phi)q^2) dx. \end{aligned} \quad (2.10)$$

The first integral in (2.10) is estimated as in lemma 2.1, i.e.

$$\int_0^1 ((q')^2 - \beta(\psi^0)'(\sin \phi^0)q^2) dx \geq \min\{1, \beta \sin \xi\} \left(q(x_1^0)^2 + \int_0^1 (q')^2 dx \right). \quad (2.11)$$

The second integral in (2.10) is estimated as follows:

$$\begin{aligned} \left| \int_0^1 ((\psi^0)'(\sin \phi^0)q^2 - \psi'(\sin \phi)q^2) dx \right| &\leq \int_0^1 |-\psi^0(\cos \phi^0)(\phi^0)' + \psi(\cos \phi)\phi'|q^2 dx \\ &\quad + \int_0^1 |-2\psi^0(\sin \phi^0)qq' + \psi(\sin \phi)qq'| dx \\ &\leq C(\|\psi^0 - \psi\|_2^2 + \|\phi^0 - \phi\|_{1,2}^2)^{1/2} \left(q(x_1^0)^2 + \int_0^1 (q')^2 dx \right). \end{aligned} \tag{2.12}$$

Equations (2.4), (2.9), (2.10), (2.11) and (2.12) yield a positive lower bound for λ if (ψ, ϕ) is close enough to (ψ^0, ϕ^0) . \square

We define

$$X = \left\{ \psi \in L^2 \left(\frac{\mathbf{R}}{\mathbf{Z}} \right) : \|\psi - \psi^0(\cdot - y)\|_2 \leq \delta, \text{ for some } y \in \frac{\mathbf{R}}{\mathbf{Z}} \right\}. \tag{2.13}$$

In (2.13), δ is small enough so that every $\psi \in X$ with $\|\psi - \psi^0\| \leq \delta$ is in the neighbourhood of ψ^0 of lemma 2.2. For each ψ there, denote the corresponding solution, ϕ , of (1.14) constructed in lemma 2.2 by ϕ_ψ . Set

$$E(\psi) = \int_0^1 \left(\frac{1}{2}(\phi'_\psi)^2 + \beta\psi' \sin \phi_\psi \right) dx, \quad \psi \in X. \tag{2.14}$$

Define, in $W^{1,2}(\mathbf{R}/\mathbf{Z}) \cap X$,

$$I_\epsilon(\psi) = \int_0^1 \left(\frac{\epsilon^2}{2}(\psi')^2 + W(\psi) \right) dx + \epsilon\gamma E(\psi). \tag{2.15}$$

Note that

$$I_\epsilon(\psi) = F_\epsilon(\psi, \phi_\psi). \tag{2.16}$$

Hence, to prove theorem 1.1 we look for a local minimizer, ψ , of I_ϵ in the interior of X because by lemma 2.2 (ψ, ϕ_ψ) would be a local minimizer of F_ϵ . To this end, we apply the Γ -convergence theory to I_ϵ .

For technical reasons, we extend the domain of I_ϵ to X by setting $I(\psi) = \infty$ if $\psi \notin W^{1,2}(\mathbf{R}/\mathbf{Z})$. The singular limit (the Γ -limit) of $\epsilon^{-1}I_\epsilon$, denoted by J in this paper, is a variational problem initially defined in

$$BV \left(\frac{\mathbf{R}}{\mathbf{Z}}, \left\{ \pm \frac{1}{2} \right\} \right) \cap X. \tag{2.17}$$

$BV(\mathbf{R}/\mathbf{Z}, \{\pm 1/2\})$ is the class of periodic functions of bounded variation whose values are $-\frac{1}{2}$ or $\frac{1}{2}$. Each function there has a finite number of jumps between $-\frac{1}{2}$ and $\frac{1}{2}$. A more formal description of these functions may be found in Evans and Gariepy [9, ch 5]. Naturally for each positive even integer N , we set

$$A_N = \left\{ \psi \in BV \left(\frac{\mathbf{R}}{\mathbf{Z}}, \left\{ \pm \frac{1}{2} \right\} \right) : \psi \text{ has } N \text{ jumps} \right\}. \tag{2.18}$$

Then we have a decomposition

$$BV \left(\frac{\mathbf{R}}{\mathbf{Z}}, \left\{ \pm \frac{1}{2} \right\} \right) = \bigcup_{N=2, \text{even}}^\infty A_N. \tag{2.19}$$

For each ψ in $BV(\mathbf{R}/\mathbf{Z}, \{\pm 1/2\}) \cap X$, we define

$$J(\psi) = \tau N + \gamma E(\psi), \quad \text{if } \psi \in A_N. \tag{2.20}$$

τ is a positive constant defined by

$$\tau = \int_{-1/2}^{1/2} \sqrt{2W(u)} \, du. \quad (2.21)$$

It is called the interface tension. Again, we extend J trivially to X by taking $J(\psi) = \infty$ if $\psi \in X \setminus BV(\mathbf{R}/\mathbf{Z}, \{\pm 1/2\})$.

