Disc-disc structure in a two-species interacting system on a flat torus

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Abstract

A two-species interacting system, motivated by the triblock copolymers theory, is studied on a flat torus, the quotient space of the complex plane by a lattice. The free energy of the system, which contains both short range and long range interactions, admits disc-disc like stationary points. The relative displacement of the disc centers in a stationary point is related to Green's function of the Laplace operator on the flat torus. When restricted to disc-disc configurations with relative displacements equal to half periods, the free energy is minimized with respect to the lattice and its half periods. The resulting optimal lattice depends on a single parameter. As this parameter varies, the optimal lattice may be rectangular, square, rhombic, or hexagonal. This is in sharp contrast to single species systems where optimal lattices are always hexagonal.

Key words. Two-species interacting system, triblock copolymer, disc-disc configuration, disc-disc stationary point, rectangular lattice, square lattice, rhombic lattice, hexagonal lattice.

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1 Introduction

Pattern formation in inhibitory physical systems results from controlled growth and self-organization. Common in these systems is that a deviation from homogeneity has a strong positive feedback on its further increase, and in the meantime a longer ranging inhibition mechanism exists to limit increase and spreading. Exquisitely structured patterns arise in such systems as orderly outcomes.

An archetype of two-species interacting systems is a triblock copolymer. A triblock copolymer molecule is a subchain of type A monomers connected to a subchain of type B monomers which in turn is connected to a subchain of type C monomers. Because of the repulsion between the unlike monomers, the different type subchains tend to segregate. However since subchains are chemically bonded in molecules, segregation cannot lead to a macroscopic phase separation; only micro-domains rich in individual type monomers emerge, forming morphological phases [3]. Here we treat two of the three monomer types in a triblock copolymer as species and view the third type as the surrounding environment, dependent on the two species. This way a triblock copolymer is a two-species interacting system.

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Two dimensional periodic structures are the focus of our study, and we take the sample space to be a flat torus, namely the complex plane divided by a lattice. A lattice Λ on the complex plane $\mathbb{C}$ is generated by two nonzero complex numbers α_1 and α_2, with $\mathrm{Im}(\alpha_2/\alpha_1) > 0$,

$$\Lambda = \{j_1\alpha_1 + j_2\alpha_2 : \ j_1, j_2 \in \mathbb{Z}\}. \tag{1.1}$$

Let $\alpha = (\alpha_1, \alpha_2)$ be a basis of Λ. The parallelogram,

$$P_\alpha = \{t_1\alpha_1 + t_2\alpha_2 : \ 0 \le t_j < 1, \ j = 1, 2\}, \tag{1.2}$$

is termed the fundamental parallelogram associated with α.

The flat torus $\mathbb{C}/\Lambda$ is the space of Λ-equivalence classes of points in $\mathbb{C}$. Two numbers in $\mathbb{C}$ are Λ-equivalent if their difference is in Λ. When no confusion exists, a point in $\mathbb{C}$ and its Λ-equivalence class in $\mathbb{C}/\Lambda$ are not distinguished. Functions on $\mathbb{C}/\Lambda$ can be identified with Λ-periodic functions on $\mathbb{C}$; subsets of $\mathbb{C}/\Lambda$ can be thought as Λ-periodic subsets of $\mathbb{C}$. If a basis $\alpha = (\alpha_1, \alpha_2)$ is given, $\mathbb{C}/\Lambda$ may be represented by the fundamental parallelogram P_α. The area, or the Lebesgue measure, of $\mathbb{C}/\Lambda$ is $|\mathbb{C}/\Lambda| = \mathrm{Im}(\overline{\alpha_1}\alpha_2)$. However $|\mathbb{C}/\Lambda|$ actually depends on the lattice Λ, not any particular basis α. For simplicity we write $|\Lambda|$ for $|\mathbb{C}/\Lambda|$, i.e.

$$|\Lambda| = |\mathbb{C}/\Lambda| = \mathrm{Im}(\overline{\alpha_1}\alpha_2). \tag{1.3}$$

Our model originates from Nakazawa and Ohta's density functional theory for triblock copolymers [13] with simplifications by Ren and Wei [20]. It is a variational problem defined on pairs of subsets of $\mathbb{C}/\Lambda$ with prescribed area. There are two sets of parameters in this model. The first consists of two numbers ω_1 and ω_2 satisfying

$$0 < \omega_1, \ \omega_2 < 1, \ \text{and } \omega_1 + \omega_2 < 1. \tag{1.4}$$

The second set of parameters form a two by two symmetric matrix γ,

$$\gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{bmatrix}, \ \gamma_{12} = \gamma_{21}. \tag{1.5}$$

A pair (Ω_1, Ω_2) of subsets of $\mathbb{C}/\Lambda$ is admissible if the following conditions hold. Both Ω_1 and Ω_2 are Lebesgure measurable; Ω_1 and Ω_2 do not overlap in the sense that

$$|\Omega_1 \cap \Omega_2| = 0; \tag{1.6}$$

the Lebesgue measure of Ω_j is fixed by ω_j, i.e.

$$|\Omega_1| = \omega_1|\Lambda|, \ |\Omega_2| = \omega_2|\Lambda|. \tag{1.7}$$

In (1.6) and (1.7), $|\cdot|$ denotes the Lebesgue measure on $\mathbb{C}/\Lambda$, and $|\Lambda|$ is the Lebesgue measure of the torus $\mathbb{C}/\Lambda$, as in (1.3).

Given an admissible pair (Ω_1, Ω_2), let $\Omega_3 = (\mathbb{C}/\Lambda)\backslash(\Omega_1 \cup \Omega_2)$. Define a functional $\mathcal{J}_\Lambda$ to be the free energy of (Ω_1, Ω_2) by

$$\mathcal{J}_\Lambda(\Omega_1, \Omega_2) = \frac{1}{2}\sum_{j=1}^{3} \mathcal{P}_{\mathbb{C}/\Lambda}(\Omega_j) + \sum_{j,k=1}^{2} \frac{\gamma_{jk}}{2} \int_{\mathbb{C}/\Lambda} \nabla I_\Lambda(\Omega_j)(z) \cdot \nabla I_\Lambda(\Omega_k)(z)\, dA(z). \tag{1.8}$$

Here $\mathcal{P}_{\mathbb{C}/\Lambda}(\Omega_j)$ is the perimeter of Ω_j in $\mathbb{C}/\Lambda$ given by

$$\mathcal{P}_{\mathbb{C}/\Lambda}(\Omega_j) = \sup\Big\{ \int_{\Omega_j} \mathrm{div}\, g(z)\, dA(z) : \ g \in C^1(\mathbb{C}/\Lambda, \mathbb{R}^2), \ |g(z)| \le 1 \ \forall z \in \mathbb{C}/\Lambda \Big\}. \tag{1.9}$$

In (1.9), and (1.8), the integrals are taken against the Lebesgue measure, denoted $\int \ldots dA$. If Ω_j has piecewise C^1 boundary, then $\mathcal{P}_{\mathbb{C}/\Lambda}(\Omega_j)$ is the length of the boundary. The first term in (1.8) is the total length of the curves separating Ω_1, Ω_2, and Ω_3. In $\sum_{j=1}^{3} \mathcal{P}_{\mathbb{C}/\Lambda}(\Omega_j)$ each boundary curve separating a Ω_j from a Ω_k, $j,k = 1,2,3,\ j \neq k$, is counted twice. The constant $\frac{1}{2}$ in the front takes care of the double counting. The function $I_\Lambda(\Omega_j)$ is the solution of Poisson's equation

$$-\Delta I_\Lambda(\Omega_j)(z) = \chi_{\Omega_j}(z) - \omega_j \text{ in } \mathbb{C}/\Lambda, \quad \int_{\mathbb{C}/\Lambda} I_\Lambda(\Omega_j)(z)\,dA(z) = 0, \tag{1.10}$$

where χ_{Ω_j} is the characteristic function of Ω_j. Equation (1.10) can be solved in terms of Green's function G_Λ of $-\Delta$; G_Λ satisfies

$$-\Delta G_\Lambda = \delta - \frac{1}{|\Lambda|}, \quad \int_{\mathbb{C}/\Lambda} G_\Lambda(z)\,dA(z) = 0, \tag{1.11}$$

where δ is the δ-measure centered at the Λ-equivalence class of the origin. Then

$$I_\Lambda(\Omega_j)(z) = \int_{\Omega_j} G_\Lambda(z-\zeta)\,dA(\zeta). \tag{1.12}$$

A stationary point (Ω_1,Ω_2) of $\mathcal{J}_\Lambda$ is an admissible pair at which the first variation of $\mathcal{J}_\Lambda$ vanishes. It is a solution to the following equations of a free boundary problem:

$$\kappa_{13} + \gamma_{11} I_\Lambda(\Omega_1) + \gamma_{12} I_\Lambda(\Omega_2) = \mu_1 \ \text{ on } \partial\Omega_1 \cap \partial\Omega_3 \tag{1.13}$$

$$\kappa_{23} + \gamma_{12} I_\Lambda(\Omega_1) + \gamma_{22} I_\Lambda(\Omega_2) = \mu_2 \ \text{ on } \partial\Omega_2 \cap \partial\Omega_3 \tag{1.14}$$

$$\kappa_{12} + (\gamma_{11}-\gamma_{12}) I_\Lambda(\Omega_1) + (\gamma_{12}-\gamma_{22}) I_\Lambda(\Omega_2) = \mu_1 - \mu_2 \ \text{ on } \partial\Omega_1 \cap \partial\Omega_2 \tag{1.15}$$

$$T_{13} + T_{23} + T_{12} = \vec{0} \ \text{ at } \partial\Omega_1 \cap \partial\Omega_2 \cap \partial\Omega_3. \tag{1.16}$$

In (1.13)-(1.15) κ_{13}, κ_{23}, and κ_{12} are the curvatures of the curves $\partial\Omega_1 \cap \partial\Omega_3$, $\partial\Omega_2 \cap \partial\Omega_3$, and $\partial\Omega_1 \cap \partial\Omega_2$, respectively. The unknown constants μ_1 and μ_2 are Lagrange multipliers associated with the constraints (1.7) for Ω_1 and Ω_2 respectively. The three interfaces, $\partial\Omega_1 \cap \partial\Omega_3$, $\partial\Omega_2 \cap \partial\Omega_3$ and $\partial\Omega_1 \cap \partial\Omega_2$, may meet at a common point in $\mathbb{C}/\Lambda$, which is termed a triple junction point. In (1.16), T_{13}, T_{23} and T_{12} are respectively the unit tangent vectors of these curves at a triple junction point. This equation simply says that at every triple junction point three curves meet at 120 degree angles.

The disc-disc structure is one of the simplest patterns on a flat torus. An admissible pair (Ω_1,Ω_2) is called a disc-disc configuration if each species Ω_j forms a closed disc in $\mathbb{C}/\Lambda$, $j=1,2$; namely

$$\Omega_j = B(\xi_j, r_j) = \{z \in \mathbb{C}/\Lambda :\ |z-\xi_j| \le r_j\},\ j=1,2, \tag{1.17}$$

where $\xi_j \in \mathbb{C}/\Lambda$ are the centers, r_j are the radii, and $r_1 + r_2 < |\xi_2 - \xi_1|$. Due to the translation invariance, the relative displacement

$$\zeta = \xi_2 - \xi_1, \tag{1.18}$$

rather than the centers ξ_1 and ξ_2, is the quantity that characterizes the configuration. However a disc-disc configuration is not a stationary point of $\mathcal{J}_\Lambda$. In this paper, we find stationary points that are close to disc-disc configurations. Such a stationary point (Ω_1,Ω_2) is called a disc-disc stationary point. Each Ω_j is a small perturbation of a disc $B(\xi_j, r_j)$, and

$$|\Omega_1| = \pi r_1^2,\ |\Omega_2| = \pi r_2^2. \tag{1.19}$$

Henceforth, r_1 and r_2 replace ω_1 and ω_2 as the first set of parameters. The area constraint (1.7) becomes (1.19). We set

$$\rho = \sqrt{r_1^2 + r_2^2}. \tag{1.20}$$

For the second set of parameters, let

$$|\gamma| = \sqrt{\sum_{j,k=1}^{2} \gamma_{jk}^2} \tag{1.21}$$

be the norm of γ.