Lemma 2.3. *Let X be equipped with the L^2 -metric.*

1. As $\epsilon \rightarrow 0$, $\epsilon^{-1}I_\epsilon$ Γ -converges to J in the following sense.
 - (a) For every family $\{\psi_\epsilon\} \subset X$ with $\lim_{\epsilon \rightarrow 0} \psi_\epsilon = \psi$, $\liminf_{\epsilon \rightarrow 0} \epsilon^{-1}I_\epsilon(\psi_\epsilon) \geq J(\psi)$.
 - (b) For every $\psi \in X$, there is a family $\{\psi_\epsilon\} \subset X$ such that $\lim_{\epsilon \rightarrow 0} \psi_\epsilon = \psi$ and $\limsup_{\epsilon \rightarrow 0} \epsilon^{-1}I_\epsilon(\psi_\epsilon) \leq J(\psi)$.
2. Let ϵ_j be a sequence of positive numbers converging to 0, and $\{\psi_j\}$ be a sequence in X . If $\epsilon_j^{-1}I_{\epsilon_j}(\psi_j)$ is bounded above in j , then $\{\psi_j\}$ is relatively compact in X and its cluster points belong to $BV(\mathbf{R}/\mathbf{Z}, \{\pm 1/2\})$.

Proof. We view $\epsilon^{-1}I_\epsilon$ as a sum of a local part,

$$H_\epsilon(\psi) := \int_0^1 \left(\frac{\epsilon}{2}(\psi')^2 + \frac{1}{\epsilon}W(\psi) \right) dx \quad (2.22)$$

and an ϵ -independent, perturbative, nonlocal part, γE . Regarding E , we note that $\psi \rightarrow E(\psi)$ is continuous from X to \mathbf{R} .

After making some minor modifications (change L^1 to L^2) in the proof of propositions 1 and 2 of Modica [14], we find that H_ϵ Γ -converges to H_0 . Here,

$$H_0(\psi) := \tau N \quad \text{if } \psi \in A_N. \quad (2.23)$$

Because $E : X \rightarrow \mathbf{R}$ is a continuous functional, by the definition of Γ -convergence, $\epsilon^{-1}I_\epsilon = H_\epsilon + \gamma E$ Γ -converges to $J = H_0 + \gamma E$.

Statement 2 of the lemma is a kind of uniform coercivity property. The proof is similar to the one in [17, appendix]. \square

The next result proved by Kohn and Sternberg [12] asserts that as a corollary of lemma 2.3, near every isolated local minimizer of J there exists a local minimizer of I_ϵ . The original result [12, theorem 2.1] deals with a domain with boundary. Here on \mathbf{R}/\mathbf{Z} we must take the translation invariance of I_ϵ into consideration and state the result a little differently. Define a manifold of translates of ψ_0 ,

$$M(\psi_0) := \left\{ \psi \in X : \psi(\cdot) = \psi_0(\cdot - y), y \in \frac{\mathbf{R}}{\mathbf{Z}} \right\}$$

and a tube-like neighbourhood of $M(\psi_0)$,

$$N_\delta(\psi_0) := \left\{ \psi \in X : \|\psi(\cdot) - \psi_0(\cdot - y)\| < \delta, \text{ for some } y \text{ in } \frac{\mathbf{R}}{\mathbf{Z}} \right\}.$$

Corollary 2.4. *Let $\delta > 0$ and $\psi_0 \in X_m$ be such that $J(\psi_0) < J(\psi)$ for all $\psi \in N_\delta(\psi_0) \setminus M(\psi_0)$. Then there exist $\epsilon_0 > 0$ and $\psi_\epsilon \in N_{\delta/2}(\psi_0)$ for all $\epsilon < \epsilon_0$ such that $I_\epsilon(\psi_\epsilon) \leq I_\epsilon(\psi)$ for all $\psi \in N_{\delta/2}(\psi_0)$. In addition, $\psi_\epsilon \rightarrow \psi_0$ up to translation.*

Lemma 2.5. *If ψ strictly minimizes J in A_K locally, up to translation, then the corresponding ψ is a strict local minimizer of J in X , modulo translation.*

Proof. Suppose that the conclusion is false. There would be a sequence of ψ_j such that $\psi_j \neq \psi$ modulo translation, $\psi_j \rightarrow \psi$ and $J(\psi_j) \leq J(\psi)$. The L^2 -continuity of E implies $\lim_{j \rightarrow \infty} E(\psi_j) = E(\psi)$. Therefore,

$$\limsup_{j \rightarrow \infty} H_0(\psi_j) \leq H_0(\psi).$$

On the other hand, the lower semicontinuity theorem of BV functions ([9], theorem 1, p 172) states that

$$\liminf_{j \rightarrow \infty} H_0(\psi_j) \geq H_0(\psi).$$

We deduce that

$$\lim_{j \rightarrow \infty} H_0(\psi_j) = H_0(\psi). \tag{2.24}$$

Hence, for large j , ψ_j has exactly K jumps and is in A_K . But this is inconsistent with $\psi_j \rightarrow \psi$, $J(\psi_j) \leq J(\psi)$ and the assumption of the lemma. \square

In A_K , H_0 is constant. So minimizing J means minimizing E . Now the study of J in X is reduced to the study of E in $A_K \cap X$. Because of lemma 2.2, corollary 2.4 and lemma 2.5, to prove theorem 1.1 we need to show that ψ^0 is a strict local minimizer of E in $A_K \cap X$ up to translation.

3. ψ^0 as a local minimizer of E

When ψ is in $A_K \cap X$, it is a step function determined by the jumps in \mathbf{R}/\mathbf{Z} which we denote by x_1, x_2, \dots, x_K , namely

$$\psi(x) = \begin{cases} \frac{1}{2} & \text{if } x \in (0, x_1), \\ -\frac{1}{2} & \text{if } x \in (x_1, x_2), \\ \frac{1}{2} & \text{if } x \in (x_2, x_3), \\ \dots & \dots \\ -\frac{1}{2} & \text{if } x \in (x_{K-1}, x_K), \\ \frac{1}{2} & \text{if } x \in (x_K, 1) \end{cases} \tag{3.1}$$

and

$$\psi' = \sum_{j=1}^K (-1)^j \delta(\cdot - x_j). \tag{3.2}$$

Let

$$g(z) = \frac{z^2}{2} - \frac{z}{2} + \frac{1}{12} \text{ in } (0, 1), \text{ periodically extended to } \mathbf{R}. \tag{3.3}$$

Then the Green function of $-d^2/dx^2$ with the periodic boundary condition is $g(x - y)$. This allows us to rewrite (1.14) as

$$\phi(x) - \bar{\phi} = \beta \sum_{j=1}^K (-1)^{j+1} g(x - x_j) \cos \phi_j \tag{3.4}$$

for the corresponding ϕ_ψ . We write this ϕ_ψ as ϕ for simplicity. Here, $\bar{\phi} = \int_0^1 \phi(x) dx$ is the average of ϕ . We have also introduced the abbreviation $\phi_j = \phi(x_j)$. By integrating (1.14), we find

$$\sum_{j=1}^K (-1)^j \cos \phi_j = 0. \quad (3.5)$$

Because, by (1.14), (3.5) and (3.4),

$$\begin{aligned} \int_0^1 (\phi')^2 dx &= -\beta \int_0^1 \psi' \phi \cos \phi dx = -\beta \int_0^1 \psi' \cos \phi (\bar{\phi} + \beta \sum_{j=1}^K (-1)^{j+1} g(x - x_j)) dx \\ &= \beta^2 \sum_{i,j=1}^K (-1)^{i+j} g(x_i - x_j) \cos \phi_i \cos \phi_j, \end{aligned}$$

we can now rewrite (2.14) as

$$E(x_1, x_2, \dots, x_K) = \frac{\beta^2}{2} \sum_{i,j=1}^K (-1)^{i+j} g(x_i - x_j) \cos \phi_i \cos \phi_j + \beta \sum_{j=1}^K (-1)^j \sin \phi_j. \quad (3.6)$$

We proceed to compute the derivatives of E . Denote $\partial E / \partial x_k$ by E_k .

$$\begin{aligned} E_k &= \beta^2 \sum_{j \neq k} (-1)^{k+j} g'(x_k - x_j) \cos \phi_j \cos \phi_k + \frac{\beta^2}{2} \sum_{i,j=1}^K g(x_i - x_j) (\cos \phi_i \cos \phi_j)_k \\ &\quad + \beta \sum_{j=1}^K (-1)^j \cos \phi_j \phi_{j,k}. \end{aligned}$$

If we differentiate (3.4) with respect to x_k , then

$$\phi_{j,k} - (\bar{\phi})_k = \beta \sum_{i=1}^K (-1)^i g(x_i - x_j) \sin \phi_i \phi_{i,k} - \beta \begin{cases} (-1)^k g'(x_k - x_j) \cos \phi_k & \text{if } k \neq j, \\ \sum_{i \neq k} (-1)^i g'(x_k - x_i) \cos \phi_i & \text{if } k = j. \end{cases} \quad (3.7)$$