A disc-disc stationary point satisfies equations (1.13) and (1.14), The other two equations, (1.15) and (1.16), are not needed, because in a disc-disc stationary point (Ω_1, Ω_2), the subsets Ω_1 and Ω_2 are separated by a positive distance. Green's function G_Λ of the $-\Delta$ operator on $\mathbb{C}/\Lambda$ plays a critical role here. Since it is smooth on $\mathbb{C}/\Lambda$ except at the lattice point 0 where

$$\lim_{z \to 0} G_\Lambda(z) = \infty, \tag{1.22}$$

G_Λ admits a global minimum. Our first theorem asserts that when r_j and γ_{jk} are in a suitable range, there exists a disc-disc stationary point whose relative displacement is close to a global minimum of G_Λ.

Theorem 1.1. *Let Λ be a lattice, $A_1 > 1$, $A_2 > 1$, and $\eta > 0$. There exists $\rho_0 > 0$ depending on Λ, A_1, A_2, and η such that if*

1. $0 < \rho = \sqrt{r_1^2 + r_2^2} < \rho_0$,
2. $\frac{1}{A_1} < \frac{r_2}{r_1} < A_1$,
3. *each entry* $\gamma_{jk} > 0$ $(j, k = 1, 2)$ *and* $|\gamma| < A_2 \gamma_{12}$,
4. *each diagonal entry* $\gamma_{jj} < \frac{12-\eta}{r_j^3}$, $j = 1, 2$,

then $\mathcal{J}_\Lambda$ admits a stationary point to (1.8) *and* (1.19) *of disc-disc type.*

If $\rho \to 0$ along a sequence, then the relative displacement of the stationary point converges, possibly along a subsequence, to a global minimum of G_Λ.

This stationary point is in a sense stable. See the discussion after the proof of the theorem.

Whenever (Ω_1, Ω_2) is a stationary point, one can shift both Ω_j, $j = 1, 2$, by an arbitrary amount ι, and the resulting pair $(\Omega_1 + \iota, \Omega_2 + \iota)$ is again a stationary point. Therefore Theorem 1.1, as well as Theorem 1.2 below, gives a family of stationary points.

Theorem 1.1 will be proved by a Lyapunov-Schmidt reduction procedure tailored for this type of variational problems. It was first developed by Ren and Wei for the diblock copolymer problem [23, 22], a single species system, and later used by Ren and Wang for the triblock copolymer problem [18]. There are two steps. First one fixes a relative displacement ζ and finds a "pseudo-solution" $\varphi(\cdot, \zeta)$ parametrized by ζ. This pseudo-solution solves (1.13) and (1.14) up to a two dimendional subspace; see Lemma 5.3. In the second step one considers the function $\zeta \to \mathcal{J}_\Lambda(\varphi(\cdot, \zeta))$ and shows that every critical point ζ_c of this function gives rise to a special pseudo-solution $\varphi(\cdot, \zeta_c)$, which solves (1.13) and (1.14) exactly; see Lemma 6.1. To find such a ζ_c, one observes that the function $\zeta \to \mathcal{J}_\Lambda(\varphi(\cdot, \zeta))$ is closely related to Green's function G_Λ. Since G_Λ admits a global minimum, $\zeta \to \mathcal{J}_\Lambda(\varphi(\cdot, \zeta))$ also has a minimum. Using this minimum as the critical point ζ_c, we obtain a stationary point $\varphi(\cdot, \zeta_c)$ of $\mathcal{J}_\Lambda$, proving the theorem.

Green's function G_Λ is Λ-periodic on $\mathbb{C}$; it is also even, i.e. $G_\Lambda(\zeta) = G_\Lambda(-\zeta)$ for all $\zeta \in \mathbb{C}$. Consequently $\nabla G_\Lambda(h) = 0$ where h is a half period, i.e. $h \notin \Lambda$ but $2h \in \Lambda$. There are precisely three Λ-inequivalent half periods. If (α_1, α_2) is a basis for Λ, they are given by

$$\frac{\alpha_1}{2}, \ \frac{\alpha_2}{2}, \ \frac{\alpha_1 + \alpha_2}{2}. \tag{1.23}$$

The question whether G_Λ admits other critical points was studied by Lin and Wang in [9]. It was shown that G_Λ has either three or five critical points in $\mathbb{C}/\Lambda$, depending on Λ. Our second theorem shows that if ζ_* is a non-degenerate critical point of G_Λ, there exists a disc-disc type stationary point whose relative displacement is close to ζ_*.

Theorem 1.2. *Let Λ be a lattice, $A_1 > 1$, $A_2 > 1$, and $\eta > 0$. Suppose that G_Λ admits a non-degenerate critical point ζ_*. There exists $\rho_0 > 0$ depending on Λ, A_1, A_2, and η such that if*

1. $0 < \rho = \sqrt{r_1^2 + r_2^2} < \rho_0$,

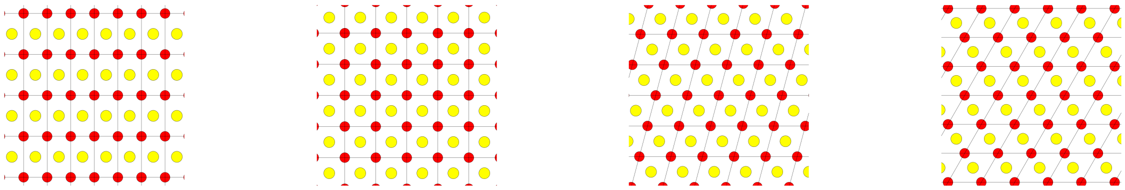

Figure 1: The fundamental parallelograms P_α and the disc-disc configurations of optimal lattices. From left to right: a rectangular lattice, a square lattice, a rhombic lattice, and a hexagonal lattice. In all cases $h = \frac{\alpha_1+\alpha_2}{2}$.

2. *$\frac{1}{A_1} < \frac{r_2}{r_1} < A_1$,*

3. *each entry $\gamma_{jk} > 0$ ($j, k = 1, 2$) and $|\gamma| < A_2\gamma_{12}$,*

4. *each diagonal entry $\gamma_{jj} < \frac{12-\eta}{r_j^3}$, $j = 1, 2$,*

then $\mathcal{J}_\Lambda$ admits a stationary point to (1.8) *and* (1.19) *of disc-disc type.*

If $\rho \to 0$, then the relative displacement of the stationary point converges to ζ_*.

The proof of Theorem 1.2 starts with the same pseudo-solution $\varphi(\cdot, \zeta)$ mentioned earlier. Instead of minimizing the function $\zeta \to \mathcal{J}_\Lambda(\varphi(\cdot, \zeta))$, consider its gradient $\nabla\mathcal{J}_\Lambda(\varphi(\cdot, \zeta))$ and show that this vector field can be approximated by ∇G_Λ, the gradient of G_Λ. Using a topological degree argument, we prove that near every non-degenerate critical point ζ_* of G_Λ there exists a critical point ζ_c of $\mathcal{J}_\Lambda(\varphi(\cdot, \zeta))$, leading to a stationary point $\varphi(\cdot, \zeta_c)$ of $\mathcal{J}_\Lambda$.

In the second part of this paper, we address the role played by the lattice Λ. From honeycomb to chicken wire fence, from graphene to carbon nanotube, the most common two dimensional lattice observed in nature is the hexagonal lattice. This lattice admits a basis (α_1, α_2) with $\frac{\alpha_2}{\alpha_1} = e^{\pi i/3}$. The reason that this lattice is called a hexagonal lattice comes from the notion of Voronoi cells. A Voronoi cell of a lattice point consists of points in $\mathbb{C}$ that are closer to the lattice point than other lattice points. For the hexagonal lattice the Voronoi cell of each lattice point is a regular hexagon.

Our finding goes against the conventional wisdom. For the two-species interacting system $\mathcal{J}_\Lambda$ the hexagonal lattice is rarely a favored structure. In section 7 it is shown that the effect of the size of a lattice can be separated from the effect of the shape of the lattice. One can determine the shape of an optimal lattice first by restricting to those lattices whos area equals 1. Then the size of the optimal lattice follows easily.

In our investigation, we are content with disc-disc configurations whose relative displacements are half periods. More precisely given $r_1, r_2 > 0$, define an admissible set $\mathcal{A}_{r_1,r_2}$ of pairs (Λ, h) such that $(\Lambda, h) \in \mathcal{A}_{r_1,r_2}$ if the following conditions hold:

1. Λ is a lattice with $|\Lambda| = 1$, and h is a half period of Λ;

2. $B(\lambda, r_1) \cap B(\lambda', r_1) = \emptyset$ whenever $\lambda, \lambda' \in \Lambda$ and $\lambda \neq \lambda'$;

3. $B(\lambda, r_2) \cap B(\lambda', r_2) = \emptyset$ whenever $\lambda, \lambda' \in \Lambda$ and $\lambda \neq \lambda'$;

4. $B(\lambda, r_1) \cap B(\lambda' + h, r_2) = \emptyset$ whenever $\lambda, \lambda' \in \Lambda$.

Here the discs $B(\lambda, r_j)$ and $B(\lambda', r_j)$ are viewed as subsets in $\mathbb{C}$. The conditions 2, 3, and 4 ensure that they are separated from each other.

Although not every pair of lattice and half period belongs to $\mathcal{A}_{r_1,r_2}$, the smaller r_1 and r_2 are, the larger the admissible set $\mathcal{A}_{r_1,r_2}$ is. In order to capture all the interesting lattices, we assume that r_1 and r_2 are sufficiently small so that $\mathcal{A}_{r_1,r_2}$ contains the following (Λ, h) pairs:

1. Λ has a rectangular fundamental parallelogram of unit area whose aspect ratio is in $[1, \sqrt{3}]$, and h is the center of the rectangle;

2. Λ has a rhombic fundamental parallelogram of unit area whose acute angles are in $[\frac{\pi}{3}, \frac{\pi}{2}]$, and h is the center of the rhombus.

Given an admissible set $\mathcal{A}_{r_1,r_2}$ satisfying the conditions 1 and 2 above we pose the minimization problem

$$\min_{(\Lambda,h)\in\mathcal{A}_{r_1,r_2}} \mathcal{J}_\Lambda(B(0,r_1), B(h,r_2)). \tag{1.24}$$

In section 8 we show that for problem (1.24), the hexagonal lattice is generally not optimal. Several lattices will appear as the most favored structures. They are illustrated in Figure 1. A rectangular lattice has a basis α whose fundamental parallelogram P_α is a rectangle. A square lattice has a square as a fundamental parallelogram. A rhombic lattice has a rhombus as a fundamental parallelogram. Finally a hexagonal lattice is a rhombus lattice with two acute angles of P_α equal to $\frac{\pi}{3}$. If

$$\tau = \frac{\alpha_2}{\alpha_1}, \tag{1.25}$$

then in terms of τ, Λ is rectangular if $\operatorname{Re}\tau = 0$, Λ is square if $\tau = \mathrm{i}$, Λ is rhombic if $|\tau| = 1$, and Λ is hexagonal if $\tau = e^{\pi\mathrm{i}/3}$. Note that these classes of lattices are not mutually exclusive. A hexagonal lattice is a rhombic lattice; a square lattice is both a rectangular lattice and a rhombic lattice. Here is the third theorem of this paper.

Theorem 1.3. *Let the parameters γ_{jk}, $j,k = 1,2$, satisfy the conditions*

$$\gamma_{11} > 0,\ \gamma_{22} > 0,\ \gamma_{12} \geq 0,\ \gamma_{11}\gamma_{22} - \gamma_{12}^2 \geq 0, \tag{1.26}$$

and set

$$b = \frac{\gamma_{11}r_1^4 + \gamma_{22}r_2^4 - 2\gamma_{12}r_1^2r_2^2}{\gamma_{11}r_1^4 + \gamma_{22}r_2^4} \tag{1.27}$$

which lies in $[0,1]$ by (1.26). Suppose that r_1 and r_2 are sufficiently small so that $\mathcal{A}_{r_1,r_2}$ satisfies the conditions 1 and 2 before (1.24).

The minimization problem (1.24) admits a minimum attained by a lattice Λ_b and a half period h_b. The lattice Λ_b has a basis $\alpha_b = (\alpha_{b,1}, \alpha_{b,2})$ such that

$$h_b = \frac{\alpha_{b,1} + \alpha_{b,2}}{2}. \tag{1.28}$$

Moreover there exists a number $B = 0.1867...$ such that the following statements hold.

1. *If $b = 0$, then P_{α_b} is a rectangle whose aspect ratio is $\sqrt{3}$.*

2. *If $b \in (0, B)$, then P_{α_b} is a rectangle whose aspect ratio is in $(1, \sqrt{3})$. As b increases from 0 to B, this ratio decreases from $\sqrt{3}$ to 1.*

3. *If $b \in [B, 1-B]$, then P_{α_b} is a square.*

4. *If $b \in (1-B, 1)$, then P_{α_b} is a rhombus. It has an acute angle in $(\frac{\pi}{3}, \frac{\pi}{2})$. As b increases from $1-B$ to 1, this angle decreases from $\frac{\pi}{2}$ to $\frac{\pi}{3}$.*

5. *If $b = 1$, then P_{α_b} is a rhombus with an acute angle equal to $\frac{\pi}{3}$.*

The minimum (Λ_b, h_b) is unique up to rotation.

As you can see, only in the case $b = 1$, the optimal lattice is hexagonal; in all other cases, the optimal lattice is not hexagonal. In contrast, there is a single species analogy to (1.8): the Ohta-Kawasaki diblock copolymer problem [15]. If considered on a flat torus, the free energy of that problem takes the form

$$\mathcal{J}_{d,\Lambda}(\Omega) = \mathcal{P}_{\mathbb{C}/\Lambda}(\Omega) + \frac{\gamma_d}{2} \int_{\mathbb{C}/\Lambda} |\nabla I_\Lambda(\Omega)(z)|^2 \, dA(z) \tag{1.29}$$

where Ω is a measurable subset of $\mathbb{C}/\Lambda$ of the prescribed area: $|\Omega| = \omega_d |\mathbb{C}/\Lambda|$, $\omega_d \in (0,1)$, and $\gamma_d > 0$. This Ω is occupied by one of the two type monomers in a diblock copolymer and is viewed as the only species of the system (the other type monomers form the environment). If one takes $\Omega = B(0,r)$ to be a disc and minimizes $\mathcal{J}_{d,\Lambda}(B(0,r))$ with respect to Λ, $|\Lambda| = 1$, Chen and Oshita showed that the minimum was achieved by a hexagonal lattice [4]; Sandier and Serfaty gave a different proof of this fact in [26].

In recent years the single species problem (1.29) has been actively studied on flat tori and bounded domains with the Neumann boundary condition for equation (1.10); see [19, 22, 8, 1, 5, 11, 7, 16] and the references therein. The study of the two-species problem (1.8) is still in the early stage. There are results on existence of stationary points in one and two dimensional domains with the Neumann boundary condition [21, 24, 25, 17, 18].

The threshold number B in Theorem 1.3 was first discovered by Luo, Ren, and Wei in their study of another configuration [10]. It comprises two discs of the first species, $B(\xi_1, r_1)$ and $B(\xi_1', r_1)$, and two discs of the second species, $B(\xi_2, r_2)$ and $B(\xi_2', r_2)$. In a fundamental parallelogram P_α the four centers are

$$\xi_1 = \frac{3}{4}\alpha_1 + \frac{1}{4}\alpha_2, \; \xi_1' = \frac{1}{4}\alpha_1 + \frac{3}{4}\alpha_2, \;\; \xi_2 = \frac{1}{4}\alpha_1 + \frac{1}{4}\alpha_2, \; \xi_2' = \frac{3}{4}\alpha_1 + \frac{3}{4}\alpha_2, \tag{1.30}$$

which form a parallelogram like a scaled down version of P_α by a factor $1/2$. When the energy of this configuration is minimized among lattices of unit area, the optimal one is rectangular, square, rhombic, or hexagonal, like in Theorem 1.3.

Another motivation for our work comes from a two-component Bose-Einstein condensates problem studied by Mueller and Ho [12]. In a Bose gas made up of two hyperfine spin states of the same atom, the vortex lattices are bound to be more intricate than those in single component condensates, as the vortices in different components can move relative to one another. The model Mueller and Ho derived, different from our two-species interacting system (1.8), also contains a parameter, and as it varies, numerical calculations show various hexagonal, rhombic, square, and rectangular lattices appearing as the optimal lattices.

2 Disc-disc configuration

Section 2 to section 6 are devoted to the proofs of Theorems 1.1 and 1.2. The constants $A_1 > 1$, $A_2 > 1$, and $\eta > 0$ in the two theorems are fixed and the conditions 2, 3, and 4 on r_j and γ_{jk} hold. Also fixed in these sections is the lattice Λ, so we drop the subscript Λ in notations like $\mathcal{J}_\Lambda$, I_Λ, and G_Λ, etc, and simply write $\mathcal{J}$, I, and G, respectively. There will be constants, such as C_0, C_1, c_2, C_2, and C_3, arising in various estimates. They depend on A_1, A_2, η, and Λ at most.

Denote the two parts of $\mathcal{J}$ by $\mathcal{J}_s$ and $\mathcal{J}_l$ for short range interaction and long rrange interaction respectively, i.e.

$$\mathcal{J}(\Omega_1, \Omega_2) = \mathcal{J}_s(\Omega_1, \Omega_2) + \mathcal{J}_l(\Omega_1, \Omega_2), \tag{2.1}$$

$$\mathcal{J}_s(\Omega_1, \Omega_2) = \frac{1}{2} \sum_{j=1}^{3} \mathcal{P}_{\mathbb{C}/\Lambda}(\Omega_j/\Lambda), \tag{2.2}$$

$$\mathcal{J}_l(\Omega_1, \Omega_2) = \sum_{j,k=1}^{2} \frac{\gamma_{jk}}{2} \int_{\mathbb{C}/\Lambda} \nabla I(\Omega_j)(z) \cdot \nabla I(\Omega_k)(z) \, dA(z). \tag{2.3}$$

Let (α_1, α_2) be a basis of Λ. It is always assumed that

$$\frac{\alpha_2}{\alpha_1} \in \mathbb{H} \tag{2.4}$$

where

$$\mathbb{H} = \{\tau \in \mathbb{C} : \ \mathrm{Im}\, \tau > 0\} \tag{2.5}$$

is the upper half plane. There is an explicit formula for Green's function G:

$$\begin{aligned} G(z) = \frac{|z|^2}{4|\Lambda|} - \frac{1}{2\pi} \log \Big| \mathrm{e}\left(\frac{z^2 \overline{\alpha_1}}{4\mathrm{i}|\Lambda|\alpha_1} - \frac{z}{2\alpha_1} + \frac{\alpha_2}{12\alpha_1}\right) \left(1 - \mathrm{e}\left(\frac{z}{\alpha_1}\right)\right) \\ \prod_{n=1}^{\infty} \left(1 - \mathrm{e}\left(n\tau + \frac{z}{\alpha_1}\right)\right) \left(1 - \mathrm{e}\left(n\tau - \frac{z}{\alpha_1}\right)\right) \Big|; \end{aligned} \tag{2.6}$$

see [4, Lemma 1] for a proof. Here

$$\mathrm{e}(w) = e^{2\pi \mathrm{i} w}. \tag{2.7}$$

Sometimes one singles out the singularity of G at 0 and decompose G into

$$G(z) = -\frac{1}{2\pi} \log \frac{2\pi|z|}{\sqrt{|\Lambda|}} + \frac{|z|^2}{4|\Lambda|} + H(z) \tag{2.8}$$

where

$$\begin{aligned} H(z) = -\frac{1}{2\pi} \log \Big| \mathrm{e}\left(\frac{z^2 \overline{\alpha_1}}{4\mathrm{i}|\Lambda|\alpha_1} - \frac{z}{2\alpha_1} + \frac{\alpha_2}{12\alpha_1}\right) \frac{\sqrt{|\Lambda|}}{2\pi z} \left(1 - \mathrm{e}\left(\frac{z}{\alpha_1}\right)\right) \\ \prod_{n=1}^{\infty} \left(1 - \mathrm{e}\left(n\tau + \frac{z}{\alpha_1}\right)\right) \left(1 - \mathrm{e}\left(n\tau - \frac{z}{\alpha_1}\right)\right) \Big| \end{aligned} \tag{2.9}$$

is a harmonic function on $(\mathbb{C}\backslash\Lambda) \cup \{0\}$.

Note that G is a smooth function on $\mathbb{C}/\Lambda$ except at the lattice point 0, where $G(z) \to \infty$ as $z \to 0$. For sufficienlty small $\delta > 0$,

$$\min_{z \in \mathbb{C}/\Lambda} G(z) < \min_{z \in (\mathbb{C}/\Lambda)\backslash(\mathbb{C}/\Lambda)_\delta} G(z) \tag{2.10}$$

where $(\mathbb{C}/\Lambda)_\delta$ is subset of $\mathbb{C}/\Lambda$ given by

$$(\mathbb{C}/\Lambda)_\delta = \{z \in \mathbb{C}/\Lambda : \ |z| > \delta\}. \tag{2.11}$$

By (2.10), if ζ_* is a minimum of G in $\mathbb{C}/\Lambda$, then $|\zeta_*| > \delta$. Henceforth we fix one such δ.

Let $\Omega_j = B(\xi_j, r_j)$, $j = 1, 2$, be two discs in $\mathbb{C}/\Lambda$ centered at ξ_j of radii r_j:

$$B(\xi_j, r_j) = \{z \in \mathbb{C}/\Lambda : \ |z - \xi_j| \leq r_j\}, \ j = 1, 2. \tag{2.12}$$

Denote by

$$\zeta = \xi_2 - \xi_1 \tag{2.13}$$

the relative displacement from ξ_1 to ξ_2. One requires that

$$|\zeta| \geq \delta, \text{ i.e., } \zeta \in \overline{(\mathbb{C}/\Lambda)_\delta}, \tag{2.14}$$

and

$$r_j < \frac{\delta}{4}, \ j = 1, 2. \tag{2.15}$$

Under (2.14) and (2.15), for any two points $z_j \in B(\xi_j, r_j)$, $j = 1, 2$,

$$|z_2 - z_1| \geq |\xi_2 - \xi_1| - r_1 - r_2 > \delta - \frac{\delta}{4} - \frac{\delta}{4} = \frac{\delta}{2}, \tag{2.16}$$

so the two sets $B(\xi_1, r_1)$ and $B(\xi_2, r_2)$ are well separated. We call $(B(\xi_1, r_1), B(\xi_2, r_2))$ a disc-disc configuration.