$(\bar{\phi})_k$ is the derivative of $\bar{\phi}$ with respect to x_k . We multiply (3.7) by $(-1)^j \cos \phi_j$ and sum over j . Using (3.5), we discover

$$\begin{aligned} \sum_j (-1)^j \cos \phi_j \phi_{j,k} &= \beta \sum_{i,j=1}^K (-1)^{i+j} g(x_i - x_j) \sin \phi_i \cos \phi_j \phi_{i,k} \\ &\quad - \beta \sum_{j \neq k} (-1)^{k+j} g'(x_k - x_j) \cos \phi_k \cos \phi_j \\ &\quad - \beta \sum_{i \neq k} (-1)^{i+k} g'(x_k - x_i) \cos \phi_i \cos \phi_k \\ &= \beta \sum_{i,j=1}^K (-1)^{i+j} g(x_i - x_j) \sin \phi_i \cos \phi_j \phi_{i,k} \\ &\quad - 2\beta \sum_{j \neq k} (-1)^{k+j} g'(x_k - x_j) \cos \phi_k \cos \phi_j. \end{aligned}$$

After we substitute the last quantity into E_k , it is simplified to

$$E_k = -\beta^2 \sum_{j \neq k} (-1)^{k+j} g'(x_k - x_j) \cos \phi_j \cos \phi_k. \quad (3.8)$$

In this paper, we are interested in the soliton-stripe pattern with K domain walls periodically placed. They are the function ψ^0 , defined in (1.15), and its translates. ψ^0 corresponds to the vector $(x_1^0, x_2^0, \dots, x_K^0)$, where $x_j^0 = 1/2K + (j-1)/K$, $j = 1, 2, \dots, K$. If $(x_1, \dots, x_K) = (x_1^0, \dots, x_K^0)$,

$$\phi_j = \phi_j^0 = (-1)^{j+1} \xi. \quad (3.9)$$

Lemma 3.1. (x_1^0, \dots, x_K^0) and its translates are critical points of E .

Proof. This lemma simply restates the fact that ϕ^0 is a solution of (1.14) when $\psi = \psi^0$. Here, we give a proof. At (x_1^0, \dots, x_K^0) or any of its translates,

$$\begin{aligned} E_k(x_1^0, \dots, x_K^0) &= -\beta^2 (\cos \xi)^2 \sum_{j \neq k} (-1)^{k+j} g'(x_k^0 - x_j^0) \\ &= \pm \beta^2 (\cos \xi)^2 \left(-g' \left(\frac{1}{K} \right) + g' \left(\frac{2}{K} \right) - \dots - g' \left(\frac{K-1}{K} \right) \right) \\ &= \pm \beta^2 (\cos \xi)^2 \left(- \left(\frac{1}{K} - \frac{1}{2} \right) + \left(\frac{2}{K} - \frac{1}{2} \right) - \dots - \left(\frac{K-1}{K} - \frac{1}{2} \right) \right) = 0. \end{aligned}$$

□

Next we calculate the second derivatives of E . Denote $\partial^2 E / \partial x_k \partial x_l$ by E_{kl} . We separate the two cases $k \neq l$ and $k = l$. When $k \neq l$,

$$\begin{aligned} E_{kl} &= \beta^2 (-1)^{k+l} g''(x_k - x_l) \cos \phi_k \cos \phi_l + \beta^2 \sum_{j \neq k} (-1)^{k+j} g'(x_k - x_j) \\ &\quad \times (\sin \phi_k \cos \phi_j \phi_{k,l} + \cos \phi_k \sin \phi_j \phi_{j,l}) \\ &= \beta^2 (-1)^{k+l} \cos \phi_k \cos \phi_l \\ &\quad + \beta^2 \sum_{j \neq k} (-1)^{k+j} g'(x_k - x_j) (\sin \phi_k \cos \phi_j \phi_{k,l} + \cos \phi_k \sin \phi_j \phi_{j,l}). \end{aligned}$$

At $x_j = x_j^0$,

$$\begin{aligned} E_{kl}(x_1^0, \dots, x_K^0) &= \beta^2 (-1)^{k+l} (\cos \xi)^2 - \beta^2 \sin \xi \cos \xi \\ &\quad \times \sum_{j \neq k} (-1)^{k+j} g'(x_k^0 - x_j^0) ((-1)^k \phi_{k,l} + (-1)^l \phi_{j,l}) \\ &= \beta^2 (-1)^{k+l} (\cos \xi)^2 - \beta^2 \sin \xi \cos \xi \sum_{j \neq k} (-1)^k g'(x_k^0 - x_j^0) \phi_{j,l}. \end{aligned} \quad (3.10)$$

Note that we have used the fact that

$$\sum_{j \neq k} (-1)^j g'(x_k^0 - x_j^0) = 0 \quad (3.11)$$

as in the proof of lemma 3.1. If $k = l$,

$$\begin{aligned} E_{kk} &= -\beta^2 \sum_{j \neq k} (-1)^{k+j} g''(x_k - x_j) \cos \phi_j \cos \phi_k \\ &\quad + \beta^2 \sum_{j \neq k} (-1)^{k+j} g'(x_k - x_j) (\sin \phi_k \cos \phi_j \phi_{k,k} + \cos \phi_k \sin \phi_j \phi_{j,k}). \end{aligned}$$