Lemma 2.1. *Let $B(\xi, r)$ be a disc in $\mathbb{C}/\Lambda$ centered at ξ of radius r. Then*

$$I(B(\xi,r))(z) = \left\{ \begin{array}{r} -\dfrac{|z-\xi|^2}{4} + \dfrac{r^2}{4} - \dfrac{r^2}{2}\log r \quad \text{if} \ \ |z-\xi| \in [0,r] \\ -\dfrac{r^2}{2}\log|z-\xi| \quad \text{if} \ \ |z-\xi| > r \end{array} \right\}$$
$$- \frac{r^2}{2}\log\frac{2\pi}{\sqrt{|\Lambda|}} + \frac{1}{4|\Lambda|}\left(\pi r^2 |z-\xi|^2 + \frac{\pi r^4}{2}\right) + \pi r^2 H(z-\xi).$$

Proof. Without the loss of generality, assume $\xi = 0$. Since the integral of G on $\mathbb{C}/\Lambda$ is zero, by (2.8)

$$\begin{aligned} I(B(0,r))(z) &= \int_{\mathbb{C}/\Lambda} G(z-\zeta)\left(\chi_{B(0,r)}(\zeta) - \frac{\pi r^2}{|\Lambda|}\right) dA(\zeta) \\ &= \int_{B(0,r)} G(z-\zeta)\, dA(\zeta) \\ &= \int_{B(0,r)} \left(-\frac{1}{2\pi}\right)\log|z-\zeta|\, dA(\zeta) + \int_{B(0,r)} \left(-\frac{1}{2\pi}\right)\log\frac{2\pi}{\sqrt{|\Lambda|}}\, dA(\zeta) \\ &\quad + \int_{B(0,r)} \frac{|z-\zeta|^2}{4|\Lambda|}\, dA(\zeta) + \int_{B(0,r)} H(z-\zeta)\, dA(\zeta). \end{aligned} \tag{2.17}$$

Let

$$v(z) = \int_{B(0,r)} \left(-\frac{1}{2\pi}\right)\log|z-\zeta|\, dA(\zeta),$$

which is a radially symmetric solution of $-\Delta v = \chi_{B(0,r)}$ in $\mathbb{R}^2 = \mathbb{C}$. With $t = |z|$,

$$-v_{tt} - \frac{1}{t}v_t = \chi_{(0,r)}, \ \ v_t(0) = 0.$$

Solving this equation, one finds

$$v(t) = \left\{ \begin{array}{r} -\dfrac{t^2}{4} + \dfrac{r^2}{4} - \dfrac{r^2}{2}\log r \quad \text{if } t \in [0,r] \\ -\dfrac{r^2}{2}\log t \quad \text{if } t > r \end{array} \right\} + C.$$

To determine the constant C, let $t = 0$ and then $v(0) = \frac{r^2}{4} - \frac{r^2}{2}\log r + C$. On the other hand

$$v(0) = \int_{B(0,r)} \left(-\frac{1}{2\pi}\right)\log|0-\zeta|\, dA(\zeta) = \frac{r^2}{4} - \frac{r^2}{2}\log r.$$

Hence $C = 0$. For the remaining terms in (2.17), one finds

$$\begin{aligned} \int_{B(0,r)} \left(-\frac{1}{2\pi}\right)\log\frac{2\pi}{\sqrt{|\Lambda|}}\, dA(\zeta) &= -\frac{r^2}{2}\log\frac{2\pi}{\sqrt{|\Lambda|}} \\ \int_{B(0,r)} \frac{|z-\zeta|^2}{4|\Lambda|}\, dA(\zeta) &= \frac{1}{4|\Lambda|}\left(\pi r^2|z|^2 + \frac{\pi r^4}{2}\right) \\ \int_{B(0,r)} H(z-\zeta)\, dA(\zeta) &= \pi r^2 H(z) \end{aligned}$$

where the last one follows from the mean value property for harmonic functions. $\square$

Lemma 2.2. *Let $(B(\xi_1, r_1), B(\xi_2, r_2))$ be a disc-disc configuration. Then*

$$\mathcal{J}\left(B(\xi_1,r_1),B(\xi_2,r_2)\right)=\sum_{j=1}^{2}2\pi r_j+\sum_{j=1}^{2}\frac{\gamma_{jj}}{2}\left(-\frac{\pi r_j^4}{2}\log\frac{2\pi r_j}{\sqrt{|\Lambda|}}+\frac{\pi r_j^4}{8}+\pi^2 r_j^4 H(0)+\frac{\pi^2 r_j^6}{4|\Lambda|}\right)$$
$$+\gamma_{12}\left(\pi^2 r_1^2 r_2^2 G(\zeta)+\frac{\pi^2(r_1^2 r_2^4+r_2^2 r_1^4)}{8|\Lambda|}\right)$$

where $\zeta = \xi_2 - \xi_1$.

Proof. Obviously

$$\mathcal{J}_s(B(\xi_1,r_1),B(\xi_2,r_2))=\sum_{j=1}^{2}2\pi r_j. \tag{2.18}$$

To find $\mathcal{J}_l(B(\xi_1, r_1), B(\xi_2, r_2))$, note that, by Lemma 2.1,

$$\begin{aligned}\int_{B(\xi_j,r_j)}\int_{B(\xi_j,r_j)}G(z-w)\,dA(w)dA(z)&=\int_{B(\xi_j,r_j)}I(B(\xi_j,r_j))(z)\,dA(z)\\&=-\frac{\pi r_j^4}{2}\log\frac{2\pi r_j}{\sqrt{|\Lambda|}}+\frac{\pi r_j^4}{8}+\pi^2 r_j^4 H(0)+\frac{\pi^2 r_j^6}{4|\Lambda|},\ j=1,2,\end{aligned} \tag{2.19}$$

$$\begin{aligned}\int_{B(\xi_j,r_j)}\int_{B(\xi_k,r_k)}G(z-w)\,dA(w)dA(z)&=\int_{B(\xi_j,r_j)}I(B(\xi_k,r_k))(z)\,dA(z)\\&=\pi^2 r_j^2 r_k^2 G(\xi_j-\xi_k)+\frac{\pi^2(r_j^2 r_k^4+r_k^2 r_j^4)}{8|\Lambda|},\ j\neq k.\end{aligned} \tag{2.20}$$

The lemma follows from (2.18), (2.19), and (2.20). □

3 First variation

Define a Hilbert space $\mathcal{Z}$ of functions whose values are in $\mathbb{R}^2$,

$$\mathcal{Z}=\left\{\phi=(\phi_1,\phi_2):\ \phi_j\in L^2(S^1),\ \int_0^{2\pi}\phi_j(\theta)d\theta=0,\ j=1,2\right\}. \tag{3.1}$$

Here S^1 is the unit circle identified with $[0, 2\pi]$, and $L^2(S^1)$ is the real valued L^2-space on S^1. The inner product on $\mathcal{Z}$ is

$$\langle\phi,\psi\rangle=\sum_{j=1}^{2}\int_0^{2\pi}\phi_j(\theta)\psi_j(\theta)\,d\theta. \tag{3.2}$$

Next let

$$\mathcal{Y}=\left\{\phi=(\phi_1,\phi_2)\in\mathcal{Z}:\ \phi_j\in H^1(S^1),\ j=1,2\right\} \tag{3.3}$$

be a subspace of $\mathcal{Z}$. Here $H^1(S^1)$ is the usual H^1 Sobolev space on S^1. The norm of $\mathcal{Y}$ is given by

$$||\phi||_{\mathcal{Y}}^2=\sum_{j=1}^{2}\int_0^{2\pi}\left((\phi_j')^2+\phi_j^2\right)\,d\theta. \tag{3.4}$$

Finally define

$$\mathcal{X}=\left\{\phi=(\phi_1,\phi_2)\in\mathcal{Z}:\ \phi_j\in H^2(S^1),\ j=1,2\right\} \tag{3.5}$$

where $H^2(S^1)$ is the H^2 Sobolev space on S^1 and the norm of $\mathscr{X}$ is given by

$$||\phi||^2_{\mathscr{X}} = \sum_{j=1}^{2} \int_0^{2\pi} \left((\phi_j'')^2 + (\phi_j')^2 + \phi_j^2\right) d\theta. \tag{3.6}$$

Clearly $\mathscr{X} \subset \mathscr{Y} \subset \mathscr{Z}$.

Fix ξ_j and r_j, $j = 1, 2$, subject to the conditions (2.14) and (2.15). If ϕ_1 and ϕ_2 are 2π-periodic continuous functions, collectively denoted $\phi = (\phi_1, \phi_2)$, then they specify two subsets of $\mathbb{C}/\Lambda$ as

$$\Omega_j = \left\{\xi_j + te^{\mathrm{i}\theta} \in \mathbb{C}/\Lambda : \theta \in [0, 2\pi],\ t \in \left[0, \sqrt{r_j^2 + 2\phi_j(\theta)}\,\right]\right\},\ j = 1, 2, \tag{3.7}$$

provided $r_j^2 + 2\phi_j(\theta) > 0$ for all $\theta \in [0, 2\pi]$, $j = 1, 2$. In particular $\phi = (0, 0)$ corresponds to the the disc-disc configuration $(B(\xi_1, r_1), B(\xi_2, r_2))$.

One views $\mathscr{J}$ as a functional of ϕ and the domain of $\mathscr{J}$ is taken to be

$$\mathrm{Dom}(\mathscr{J}) = \left\{\phi \in \mathscr{Y} :\ \|\phi\|_{\mathscr{Y}} < d\rho^2\right\}. \tag{3.8}$$

In (3.8), d is a positive number small enough so that

$$2Cd(1 + A_1^2) < \frac{1}{2} \tag{3.9}$$

where $C > 0$ is a constant in the imbedding $H^1(S^1) \to C(S^1)$, i.e.,

$$\|f\|_{L^\infty(S^1)} \le C\|f\|_{H^1(S^1)},\ \text{for all } f \in H^1(S^1), \tag{3.10}$$

and $A_1 > 1$ is the constant in condition 2 of Theorems 1.1 and 1.2.

Note that under (3.8) and (3.9),

$$\|2\phi_j\|_{L^\infty(S^1)} \le 2C\|\phi_j\|_{H^1(S^1)} \le 2Cd\rho^2 \le 2Cd(1 + A_1^2)r_j^2 < \frac{r_j^2}{2}, \tag{3.11}$$

and hence

$$0 < \frac{r_j^2}{2} < r_j^2 + 2\phi_j(\theta) < \frac{3r_j^2}{2},\ \text{for all } \theta \in S^1. \tag{3.12}$$

The lower bound in (3.12) shows that Ω_j is well-defined by (3.7). The upper bound in (3.12) implies that Ω_1 and Ω_2 are well separated, for if $z_j \in \Omega_j$, $j = 1, 2$, then by (2.14) and (2.15),

$$|z_2 - z_1| > |\xi_2 - \xi_1| - \sqrt{\frac{3}{2}}r_1 - \sqrt{\frac{3}{2}}r_2 > \delta - \sqrt{\frac{3}{2}}\left(\frac{\delta}{4}\right) - \sqrt{\frac{3}{2}}\left(\frac{\delta}{4}\right) = 0.3876... \times \delta. \tag{3.13}$$

The area of Ω_j is

$$|\Omega_j| = \int_0^{2\pi} \int_0^{\sqrt{r_j^2+2\phi_j(\theta)}} t\,dt\,d\theta = \pi r_j^2 + \int_0^{2\pi} \phi_j(\theta)d\theta. \tag{3.14}$$

Since $\phi \in \mathscr{Z}$, $\int_0^{2\pi} \phi_j(\theta)\,d\theta = 0$. Hence

$$|\Omega_j| = \pi r_j^2,\ j = 1, 2, \tag{3.15}$$

satisfying the area constraint (1.19).

Like $\mathscr{J}$, $\mathscr{J}_s$ and $\mathscr{J}_l$ defined in (2.2) and (2.3) are also functionals of ϕ. More explicitly

$$\mathscr{J}_s(\phi) = \sum_{j=1}^{2} \int_0^{2\pi} \sqrt{r_j^2 + 2\phi_j + \frac{\left(\phi_j'\right)^2}{r_j^2 + 2\phi_j}}\ d\theta \tag{3.16}$$

$$\begin{aligned}\mathscr{J}_l(\phi) = &\sum_{k=1}^{2} \frac{\gamma_{jj}}{2} \int_0^{2\pi} \int_0^{\sqrt{r_j^2+2\phi_j(\theta)}} \int_0^{2\pi} \int_0^{\sqrt{r_j^2+2\phi_j(\eta)}} G(te^{\mathrm{i}\theta} - qe^{\mathrm{i}\eta})tq\ dq d\eta\, dt d\theta \\ &+ \gamma_{12} \int_0^{2\pi} \int_0^{\sqrt{r_1^2+2\phi_1(\theta)}} \int_0^{2\pi} \int_0^{\sqrt{r_2^2+2\phi_2(\eta)}} G(-\zeta + te^{\mathrm{i}\theta} - qe^{\mathrm{i}\eta})tq\ dq d\eta\, dt d\theta.\end{aligned} \tag{3.17}$$

Remark 3.1. *In addition to ϕ, $\mathcal{J}_l(\phi)$, and $\mathcal{J}(\phi)$ consequently, also depend on ζ, as seen in the last line of* (3.17)*. However ζ is fixed until section 5.*

The first variation of $\mathcal{J}$ is a nonlinear integro-differential operator $\mathcal{S}$ such that

$$\left.\frac{d\mathcal{J}(\phi+\varepsilon\psi)}{d\varepsilon}\right|_{\varepsilon=0} = \langle \mathcal{S}(\phi), \psi\rangle. \tag{3.18}$$

The operator $\mathcal{S}$ maps a neighborhood of 0 in $\mathcal{X}$ into $\mathcal{Z}$. More precisely the domain of $\mathcal{S}$ is set to be

$$\mathrm{Dom}(\mathcal{S}) = \left\{\phi \in \mathcal{X} : \ \|\phi\|_{\mathcal{X}} < d\rho^2\right\}. \tag{3.19}$$

Note that the d in (3.19) is the same as the d in (3.8), so $\mathrm{Dom}(\mathcal{S})$ is a subset of $\mathrm{Dom}(\mathcal{J})$.

For functions f_1 and f_2 defined on S^1, it is convenient to use the notation

$$f_1 \simeq f_2 \tag{3.20}$$

if f_1 and f_2 differ by a constant. One can find the constant by averaging, i.e.

$$f_1 - f_2 = \frac{1}{2\pi}\int_0^{2\pi} f_1(\theta)\,d\theta - \frac{1}{2\pi}\int_0^{2\pi} f_2(\theta)\,d\theta. \tag{3.21}$$

Write $\mathcal{S} = \mathcal{S}_s + \mathcal{S}_l$ where $\mathcal{S}_s$ and $\mathcal{S}_l$ are the first variations of $\mathcal{J}_s$ and $\mathcal{J}_l$ respectively. They are also operators from $\mathrm{Dom}(\mathcal{S})$ to $\mathcal{Z}$, and calculations show

$$\mathcal{S}_{s,j}(\phi)(\theta) \simeq \frac{r_j^2 + 2\phi_j(\theta) + \frac{3(\phi_j'(\theta))^2}{r_j^2+2\phi_j(\theta)} - \phi_j''(\theta)}{\left(r_j^2 + 2\phi_j(\theta) + \frac{(\phi_j'(\theta))^2}{r_j^2+2\phi_j(\theta)}\right)^{3/2}}, \ j = 1, 2. \tag{3.22}$$

where the right side is the curvature of $\partial\Omega_j$. Note that $\simeq$, instead of $=$, is used in (3.22). This is because the first variation of $\mathcal{J}_s$ is calculated in the space $\mathcal{Y}$, where functions have zero average. The left side $\mathcal{S}_{s,j}(\phi)$ must have zero average but the curvature on the right side does not, and the two differ by a constant. If we denote the right side of (3.22) by $\mathcal{K}_j(\phi_j)$, then

$$\mathcal{S}_{s,j}(\phi) = \mathcal{K}_j(\phi_j) - \overline{\mathcal{K}_j(\phi_j)} \tag{3.23}$$

where $\overline{\mathcal{K}_j(\phi_j)}$ denotes the average of $\mathcal{K}_j(\phi_j)$,

$$\overline{\mathcal{K}_j(\phi_j)} = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{K}_j(\phi_j)(\theta)\,d\theta. \tag{3.24}$$

The components of $\mathcal{S}_l(\phi)$ are

$$\mathcal{S}_{l,1}(\phi)(\theta) \ \simeq \ \gamma_{11} I(\Omega_1)\left(\xi_1 + \sqrt{r_1^2 + 2\phi_1(\theta)}e^{\mathrm{i}\theta}\right) + \gamma_{12} I(\Omega_2)\left(\xi_1 + \sqrt{r_1^2 + 2\phi_1(\theta)}e^{\mathrm{i}\theta}\right), \tag{3.25}$$

$$\mathcal{S}_{l,2}(\phi)(\theta) \ \simeq \ \gamma_{12} I(\Omega_1)\left(\xi_2 + \sqrt{r_2^2 + 2\phi_2(\theta)}e^{\mathrm{i}\theta}\right) + \gamma_{22} I(\Omega_2)\left(\xi_2 + \sqrt{r_2^2 + 2\phi_2(\theta)}e^{\mathrm{i}\theta}\right). \tag{3.26}$$

Again the left side and the right side in each of (3.25) and (3.26) differ by a constant. Let

$$G(z) = -\frac{1}{2\pi}\log|z| + R(z) \tag{3.27}$$

where by (2.8)

$$R(z) = -\frac{1}{2\pi}\log\frac{2\pi}{\sqrt{|\Lambda|}} + \frac{|z|^2}{4|\Lambda|} + H(z) \tag{3.28}$$

is smooth on $(\mathbb{C}\backslash\Lambda)\cup\{0\}$. The $\mathcal{S}_l$ operator may be written in a more explicit form:

$$\begin{aligned}
\mathcal{S}_{l,1}(\phi)(\theta) \simeq & -\frac{\gamma_{11}}{2\pi}\int_0^{2\pi}\int_0^{\sqrt{r_1^2+2\phi_1(\eta)}} \log\left|\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta}-te^{i\eta}\right| t\,dtd\eta \\
& + \gamma_{11}\int_0^{2\pi}\int_0^{\sqrt{r_1^2+2\phi_1(\eta)}} R\left(\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta}-te^{i\eta}\right) t\,dtd\eta \\
& + \gamma_{12}\int_0^{2\pi}\int_0^{\sqrt{r_2^2+2\phi_2(\eta)}} G\left(-\zeta+\sqrt{r_1^2+2\phi_1(\theta)}e^{i\theta}-te^{i\eta}\right) t\,dtd\eta \qquad (3.29)\\
\mathcal{S}_{l,2}(\phi)(\theta) \simeq & -\frac{\gamma_{22}}{2\pi}\int_0^{2\pi}\int_0^{\sqrt{r_2^2+2\phi_2(\eta)}} \log\left|\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta}-te^{i\eta}\right| t\,dtd\eta \\
& + \gamma_{22}\int_0^{2\pi}\int_0^{\sqrt{r_2^2+2\phi_2(\eta)}} R\left(\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta}-te^{i\eta}\right) t\,dtd\eta \\
& + \gamma_{12}\int_0^{2\pi}\int_0^{\sqrt{r_1^2+2\phi_1(\eta)}} G\left(\zeta+\sqrt{r_2^2+2\phi_2(\theta)}e^{i\theta}-te^{i\eta}\right) t\,dtd\eta. \qquad (3.30)
\end{aligned}$$

Comparing (3.22), (3.25), and (3.26) to (1.13) and (1.14), we see that if

$$\mathcal{S}(\phi)=0, \qquad (3.31)$$

then the pair (Ω_1,Ω_2) described by ϕ is a stationary point of $\mathcal{J}$.

In this paper, the $O(\cdot)$ notation is used. For instance $O(|\gamma|\rho^4)$ in the next lemma stands for a quantity that can be bounded by $C|\gamma|\rho^4$ uniformly in θ for some $C>0$ where C is independent of γ, r_j, and ζ.

Lemma 3.2. *The first variation at the disc-disc configuration, represented by $\phi=(0,0)$, is*

$$\begin{aligned}
\mathcal{S}_1(0,0)(\theta) &= \gamma_{11}\pi r_1^2\nabla H(0)\cdot r_1e^{i\theta}+\gamma_{12}\pi r_2^2\nabla G(-\zeta)\cdot r_1e^{i\theta}+O(|\gamma|\rho^4)\\
\mathcal{S}_2(0,0)(\theta) &= \gamma_{22}\pi r_2^2\nabla H(0)\cdot r_2e^{i\theta}+\gamma_{12}\pi r_1^2\nabla G(\zeta)\cdot r_2e^{i\theta}+O(|\gamma|\rho^4).
\end{aligned}$$

Proof. The curvature of a circle is the inverse of its radius, so

$$\mathcal{S}_{s,j}(0,0)\simeq\frac{1}{r_j},\ j=1,2. \qquad (3.32)$$

By Lemma 2.1 and dropping constant terms, one derives

$$\begin{aligned}
\mathcal{S}_{l,1}(0,0)(\theta) &\simeq \gamma_{11}\left[-\frac{r_1^2}{2}\log\frac{2\pi r_1}{\sqrt{|\Lambda|}}+\frac{3\pi r_1^4}{8|\Lambda|}+\pi r_1^2H(r_1e^{i\theta})\right]+\gamma_{12}\left[\pi r_2^2G(-\zeta+r_1e^{i\theta})+\frac{\pi r_2^4}{8|\Lambda|}\right]\\
&\simeq \gamma_{11}\pi r_1^2H(r_1e^{i\theta})+\gamma_{12}\pi r_2^2G(-\zeta+r_1e^{i\theta})\\
&\simeq \gamma_{11}\pi r_1^2\left(H(0)+\nabla H(0)\cdot r_1e^{i\theta}+O(\rho^2)\right)+\gamma_{12}\pi r_2^2\left(G(-\zeta)+\nabla G(-\zeta)\cdot r_1e^{i\theta}+O(\rho^2)\right)\\
&\simeq \gamma_{11}\pi r_1^2\nabla H(0)\cdot r_1e^{i\theta}+\gamma_{12}\pi r_2^2\nabla G(-\zeta)\cdot r_1e^{i\theta}+O(|\gamma|\rho^4) \qquad (3.33)\\
\mathcal{S}_{l,2}(0,0)(\theta) &\simeq \gamma_{22}\left[-\frac{r_2^2}{2}\log\frac{2\pi r_2}{\sqrt{|\Lambda|}}+\frac{3\pi r_2^4}{8|\Lambda|}+\pi r_2^2H(r_2e^{i\theta})\right]+\gamma_{12}\left[\pi r_1^2G(\zeta+r_2e^{i\theta})+\frac{\pi r_1^4}{8|\Lambda|}\right]\\
&\simeq \gamma_{22}\pi r_2^2\nabla H(0)\cdot r_2e^{i\theta}+\gamma_{12}\pi r_1^2\nabla G(\zeta)\cdot r_2e^{i\theta}+O(|\gamma|\rho^4). \qquad (3.34)
\end{aligned}$$

Now that

$$\begin{aligned}
\mathcal{S}_1(0,0)(\theta) &\simeq \frac{1}{r_1}+\gamma_{11}\pi r_1^2\nabla H(0)\cdot r_1e^{i\theta}+\gamma_{12}\pi r_2^2\nabla G(-\zeta)\cdot r_1e^{i\theta}+O(|\gamma|\rho^4)\\
&\simeq \gamma_{11}\pi r_1^2\nabla H(0)\cdot r_1e^{i\theta}+\gamma_{12}\pi r_2^2\nabla G(-\zeta)\cdot r_1e^{i\theta}+O(|\gamma|\rho^4)
\end{aligned}$$

by (3.32) and (3.33), there is $K \in \mathbb{R}$, independent of θ, such that

$$\mathcal{S}_1(0,0)(\theta) = \gamma_{11}\pi r_1^2 \nabla H(0)\cdot r_1 e^{\mathrm{i}\theta} + \gamma_{12}\pi r_2^2 \nabla G(-\zeta)\cdot r_1 e^{\mathrm{i}\theta} + O(|\gamma|\rho^4) + K. \tag{3.35}$$

Since $\mathcal{S}(0,0) \in \mathcal{Z}$, the average of $\mathcal{S}_1(0,0)$ vanishes. Integrating (3.35) with respect to θ yields

$$0 = 0 + 0 + O(|\gamma|\rho^4) + K,$$

which implies

$$K = O(|\gamma|\rho^4)$$

and

$$\mathcal{S}_1(0,0)(\theta) = \gamma_{11}\pi r_1^2 \nabla H(0)\cdot r_1 e^{\mathrm{i}\theta} + \gamma_{12}\pi r_2^2 \nabla G(-\zeta)\cdot r_1 e^{\mathrm{i}\theta} + O(|\gamma|\rho^4).$$

The same argument applies to $\mathcal{S}_2(0,0)$. □

4 Second variation

The Fréchet derivative of $\mathcal{S}$ at $(0,0)$, which is the second variation of $\mathcal{J}$ at $(0,0)$, is a linear operator,

$$\mathcal{S}'(0,0):\ u \in \mathcal{X} \to \mathcal{S}'(0,0)(u) \in \mathcal{Z}. \tag{4.1}$$