At (x_1^0, \dots, x_K^0) ,

$$E_{kk}(x_1^0, \dots, x_K^0) = \beta^2(\cos \xi)^2 - \beta^2 \sin \xi \cos \xi \sum_{j \neq k} (-1)^k g'(x_k^0 - x_j^0) \phi_{j,k}. \quad (3.12)$$

We write (3.10) and (3.12) in a more compact way. Define K by K matrices H , F and Φ' by

$$H_{kl} = \begin{cases} (-1)^k g'(x_k^0 - x_l^0) & \text{if } k \neq l, \\ 0 & \text{if } k = l, \end{cases} \quad F_{kl} = (-1)^{k+l}, \quad \Phi'_{kl} = \frac{\partial \phi_k}{\partial x_l}(x_1^0, \dots, x_K^0).$$

Therefore,

$$E''(x_1^0, \dots, x_K^0) = \beta^2(\cos \xi)^2 F - \beta^2(\sin \xi \cos \xi) H \Phi'. \quad (3.13)$$

It remains to compute Φ' . Define a $K \times K$ matrix $G \times G_{kl} = g(x_k^0 - x_l^0)$. At (x_1^0, \dots, x_K^0) , we find from (3.7) that

$$\Phi'_{jk} - (\bar{\phi})_k(x_1^0, \dots, x_K^0) = -\beta \sin \xi \sum_{i=1}^K G_{ji} \Phi'_{ik} - \beta \cos \xi H_{kj}. \quad (3.14)$$

By differentiating (3.5), we deduce

$$\sum_{j=1}^K (-1)^j \sin \phi_j \phi_{j,k} = 0$$

and so at (x_1^0, \dots, x_K^0) ,

$$\sum_{j=1}^K \Phi'_{jk} = 0. \quad (3.15)$$

We sum (3.14) over j , and by (3.15),

$$-K(\bar{\phi})_k(x_1^0, \dots, x_K^0) = -\beta \sin \xi \sum_{j,i=1}^K G_{ji} \Phi'_{ik} - \beta \cos \xi \sum_{j=1}^K H_{kj}. \quad (3.16)$$

The right-hand side of (3.16) is 0 since

$$\sum_{j,i=1}^K G_{ji} \Phi'_{ik} = \sum_{i=1}^K \Phi'_{ik} \sum_{j=1}^K G_{ji} = \left(\sum_{i=1}^K \Phi'_{ik} \right) \left(\sum_{j=1}^K g \left(\frac{j-1}{K} \right) \right) = \left(\sum_{i=1}^K \Phi'_{ik} \right) \frac{1}{12K} = 0 \quad (3.17)$$

and

$$\sum_{j=1}^K H_{kj} = (-1)^k \sum_{j=1}^{K-1} g' \left(\frac{j}{K} \right) = (-1)^k \sum_{j=1}^{K-1} \left(\frac{j}{K} - \frac{1}{2} \right) = 0. \quad (3.18)$$

Therefore,

$$(\bar{\phi})_k(x_1^0, \dots, x_K^0) = 0$$

and (3.14) yields

$$\Phi' = -\beta \cos \xi (\text{Id} + \beta \sin \xi G)^{-1} H^T, \quad (3.19)$$

where Id is the $K \times K$ identity matrix. To be sure that $(\text{Id} + \beta \sin \xi G)^{-1}$ exists in the last equation, we note that one eigenvalue of G is $1/(12K)$, corresponding to the eigenvector $(1, 1, \dots, 1)$ as shown in (3.17). The remaining $K - 1$ eigenvalues are also positive (see (3.24)).

Substituting (3.19) back into (3.13), we obtain

$$E''(x_1^0, \dots, x_K^0) = \beta^2(\cos \xi)^2 F + \beta^3 \sin \xi (\cos \xi)^2 H (\text{Id} + \beta \sin \xi G)^{-1} H^T. \quad (3.20)$$

Lemma 3.2. $E''(x_1^0, \dots, x_K^0)$ has one eigenvalue equal to 0 whose eigenvector is $(1, 1, \dots, 1)^T$, which corresponds to the translation action. The remaining $K - 1$ eigenvalues are all positive.

Proof. If $q = (1, 1, \dots, 1)^T$, then clearly $Fq = \vec{0}$ and $H^T q = \vec{0}$ by (3.11). Hence,

$$0, (1, 1, \dots, 1)^T \tag{3.21}$$

form an eigenpair.