Calculations show

$$\begin{aligned}
&\mathcal{S}_1'(0,0)(u)(\theta)\\
&\simeq -\frac{1}{r_1^3}\left(u_1''(\theta) + u_1(\theta)\right)\\
&\quad - \frac{\gamma_{11}}{2\pi}\int_0^{2\pi} u_1(\eta)\log|r_1 e^{\mathrm{i}\theta} - r_1 e^{\mathrm{i}\eta}|d\eta - \frac{\gamma_{11}}{2}u_1(\theta)\\
&\quad + \gamma_{11}\int_0^{2\pi} u_1(\eta) R(r_1 e^{\mathrm{i}\theta} - r_1 e^{\mathrm{i}\eta})d\eta + \gamma_{11}\frac{u_1(\theta)}{r_1}\int_{B(0,r_1)} \nabla R(r_1 e^{\mathrm{i}\theta} - z)\cdot e^{\mathrm{i}\theta} dA(z)\\
&\quad + \gamma_{12}\int_0^{2\pi} u_2(\eta) G(-\zeta + r_1 e^{\mathrm{i}\theta} - r_2 e^{\mathrm{i}\eta})d\eta + \gamma_{12}\frac{u_1(\theta)}{r_1}\int_{B(0,r_2)} \nabla G(-\zeta + r_1 e^{\mathrm{i}\theta} - z)\cdot e^{\mathrm{i}\theta}\, dA(z)
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
&\mathcal{S}_2'(0,0)(u)(\theta)\\
&\simeq -\frac{1}{r_2^3}\left(u_2''(\theta) + u_2(\theta)\right)\\
&\quad - \frac{\gamma_{22}}{2\pi}\int_0^{2\pi} u_2(\eta)\log|r_2 e^{\mathrm{i}\theta} - r_2 e^{\mathrm{i}\eta}|d\eta - \frac{\gamma_{22}}{2}u_2(\theta)\\
&\quad + \gamma_{22}\int_0^{2\pi} u_2(\eta) R(r_2 e^{\mathrm{i}\theta} - r_2 e^{\mathrm{i}\eta})d\eta + \gamma_{22}\frac{u_2(\theta)}{r_2}\int_{B(0,r_2)} \nabla R(r_2 e^{\mathrm{i}\theta} - z)\cdot e^{\mathrm{i}\theta} dA(z)\\
&\quad + \gamma_{12}\int_0^{2\pi} u_1(\eta) G(\zeta + r_2 e^{\mathrm{i}\theta} - r_1 e^{\mathrm{i}\eta})d\eta + \gamma_{12}\frac{u_2(\theta)}{r_2}\int_{B(0,r_1)} \nabla G(\zeta + r_2 e^{\mathrm{i}\theta} - z)\cdot e^{\mathrm{i}\theta}\, dA(z).
\end{aligned} \tag{4.3}$$

Write $\mathcal{S}'(0,0)$ as a sum of two parts: $\mathcal{E}$ and $\mathcal{F}$ where

$$\mathcal{E}_j u(\theta) \simeq -\frac{1}{r_j^3}\left(u_j''(\theta) + u_j(\theta)\right) - \frac{\gamma_{jj}}{2\pi}\int_0^{2\pi} u_j(\eta)\log|r_j e^{\mathrm{i}\theta} - r_j e^{\mathrm{i}\eta}|\, d\eta - \frac{\gamma_{jj}}{2}u_j\,(\theta)\,,\ j = 1, 2, \tag{4.4}$$

is the major part, and

$$
\begin{aligned}
\mathcal{F}_1 u(\theta) &\simeq \gamma_{11}\int_0^{2\pi} u_1(\eta) R(r_1 e^{\mathrm{i}\theta} - r_1 e^{\mathrm{i}\eta})d\eta + \gamma_{11}\frac{u_1(\theta)}{r_1}\int_{B(0,r_1)} \nabla R(r_1 e^{\mathrm{i}\theta} - z)\cdot e^{\mathrm{i}\theta} dA(z) \\
&\quad + \gamma_{12}\int_0^{2\pi} u_2(\eta) G(-\zeta + r_1 e^{\mathrm{i}\theta} - r_2 e^{\mathrm{i}\eta})d\eta + \gamma_{12}\frac{u_1(\theta)}{r_1}\int_{B(0,r_2)} \nabla G(-\zeta + r_1 e^{\mathrm{i}\theta} - z)\cdot e^{\mathrm{i}\theta}\, dA(z) \quad (4.5)\\
\mathcal{F}_2 u(\theta) &\simeq \gamma_{22}\int_0^{2\pi} u_2(\eta) R(r_2 e^{\mathrm{i}\theta} - r_2 e^{\mathrm{i}\eta})d\eta + \gamma_{22}\frac{u_2(\theta)}{r_2}\int_{B(0,r_2)} \nabla R(r_2 e^{\mathrm{i}\theta} - z)\cdot e^{\mathrm{i}\theta} dA(z) \\
&\quad + \gamma_{12}\int_0^{2\pi} u_1(\eta) G(\zeta + r_2 e^{\mathrm{i}\theta} - r_1 e^{\mathrm{i}\eta})d\eta + \gamma_{12}\frac{u_2(\theta)}{r_2}\int_{B(0,r_1)} \nabla G(\zeta + r_2 e^{\mathrm{i}\theta} - z)\cdot e^{\mathrm{i}\theta}\, dA(z) \quad (4.6)
\end{aligned}
$$

is the minor part.

Note that $\mathcal{E}_1$ and $\mathcal{E}_2$ are decoupled in $\mathcal{E}u$: $\mathcal{E}_1$ acts on u_1 only and $\mathcal{E}_2$ acts on u_2 only. To determine the spectrum of $\mathcal{E}$, decompose

$$
\mathcal{Z} = \bigoplus_{n=1}^{\infty} \mathcal{Z}(n), \quad \text{where } \mathcal{Z}(n) = \Big\{A\cos n\theta + B\sin n\theta : A, B \in \mathbb{R}^2\Big\},\ n = 1, 2, ... \tag{4.7}
$$

Since

$$
\log|1 - e^{\mathrm{i}\theta}| = -\sum_{k=1}^{\infty} \frac{\cos k\theta}{k},
$$

we deduce

$$
\mathcal{E}_j(e^{\mathrm{i}n\theta}) \quad = \quad \frac{n^2-1}{r_j^3} e^{\mathrm{i}n\theta} + \frac{\gamma_{jj}}{2n} e^{\mathrm{i}n\theta} - \frac{\gamma_{jj}}{2} e^{\mathrm{i}n\theta},\ n = 1, 2, ...,\ j = 1, 2. \tag{4.8}
$$

This means that, the eigenvalues of $\mathcal{E}$ are

$$
\mu_{n,j} = \frac{n^2-1}{r_j^3} + \frac{\gamma_{jj}}{2n} - \frac{\gamma_{jj}}{2},\ \text{for } n = 1, 2, 3, ...,\ \text{and } j = 1, 2. \tag{4.9}
$$

The corresponding eigenvectors are

$$
\begin{bmatrix} \cos n\theta \\ 0 \end{bmatrix}, \begin{bmatrix} \sin n\theta \\ 0 \end{bmatrix},\ \text{if } j = 1;\quad \begin{bmatrix} 0 \\ \cos n\theta \end{bmatrix}, \begin{bmatrix} 0 \\ \sin n\theta \end{bmatrix},\ \text{if } j = 2. \tag{4.10}
$$

These four vectors generate the invariant subspace $\mathcal{Z}(n)$.

Let Π be the orthogonal projection operator from $\mathcal{Z}$ to a subspace $\mathcal{Z}_\flat$, where

$$
\mathcal{Z}_\flat = \left\{\phi = (\phi_1, \phi_2) \in \mathcal{Z} : \int_0^{2\pi} \phi_j \cos\theta\, d\theta = \int_0^{2\pi} \phi_j \sin\theta\, d\theta = 0,\ j = 1, 2,\ \right\}. \tag{4.11}
$$

Then $\mathcal{Z}_\flat$ is the orthogonal compliment of $\mathcal{Z}(1)$ and

$$
\mathcal{Z}_\flat = \bigoplus_{n=2}^{\infty} \mathcal{Z}(n). \tag{4.12}
$$

Also define

$$
\mathcal{Y}_\flat = \mathcal{Y} \cap \mathcal{Z}_\flat, \quad \mathcal{X}_\flat = \mathcal{X} \cap \mathcal{Z}_\flat. \tag{4.13}
$$

When $\phi \in \mathcal{Z}_\flat$, the perturbed disc Ω_j described by ϕ_j is considered to be centered at ξ_j.

We are more interested in $\Pi\mathcal{S}'(0,0)$ and $\Pi\mathcal{E}$ restricted to $\mathcal{X}_\flat$ instead of $\mathcal{S}'(0,0)$ and $\mathcal{E}$ on $\mathcal{X}$. Since $\mathcal{E}$ maps $\mathcal{X}_\flat$ into $\mathcal{Z}_\flat$, $\Pi\mathcal{E} = \mathcal{E}$ on $\mathcal{X}_\flat$.

Lemma 4.1. *There exists $c_2 > 0$ such that*

1. $$\langle \Pi\mathcal{E}u, u\rangle \;\geq\; 2c_2\rho^{-3}||u||_{\mathcal{Y}}^2$$

2. $$||\Pi\mathcal{E}u||_{\mathcal{Z}} \;\geq\; 2c_2\rho^{-3}||u||_{\mathcal{X}}$$

for all $u \in \mathcal{X}_\flat$.

Proof. As $\Pi\mathcal{E}$ agrees with $\mathcal{E}$ on $\mathcal{X}_\flat$, by (4.9) the eigenvalues of $\Pi\mathcal{E}$ in $\mathcal{X}_\flat$ are

$$\mu_{n,j} = \frac{(n-1)}{2nr_j^3}\left(2n(n+1) - \gamma_{jj}r_j^3\right),\;\; n = 2, 3, ...,\;\; j = 1, 2.$$

According to condition 4 of Theorems 1.1 and 1.2, $\gamma_{jj}r_j^3 < 12 - \eta$. Hence

$$\mu_{n,j} > \frac{(n-1)}{2nr_j^3}\left(2n(n+1) - 12 + \eta\right).$$

Then there exists $c > 0$ such that

$$\mu_{n,j} > c\rho^{-3}n^2,\;\; n = 2, 3, ...,\;\; j = 1, 2. \tag{4.14}$$

Given $u \in \mathcal{X}_\flat \subset \mathcal{Z}_\flat$, one can expand it with respect to the eigenvectors (4.10) so that

$$u = \sum_{n=2}^{\infty}\left(A_{n,1}\begin{bmatrix}\cos n\theta\\ 0\end{bmatrix} + B_{n,1}\begin{bmatrix}\sin n\theta\\ 0\end{bmatrix} + A_{n,2}\begin{bmatrix}0\\ \cos n\theta\end{bmatrix} + B_{n,2}\begin{bmatrix}0\\ \sin n\theta\end{bmatrix}\right), \tag{4.15}$$

$$\Pi\mathcal{E}u = \sum_{n=2}^{\infty}\left(A_{n,1}\mu_{n,1}\begin{bmatrix}\cos n\theta\\ 0\end{bmatrix} + B_{n,1}\mu_{n,1}\begin{bmatrix}\sin n\theta\\ 0\end{bmatrix} + A_{n,2}\mu_{n,2}\begin{bmatrix}0\\ \cos n\theta\end{bmatrix} + B_{n,2}\mu_{n,2}\begin{bmatrix}0\\ \sin n\theta\end{bmatrix}\right). \tag{4.16}$$

They imply

$$\langle \Pi\mathcal{E}u, u\rangle = \pi\sum_{n=2}^{\infty}\sum_{j=1}^{2}\left(A_{n,j}^2 + B_{n,j}^2\right)\mu_{n,j}, \tag{4.17}$$

$$\|u\|_{\mathcal{Y}}^2 = \pi\sum_{n=2}^{\infty}\sum_{j=1}^{2}\left(A_{n,j}^2 + B_{n,j}^2\right)(1 + n^2), \tag{4.18}$$

$$\|\Pi\mathcal{E}u\|_{\mathcal{Z}}^2 = \pi\sum_{n=2}^{\infty}\sum_{j=1}^{2}\left(A_{n,j}^2 + B_{n,j}^2\right)\mu_{n,j}^2, \tag{4.19}$$

$$\|u\|_{\mathcal{X}}^2 = \pi\sum_{n=2}^{\infty}\sum_{j=1}^{2}\left(A_{n,j}^2 + B_{n,j}^2\right)(1 + n^2 + n^4). \tag{4.20}$$

Both parts of the lemma follow after one sets $c_2 = c/4$. □

Lemma 4.2. *There exists $C_2 > 0$ such that*

$$\|\mathcal{F}u\|_{\mathcal{Z}} \leq C_2|\gamma|\rho\,\|u\|_{\mathcal{Z}}$$

for all $u \in \mathcal{X}_\flat$.