Another eigenvector is $q = (-1, 1, -1, 1, \dots, -1, 1)^T$. Clearly, $Fq = Kq$, and the k th element of $H^T q$ is

$$\sum_{j=1}^K H_{jk}(-1)^j = \sum_{j=1}^{K-1} g' \left(\frac{j}{K} \right) = 0,$$

as in (3.18). Hence, $E''(x_1^0, \dots, x_K^0)q = \beta^2(\cos \xi)^2 Kq$, and we have the second eigenpair,

$$\beta^2(\cos \xi)^2 K, \quad (-1, 1, \dots, -1, 1)^T. \tag{3.22}$$

To find the remaining $K - 2$ eigenpairs, for each $n = 1, 2, \dots, K - 1, n \neq K/2$, we let $\theta = 2n\pi/K$ and $\zeta = e^{\sqrt{-1}\theta}$. Define $q_\zeta = (\zeta, \zeta^2, \zeta^3, \dots, \zeta^K)^T$. We proceed to show that q_ζ is an eigenvector. It is easy to see that $Fq_\zeta = \vec{0}$. Regarding the second part in (3.20), we need the following three identities:

$$\sum_{j=1}^K \zeta^j = 0, \quad \sum_{j=1}^K j\zeta^j = -\frac{K\zeta}{1-\zeta}, \quad \sum_{j=1}^K j^2\zeta^j = -\frac{K^2\zeta}{1-\zeta} - \frac{2K\zeta}{(1-\zeta)^2}. \tag{3.23}$$

First we compute Gq_ζ . The k th entry of Gq_ζ is

$$\begin{aligned} \sum_{j=1}^K g \left(\frac{k-j}{K} \right) \zeta^j &= \zeta^k \sum_{j=1}^K g \left(\frac{j-k}{K} \right) \zeta^{j-k} = \zeta^k \sum_{j=1}^K g \left(\frac{j}{K} \right) \zeta^j \\ &= \zeta^k \sum_{j=1}^K \left(\frac{1}{2} \left(\frac{j}{K} \right)^2 - \frac{1}{2} \frac{j}{K} + \frac{1}{12} \right) \zeta^j \\ &= \zeta^k \left(\frac{1}{2K^2} \left(-\frac{K^2\zeta}{1-\zeta} - \frac{2K\zeta}{(1-\zeta)^2} \right) - \frac{1}{2K} \left(-\frac{K\zeta}{1-\zeta} \right) \right) \\ &= -\frac{\zeta}{K(1-\zeta)} \zeta^k = \frac{1}{2K(1-\cos \theta)} \zeta^k. \end{aligned}$$

Therefore,

$$Gq_\zeta = \frac{1}{2K(1-\cos \theta)} q_\zeta. \tag{3.24}$$

Hence, q_ζ is an eigenvector of G . Next we compute $H^T q_\zeta$. The k th entry of $H^T q_\zeta$ is

$$\begin{aligned} \sum_{j \neq k} (-1)^j g' \left(\frac{j-k}{K} \right) \zeta^j &= (-\zeta)^k \sum_{j \neq k} g' \left(\frac{j-k}{K} \right) \zeta^{j-k} \\ &= (-\zeta)^k \sum_{j=1}^{K-1} g' \left(\frac{j}{K} \right) (-\zeta)^j = (-\zeta)^k \sum_{j=1}^{K-1} \left(\frac{j}{K} - \frac{1}{2} \right) (-\zeta)^j \\ &= (-\zeta)^k \left(\sum_{j=1}^K \left(\frac{j}{K} - \frac{1}{2} \right) (-\zeta)^j - \left(\frac{K}{K} - \frac{1}{2} \right) (-\zeta)^K \right) \\ &= (-\zeta)^k \left(\frac{1}{K} \left(-\frac{K(-\zeta)}{1+\zeta} \right) - \frac{1}{2} \right) = \frac{\zeta-1}{2(1+\zeta)} (-\zeta)^k = \frac{\sqrt{-1} \sin \theta}{2(1+\cos \theta)} (-\zeta)^k. \end{aligned}$$

q_ξ is not an eigenvector of H^T , but

$$H^T q_\xi = \frac{\sqrt{-1} \sin \theta}{2(1 + \cos \theta)} q_{-\xi}. \tag{3.25}$$

In other words, H^T sends q_ξ to a scalar multiple of $q_{-\xi}$ which is made from a different angle, $\theta + \pi$. Combining (3.25) and (3.24), we obtain

$$(\text{Id} + \beta \sin \xi G)^{-1} H^T q_\xi = \left(1 + \frac{\beta \sin \xi}{2K(1 - \cos(\theta + \pi))}\right)^{-1} \left(\frac{\sqrt{-1} \sin \theta}{2(1 + \cos \theta)}\right) q_{-\xi}. \tag{3.26}$$