Proof. Note, for example,

$$\int_0^{2\pi} u_1(\eta) R\left(r_1 e^{\mathrm{i}\theta} - r_1 e^{\mathrm{i}\eta}\right) d\eta = \int_0^{2\pi} u_1(\eta)\left(R(0) + \nabla R(0)\cdot\left(r_1 e^{\mathrm{i}\theta} - r_1 e^{\mathrm{i}\eta}\right)\right) d\eta + \int_0^{2\pi} u_1(\eta) O(\rho^2)\, d\eta$$
$$= \int_0^{2\pi} u_1(\eta) O(\rho^2)\, d\eta$$
$$\frac{u_1(\theta)}{r_1}\int_{B(0,r_1)} \nabla R(r_1 e^{\mathrm{i}\theta} - y)\cdot e^{\mathrm{i}\theta} dy = \frac{u_1(\theta)}{r_1} O(\rho^2) = u_1(\theta)\, O(\rho),$$

by the facts that, since $u \in \mathcal{Z}_\flat$,

$$\int_0^{2\pi} u_j(\eta)\, d\eta = \int_0^{2\pi} u_j(\eta)\cos\eta\, d\eta = \int_0^{2\pi} u_j(\eta)\sin\eta\, d\eta = 0,\ \ j = 1, 2,$$

and the area of $B(0, r_1)$ is of order ρ^2. Similarly estimates hold for the other terms in $\mathcal{F}_1(u)(\theta)$ and $\mathcal{F}_2(u)(\theta)$, and the lemma follows. $\square$

Lemma 4.3. *There exists $c_2 > 0$ such that when ρ is sufficiently small,*

1. $$\langle \Pi\mathcal{S}'(0,0)(u), u\rangle \geq c_2\rho^{-3}\|u\|_{\mathcal{Y}}^2,\ \textit{for all } u \in \mathcal{X}_\flat,$$

2. $$\|\Pi\mathcal{S}'(0,0)(u)\|_{\mathcal{Z}} \geq c_2\rho^{-3}\|u\|_{\mathcal{X}},\ \textit{for all } u \in \mathcal{X}_\flat,$$

3. *the operator $\Pi\mathcal{S}'(0,0)$ is one-to-one and onto from $\mathcal{X}_\flat$ to $\mathcal{Z}_\flat$.*

Proof. In this proof and later let $C_0 > 0$ such that

$$\rho^3|\gamma| \leq C_0, \tag{4.21}$$

by condition 4 of Theorems 1.1 and 1.2.

By Lemma 4.2, for all $u \in \mathcal{X}_\flat$ and sufficiently small ρ,

$$\|\mathcal{F}u\|_{\mathcal{Z}} \leq C_2|\gamma|\rho\,\|u\|_{\mathcal{Z}} \leq C_2 C_0\rho^{-2}\,\|u\|_{\mathcal{Z}} \leq c_2\rho^{-3}\|u\|_{\mathcal{Z}}, \tag{4.22}$$

and by Lemma 4.1.1,

$$\begin{aligned} \langle \Pi\mathcal{S}'(0,0)(u), u\rangle &= \langle \Pi\mathcal{E}u, u\rangle + \langle \Pi\mathcal{F}u, u\rangle \\ &\geq 2c_2\rho^{-3}\|u\|_{\mathcal{Y}}^2 - c_2\rho^{-3}\|u\|_{\mathcal{Z}}^2 \\ &\geq c_2\rho^{-3}\|u\|_{\mathcal{Y}}^2, \end{aligned}$$

which proves part 1 of the lemma. For part 2, by Lemma 4.1.2 and (4.22),

$$\begin{aligned} \|\Pi\mathcal{S}'(0,0)(u)\|_{\mathcal{Z}} &\geq \|\Pi\mathcal{E}u\|_{\mathcal{Z}} - \|\Pi\mathcal{F}u\|_{\mathcal{Z}} \\ &\geq 2c_2\rho^{-3}\|u\|_{\mathcal{X}} - c_2\rho^{-3}\|u\|_{\mathcal{Z}} \\ &\geq c_2\rho^{-3}\|u\|_{\mathcal{X}}, \end{aligned}$$

for all $u \in \mathcal{X}_\flat$.

For part 3, a weaker version of part 2,

$$\|\Pi\mathcal{S}'(0,0)(u)\|_{\mathcal{Z}} \geq c_2\rho^{-3}\|u\|_{\mathcal{Z}},\ \text{for all } u \in \mathcal{X}_\flat, \tag{4.23}$$

implies that $\Pi\mathcal{S}'(0,0)$ is one-to-one.

Let $v \in \mathcal{Z}_\flat$ be perpendicular to the range of $\Pi\mathcal{S}'(0,0)$, i.e. $\langle \Pi\mathcal{S}'(0,0)(u), v\rangle = 0$ for all $u \in \mathcal{X}_\flat$. Since $\Pi\mathcal{S}'(0,0)$ is a self-adjoint operator on $\mathcal{Z}_\flat$ with the domain $\mathcal{X}_\flat \subset \mathcal{Z}_\flat$, one deduces that $v \in \mathcal{X}_\flat$ and $\Pi\mathcal{S}'(0,0)(v) = 0$. By the injectivity of $\Pi\mathcal{S}'(0,0)$, $v = 0$. Hence the range of $\Pi\mathcal{S}'(0,0)$ is dense in $\mathcal{Z}_\flat$.

To show that $\Pi\mathcal{S}'(0,0)$ is surjective, let $w \in \mathcal{Z}_\flat$. There exist $u_n \in \mathcal{X}_\flat$ such that $\Pi\mathcal{S}'(0,0)(u_n) \to w$ in $\mathcal{Z}_\flat$. Therefore $\Pi\mathcal{S}'(0,0)(u_n)$ is a Cauchy sequence in $\mathcal{Z}_\flat$. By (4.23), u_n is also a Cauchy seqence in $\mathcal{Z}_\flat$. There exists $u \in \mathcal{Z}_b$ such that $u_n \to u$ in $\mathcal{Z}_b$. As a self-adjoint operator, $\Pi\mathcal{S}'(0,0)$ has a closed graph in $\mathcal{Z}_\flat \times \mathcal{Z}_\flat$, so (u, w) is on this graph. This means that $u \in \mathcal{X}_b$ and $\Pi\mathcal{S}'(0,0)(u) = w$. ☐

5 Reduction

It is not possible to solve $\mathcal{S}(\phi) = 0$, (3.31), for every $\zeta \in \overline{(\mathbb{C}/\Lambda)_\delta}$. Instead we solve the equation

$$\Pi\mathcal{S}(\phi) = 0 \tag{5.1}$$

for each ζ in this section. A solution of (5.1) is called a pseudo-solution of (3.31).

Lemma 5.1. *There exists $C_1 > 0$ such that*

$$\|\Pi\mathcal{S}(0,0)\|_{\mathcal{Z}} \leq C_1|\gamma|\rho^4. \tag{5.2}$$

Proof. This follows from Lemma 3.2 and the definition of the projection operator Π. ☐

Also needed is an estimate on the second Fréchet derivative of $\mathcal{S}$, i.e. the third variation of $\mathcal{J}$.

Lemma 5.2. *There exists $C_3 > 0$ such that for all $\phi \in Dom(\mathcal{S})$, the following estimates hold for all $u \in \mathcal{X}$ and $v \in \mathcal{X}$,*

1. $$|\langle \mathcal{S}''(\phi)(u,v), v\rangle| \leq C_3\left(\rho^{-5} + |\gamma|\rho^{-2}\right)\|u\|_{\mathcal{X}}\|v\|^2_{\mathcal{Y}},$$

2. $$||\mathcal{S}''(\phi)(u,v)||_{\mathcal{Z}} \leq C_3\left(\rho^{-5} + |\gamma|\rho^{-2}\right)\|u\|_{\mathcal{X}}\|v\|_{\mathcal{X}}.$$

We refer to [23, Lemma 3.2] and [22, Lemma 6.1] for proofs of these results.

Lemma 5.3. *When ρ is sufficiently small, there exists $\varphi \in \mathcal{X}_\flat$ for every $\zeta \in \overline{(\mathbb{C}/\Lambda)_\delta}$ such that φ solves* (5.1) *and*

$$\|\varphi\|_{\mathcal{X}} \leq \frac{2C_1}{c_2}|\gamma|\rho^7.$$

Proof. Expand $\mathcal{S}(\phi)$ as

$$\mathcal{S}(\phi) = \mathcal{S}(0,0) + \mathcal{S}'(0,0)(\phi) + \mathcal{R}(\phi) \tag{5.3}$$

where $\mathcal{R}(\phi)$, defined by (5.3), is a higher order term. Turn (5.1) to a fixed point form:

$$\phi = \mathcal{T}(\phi) \tag{5.4}$$

where

$$\mathcal{T}(\phi) = -(\Pi\mathcal{S}'(0,0))^{-1}(\Pi\mathcal{S}(0,0) + \Pi\mathcal{R}(\phi)). \tag{5.5}$$

is an operator defined on $\mathcal{W} = \{\phi \in \mathcal{X}_\flat : \|\phi\|_{\mathcal{X}} \leq \epsilon\rho^2\} \subset \mathrm{Dom}(\mathcal{S})$. Here $\epsilon > 0$ is to be determined.

By Lemmas 5.1 and 4.3.2,

$$\|(\Pi\mathcal{S}'(0,0))^{-1}\Pi\mathcal{S}(0,0)\|_{\mathcal{X}} \leq \frac{C_1}{c_2}|\gamma|\rho^7. \tag{5.6}$$

Lemma 5.2.2 implies that

$$\|\mathcal{R}(\phi)\|_{\mathcal{Z}} \le C_3\left(\rho^{-5}+|\gamma|\rho^{-2}\right)\|\phi\|_{\mathcal{X}}^2. \tag{5.7}$$

By Lemma 4.3.2,

$$\|(\Pi\mathcal{S}'(0,0))^{-1}\Pi\mathcal{R}(\phi)\|_{\mathcal{X}} \quad \le \quad \frac{C_3}{c_2}\left(\rho^{-2}+|\gamma|\rho\right)\|\phi\|_{\mathcal{X}}^2. \tag{5.8}$$

For $\phi\in\mathcal{W}$, by (4.21), (5.5), (5.6), and (5.8) one deduces

$$\|\mathcal{T}(\phi)\|_{\mathcal{X}} \le \frac{C_1}{c_2}|\gamma|\rho^7+\frac{C_3}{c_2}\left(\rho^2+|\gamma|\rho^5\right)\epsilon^2 \le \Big(\frac{C_1}{c_2}C_0\rho^2+\frac{C_3}{c_2}\epsilon^2+\frac{C_3}{c_2}C_0\epsilon^2\Big)\rho^2.$$

Take

$$\epsilon=\min\left\{\frac{c_2}{4C_3(1+C_0)},\frac{d}{2}\right\} \tag{5.9}$$

where d comes from (3.19), the domain of $\mathcal{S}$. Let ρ be small enough such that $\frac{C_1}{c_2}C_0\rho^2<\frac{\epsilon}{2}$ and Lemma 4.3 holds. Then

$$\|\mathcal{T}(\phi)\|_{\mathcal{X}}\le\epsilon\rho^2.$$

Therefore $\mathcal{T}$ maps $\mathcal{W}$ into itself.