Finally, we compute $Hq_{-\xi}$. The k th entry of $Hq_{-\xi}$ is

$$\begin{aligned} \sum_{j \neq k} (-1)^k g' \left(\frac{k-j}{K}\right) (-\xi)^j &= \xi^k \sum_{j \neq k} g' \left(\frac{k-j}{K}\right) \left(-\frac{1}{\xi}\right)^{k-j} \\ &= \xi^k \sum_{j=1}^{K-1} g' \left(\frac{j}{K}\right) \left(-\frac{1}{\xi}\right)^j = \xi^k \sum_{j=1}^{K-1} \left(\frac{j}{K} - \frac{1}{2}\right) \left(-\frac{1}{\xi}\right)^j \\ &= \xi^k \left(\sum_{j=1}^K \left(\frac{j}{K} - \frac{1}{2}\right) \left(-\frac{1}{\xi}\right)^j - \left(\frac{K}{K} - \frac{1}{2}\right) \left(-\frac{1}{\xi}\right)^K\right) \\ &= \xi^k \left(\frac{1}{K} \left(-\frac{K(-1/\xi)}{1 + 1/\xi}\right) - \frac{1}{2}\right) = \frac{1 - \xi}{2(1 + \xi)} \xi^k = -\frac{\sqrt{-1} \sin \theta}{2(1 + \cos \theta)} \xi^k. \end{aligned}$$

So H transforms $q_{-\xi}$ back to a scalar multiple of q_ξ , i.e.

$$Hq_{-\xi} = -\frac{\sqrt{-1} \sin \theta}{2(1 + \cos \theta)} q_\xi. \tag{3.27}$$

Together with (3.26) and the fact that $Fq_\xi = \vec{0}$, we deduce

$$\begin{aligned} E''(x_1^0, \dots, x_K^0) q_\xi &= \beta^3 \sin \xi (\cos \xi)^2 \frac{-\sqrt{-1} \sin \theta}{2(1 + \cos \theta)} \left(1 + \frac{\beta \sin \xi}{2K(1 + \cos \theta)}\right)^{-1} \frac{\sqrt{-1} \sin \theta}{2(1 + \cos \theta)} q_\xi \\ &= \frac{\beta^3 \sin \xi (\cos \xi)^2 (1 - \cos \theta) K}{4K(1 + \cos \theta) + 2\beta \sin \xi} q_\xi. \end{aligned} \tag{3.28}$$

We thus find $K - 2$ eigenpairs,

$$\frac{\beta^3 \sin \xi (\cos \xi)^2 (1 - \cos \theta) K}{4K(1 + \cos \theta) + 2\beta \sin \xi}, \quad q_\xi, \tag{3.29}$$

where $\theta = 2\pi/K, 4\pi/K, \dots, (K - 2)\pi/K, (K + 2)\pi/K, \dots, (2K - 2)\pi/K$. The eigenvalues in (3.29) are all positive since $\xi \in (0, \pi/2)$ and $\theta \in (0, 2\pi)$. Note that if θ in (3.29) is set to be 0 or π , we recover (3.21) and (3.22), respectively. \square

The proof of theorem 1.1 is complete.

4. Closing remarks

According to (2.14) at (x_1^0, \dots, x_K^0) or any of its translates,

$$E(x_1^0, \dots, x_K^0) = K \int_0^{1/K} \frac{(\phi')^2}{2} dx - K\beta \sin \xi = 2K^2 \xi^2 - K\beta \sin \xi.$$

For large K , we find the expansion

$$\xi = \frac{\beta}{4K} - \frac{1}{2} \left(\frac{\beta}{4K} \right)^3 \dots \quad (4.1)$$

from (1.17). Equation (4.1) gives rise to an approximation,

$$E(x_1^0, \dots, x_K^0) \approx -\frac{\beta^2}{8} + \frac{\beta^4}{384K^2}. \quad (4.2)$$

Therefore, for large K we find

$$J(\psi_0) + \frac{\gamma\beta^2}{8} \approx \tau K + \frac{\gamma\beta^4}{K^2} = \tau K + \frac{CL^3}{K^2}. \quad (4.3)$$

In the last expression we have returned to the original parameters, so that C depends on κ , t , s , \tilde{K} and λ but not on L . This asymptotic formula, which leads to the optimal spacing, shows up in many other physical systems, including di- and tri-block copolymers [18, 23] and the Seul–Andelman membrane problem [24, 28]. It is minimized at $K = (2C/\tau)^{1/3}L$. Even though the optimal spacing,

$$\frac{L}{K} = \left(\frac{\tau}{2C} \right)^{1/3},$$

is only an approximation here, it is actually the exact value in the thermodynamic limit.

Another important formula that leads to optimal spacing is

$$\tau K + \frac{CL^2}{K}, \quad (4.4)$$

which is minimized at $K = (C/\tau)^{1/2}L$. The difference between the exponents $\frac{1}{3}$ and $\frac{1}{2}$ may be significant. In [25], we showed that (4.4) appears in a charged Langmuir monolayer problem proposed by Andelman *et al* [1]. It is also found in studies of domain structures of superconductors in the intermediate state [29] and in ferromagnets [13].

It is natural to extend the one-dimensional soliton-stripe solutions trivially to a two-dimensional square. They continue to be solutions of the Euler–Lagrange equations of (1.5). Whether they are still local minimizers in two dimensions is not clear. The techniques we developed for the diblock copolymer problem in [21, 26] should help in a future study.

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References

- [1] Andelman D, Brochard F and Joanny J-F 1987 Phase transitions in Langmuir monolayers of polar molecules *J. Chem. Phys.* **86** 3673–81
- [2] Chmaj A and Ren X 2000 Multiple layered solutions of the nonlocal bistable equation *Physica D* **147** 135–54
- [3] Choksi R 2001 Scaling laws in microphase separation of diblock copolymers *J. Nonlinear Sci.* **11** 223–36
- [4] Choksi R and Ren X 2003 On the derivation of a density functional theory for microphase separation of diblock copolymers *J. Statist. Phys.* **113** 151–76
- [5] Dal Maso G 1992 *Introduction to Gamma-Convergence. Progress in Nonlinear Differential Equations and Their Applications* vol 8 (Boston, MA: Birkhäuser)
- [6] De Gennes P G 1972 An analogy between superconductivity and smectics a *Solid State Commun.* **10** 753
- [7] De Gennes P G and Prost J 1995 *The Physics of Liquid Crystals* 2nd edn (Oxford: Oxford University Press)
- [8] De Giorgi E 1975 Sulla convergenza di alcune successioni di integrali del tipo della'area *Rend. Mat.* **8** 277–94

- [9] Evans L C and Garipey R F 1992 *Measure Theory and Fine Properties of Functions* (Boca Raton, FL: CRC Press)
- [10] Fife P C and Hilhorst D 2001 The Nishiura–Ohnishi free boundary problem in the 1D case *SIAM J. Math. Anal.* **33** 589–606
- [11] Henry M 2001 Singular limit of a fourth order problem arising in the micro-phase separation of diblock copolymers *Adv. Diff. Eqns* **6** 1049–114
- [12] Kohn R and Sternberg P 1989 Local minimisers and singular perturbations *Proc. R. Soc. Edin. A* **111** 69–84
- [13] Landau L D, Lifshitz E M and Pitaevskii L P 1984 *Electrodynamics of Continuous Media, Course of Theoretical Physics* vol 8 2nd edn (Portsmouth, NH: Butterworth-Heinemann)
- [14] Modica L 1987 The gradient theory of phase transitions and the minimal interface criterion *Arch. Rat. Mech. Anal.* **98** 357–83
- [15] Nishiura Y and Ohnishi I 1995 Some mathematical aspects of the microphase separation in diblock copolymers *Physica D* **84** 31–9
- [16] Ohnishi I, Nishiura Y, Imai M and Matsushita Y 1999 Analytical solutions describing the phase separation driven by a free energy functional containing a long-range interaction term *Chaos* **9** 329–41
- [17] Ren X and Truskinovsky L 2000 Finite scale microstructures in nonlocal elasticity. In recognition of the sixtieth birthday of Roger L Fosdick (Blacksburg, VA, 1999) *J. Elasticity* **59** 319–55
- [18] Ren X and Wei J 2000 On the multiplicity of solutions of two nonlocal variational problems *SIAM J. Math. Anal.* **31** 909–24
- [19] Ren X and Wei J 2002 Concentrically layered energy equilibria of the di-block copolymer problem *Eur. J. Appl. Math.* **13** 479–96
- [20] Ren X and Wei J 2003 On energy minimizers of the di-block copolymer problem *Interface. Free Bound.* **5** 193–238
- [21] Ren X and Wei J 2003 On the spectra of 3-D lamellar solutions of the diblock copolymer problem *SIAM J. Math. Anal.* **35** 1–32
- [22] Ren X and Wei J 2003 Triblock copolymer theory: free energy, disordered phase and weak segregation *Physica D* **178** 103–17
- [23] Ren X and Wei J 2003 Triblock copolymer theory: ordered ABC lamellar phase *J. Nonlinear Sci.* **45** 175–208
- [24] Ren X and Wei J The soliton-stripe pattern in the Seul–Andelman membrane *Physica D* at press
- [25] Ren X and Wei J Soliton-stripe patterns in charged Langmuir monolayers *J. Nonlinear Sci.* at press
- [26] Ren X and Wei J Wiggled lamellar solutions and their stability in the diblock copolymer problem *Preprint*
- [27] Selinger J V, Wang Z-G, Bruinsma R F and Knobler C M 1993 Chiral symmetry breaking in Langmuir monolayers and smectic films *Phys. Rev. Lett.* **70** 1139–42
- [28] Seul M and Andelman D 1995 Domain shapes and patterns: the phenomenology of modulated phases *Science* **267** 476–83
- [29] Tinkham M 1995 *Introduction to Superconductivity* 2nd edn (New York: McGraw-Hill)