Next show that $\mathcal{T}$ is a contraction. Let $\phi,\psi\in\mathcal{W}$. First note that

$$\mathcal{T}(\phi)-\mathcal{T}(\psi)=-(\Pi\mathcal{S}'(0,0))^{-1}(\Pi)\left(\mathcal{R}(\phi)-\mathcal{R}(\psi)\right). \tag{5.10}$$

Because

$$\mathcal{R}(\phi)-\mathcal{R}(\psi)=\mathcal{S}(\phi)-\mathcal{S}(\psi)-\mathcal{S}'(0,0)(\phi-\psi), \tag{5.11}$$

one finds, with the help of Lemma 5.2.2, that

$$\begin{aligned}\|\mathcal{R}(\phi)-\mathcal{R}(\psi)\|_{\mathcal{Z}} &\le \|\mathcal{S}'(\psi)(\phi-\psi)-\mathcal{S}'(0,0)(\phi-\psi)\|_{\mathcal{Z}}+\frac{C_3}{2}\left(\rho^{-5}+|\gamma|\rho^{-2}\right)\|\phi-\psi\|_{\mathcal{X}}^2\\ &\le C_3\left(\rho^{-5}+|\gamma|\rho^{-2}\right)\|\psi\|_{\mathcal{X}}\|\phi-\psi\|_{\mathcal{X}}+\frac{C_3}{2}\left(\rho^{-5}+|\gamma|\rho^{-2}\right)\|\phi-\psi\|_{\mathcal{X}}^2\\ &\le C_3\left(\rho^{-5}+|\gamma|\rho^{-2}\right)(\epsilon+\epsilon)\,\rho^2\|\phi-\psi\|_{\mathcal{X}}\\ &\le 2\epsilon\, C_3\left(\rho^{-3}+|\gamma|\right)\|\phi-\psi\|_{\mathcal{X}}.\end{aligned}$$

Then Lemma 4.3.2, (4.21) and (5.9) imply that

$$\|\mathcal{T}(\phi)-\mathcal{T}(\psi)\|_{\mathcal{X}}\le\frac{2\epsilon\, C_3}{c_2}\left(1+C_0\right)\|\phi-\psi\|_{\mathcal{X}}\le\frac{1}{2}\|\phi-\psi\|_{\mathcal{X}}. \tag{5.12}$$

Hence $\mathcal{T}$ is a contraction mapping, and a unique fixed point φ exists in $\mathcal{W}$.

By the definition of $\mathcal{W}$, $\|\varphi\|_{\mathcal{X}}=O(\rho^2)$. However, this can be improved, if one revisits the equation $\varphi=\mathcal{T}(\varphi)$ and derives from (5.5), (5.6) and (5.8) that

$$\|\varphi\|_{\mathcal{X}}\le\|(\Pi\mathcal{S}'(0,0))^{-1}\Pi\mathcal{S}(0,0)\|_{\mathcal{X}}+\|(\Pi\mathcal{S}'(0,0))^{-1}\Pi\mathcal{R}(\varphi)\|_{\mathcal{X}}\le\frac{C_1}{c_2}|\gamma|\rho^7+\frac{C_3}{c_2}\left(\rho^{-2}+|\gamma|\rho\right)\|\varphi\|_{\mathcal{X}}^2.$$

Rewrite the above as

$$\left(1-\frac{C_3}{c_2}\left(\rho^{-2}+|\gamma|\rho\right)\|\varphi\|_{\mathcal{X}}\right)\|\varphi\|_{\mathcal{X}}\le\frac{C_1}{c_2}|\gamma|\rho^7. \tag{5.13}$$

In (5.13) estimate

$$\frac{C_3}{c_2}\left(\rho^{-2}+|\gamma|\rho\right)\|\varphi\|_{\mathcal{X}}\le\frac{C_3}{c_2}\left(1+|\gamma|\rho^3\right)\epsilon\le\frac{C_3}{c_2}\left(1+C_0\right)\epsilon\le\frac{1}{4} \tag{5.14}$$

by (4.21) and (5.9). The estimate of φ follows from (5.13) and (5.14). $\square$

The next two lemmas show some properties of the pseudo-solution φ. Lemma 5.4 says that $\Pi\mathcal{S}'(\varphi)$ is positive definite, so φ locally minimizes $\mathcal{J}$ in $\mathcal{X}_\flat$. Lemma 5.5 gives a good estimate of $\mathcal{J}(\varphi)$ which is very close to $\mathcal{J}(0,0)$, the energy of the disc-disc configuration of the same relative displacement.

Lemma 5.4. *When ρ is sufficiently small, for all $u \in \mathcal{X}_\flat$,*

1. $$\langle \Pi\mathcal{S}'(\varphi)(u), u\rangle \geq \frac{c_2}{2}\rho^{-3}\|u\|_{\mathcal{Y}}^2$$

2. $$\|\Pi\mathcal{S}'(\varphi)(u)\|_{\mathcal{Z}} \geq \frac{c_2}{2}\rho^{-3}\|u\|_{\mathcal{X}}.$$

Proof. By Lemma 5.2,

$$\begin{aligned}
\langle \Pi\mathcal{S}'(\varphi)(u), u\rangle &= \langle \Pi\mathcal{S}'(0,0)(u), u\rangle + \langle \Pi(\mathcal{S}'(\varphi) - \mathcal{S}'(0,0))u, u\rangle \\
&\geq c_2\rho^{-3}\|u\|_{\mathcal{Y}}^2 - C_3\left(\rho^{-5} + |\gamma|\rho^{-2}\right)\|\varphi\|_{\mathcal{X}}\|u\|_{\mathcal{Y}}^2 \\
&\geq \left(c_2 - \frac{2C_1C_3C_0}{c_2}(1+C_0)\rho^2\right)\rho^{-3}\|u\|_{\mathcal{Y}}^2,
\end{aligned}$$

and

$$\begin{aligned}
\|\Pi\mathcal{S}'(\varphi)(u)\|_{\mathcal{Z}} &\geq \|\Pi\mathcal{S}'(0,0)(u)\|_{\mathcal{Z}} - \|\Pi(\mathcal{S}'(\varphi) - \mathcal{S}'(0,0))(u)\|_{\mathcal{Z}} \\
&\geq c_2\rho^{-3}\|u\|_{\mathcal{X}} - C_3\left(\rho^{-5} + |\gamma|\rho^{-2}\right)\|\varphi\|_{\mathcal{X}}\|u\|_{\mathcal{X}} \\
&\geq \left(c_2 - \frac{2C_1C_3C_0}{c_2}(1+C_0)\rho^2\right)\rho^{-3}\|u\|_{\mathcal{X}}.
\end{aligned}$$

If ρ is sufficiently small, then $\frac{2C_1C_3C_0}{c_2}(1+C_0)\rho^2 \leq \frac{c_2}{2}$ and both parts of the lemma follow. $\square$

Lemma 5.5. *It holds uniformly for $\zeta \in \overline{(\mathbb{C}/\Lambda)_\delta}$ that*

$$\mathcal{J}(\varphi) = \sum_{j=1}^{2}\left(2\pi r_j + \frac{\gamma_{jj}}{2}\left(-\frac{\pi r_j^4}{2}\log\frac{2\pi r_j}{\sqrt{|\Lambda|}} + \frac{\pi r_j^4}{8} + \pi^2 r_j^4 H(0)\right)\right) + \gamma_{12}\pi^2 r_1^2 r_2^2 G(\zeta) + O(|\gamma|\rho^6).$$

Proof. Expanding $\mathcal{J}(\varphi)$ yields

$$\mathcal{J}(\varphi) = \mathcal{J}(0,0) + \langle \mathcal{S}(0,0), \varphi\rangle + \frac{1}{2}\langle \mathcal{S}'(0,0)(\varphi), \varphi\rangle + \frac{1}{6}\langle \mathcal{S}''(t\varphi)(\varphi,\varphi), \varphi\rangle \tag{5.15}$$

for some $t \in (0,1)$. On the other hand expanding $\mathcal{S}(\varphi)$, and then applying Π on both sides give

$$\|\Pi\mathcal{S}(\varphi) - \Pi\mathcal{S}(0,0) - \Pi\mathcal{S}'(0,0)(\varphi)\|_{\mathcal{Z}} \leq \sup_{t\in(0,1)}\frac{1}{2}\|\Pi\mathcal{S}''(t\varphi)(\varphi,\varphi)\|_{\mathcal{Z}}. \tag{5.16}$$

Since $\Pi\mathcal{S}(\varphi) = 0$, (5.16) shows that

$$\|\Pi\mathcal{S}(0,0) + \Pi\mathcal{S}'(0,0)(\varphi)\|_{\mathcal{Z}} \leq \sup_{t\in(0,1)}\frac{1}{2}\|\Pi\mathcal{S}''(t\varphi)(\varphi,\varphi)\|_{\mathcal{Z}},$$

which implies that

$$\|\langle \Pi\mathcal{S}(0,0), \varphi\rangle + \langle \Pi\mathcal{S}'(0,0)(\varphi), \varphi\rangle\|_{\mathcal{Z}} \leq \left(\sup_{t\in(0,1)}\frac{1}{2}\|\Pi\mathcal{S}''(t\varphi)(\varphi,\varphi)\|_{\mathcal{Z}}\right)\|\varphi\|_{\mathcal{X}}. \tag{5.17}$$

Since $\varphi \in \mathcal{X}_\flat$,

$$\langle \Pi \mathcal{S}(0,0), \varphi \rangle = \langle \mathcal{S}(0,0), \varphi \rangle, \quad \langle \Pi \mathcal{S}'(0,0)(\varphi), \varphi \rangle = \langle \mathcal{S}'(0,0)(\varphi), \varphi \rangle. \tag{5.18}$$

Then (5.17) shows that

$$\| \langle \mathcal{S}(0,0), \varphi \rangle + \langle \mathcal{S}'(0,0)(\varphi), \varphi \rangle \|_{\mathcal{Z}} \leq \left(\sup_{t \in (0,1)} \frac{1}{2} \| \Pi \mathcal{S}''(t\varphi)(\varphi, \varphi) \|_{\mathcal{Z}} \right) \|\varphi\|_{\mathcal{X}}. \tag{5.19}$$

By (5.19), (5.15) yields that

$$|\mathcal{J}(\varphi) - \mathcal{J}(0,0) - \frac{1}{2} \langle \mathcal{S}(0,0), \varphi \rangle| \leq \frac{5}{12} \left(\sup_{t \in (0,1)} \| \mathcal{S}''(t\varphi)(\varphi, \varphi) \|_{\mathcal{Z}} \right) \|\varphi\|_{\mathcal{X}}.$$

Therefore Lemma 5.1, (5.18), Lemma 5.2.2 and Lemma 5.3 imply that

$$\begin{aligned} |\mathcal{J}(\varphi) - \mathcal{J}(0,0)| &\leq \frac{1}{2} |\langle \mathcal{S}(0,0), \varphi \rangle| + \frac{5}{12} \left(\sup_{t \in (0,1)} \| \mathcal{S}''(t\varphi)(\varphi, \varphi) \|_{\mathcal{Z}} \right) \|\varphi\|_{\mathcal{X}} \\ &\leq \frac{1}{2} (C_1 |\gamma| \rho^4) \frac{2C_1}{c_2} |\gamma| \rho^7 + \frac{5}{12} C_3 \left(\rho^{-5} + |\gamma| \rho^{-2} \right) \left(\frac{2C_1}{c_2} |\gamma| \rho^7 \right)^3 \\ &= |\gamma|^2 \rho^{11} \left(\frac{C_1^2}{c_2} + \frac{10 C_3 C_1^3}{3 c_2^3} (1 + |\gamma| \rho^3) |\gamma| \rho^5 \right). \end{aligned}$$

Finally one uses Lemma 2.2 and (4.21) to complete the proof. □

6 Stationary points

We emphasize the dependence of $\mathcal{J}$ and $\mathcal{S}$ on ζ and write them as $\mathcal{J}(\cdot, \zeta)$ and $\mathcal{S}(\cdot, \zeta)$ respectively. It is proved in Lemma 5.3 that for every $\zeta \in \overline{(\mathbb{C}/\Lambda)_\delta}$, there exists $\varphi \in \mathcal{X}_\flat$ such that $\Pi \mathcal{S}(\varphi) = 0$. This pseudo-solution φ also depends on ζ, so we write it as $\varphi(\cdot, \zeta)$. In this section we prove Theorems 1.1 and 1.2 by finding a particular ζ_c such that at $\zeta = \zeta_c$,

$$\mathcal{S}(\varphi(\cdot, \zeta_c), \zeta_c) = 0. \tag{6.1}$$

Then the pair of subsets in $\mathbb{C}/\Lambda$ specified by $\varphi(\cdot, \zeta_c)$ is a stationary point of $\mathcal{J}$. Define a function

$$J(\zeta) = \mathcal{J}(\varphi(\cdot, \zeta), \zeta), \ \zeta \in \overline{(\mathbb{C}/\Lambda)_\delta}. \tag{6.2}$$

Lemma 6.1. *If $\zeta_c \in (\mathbb{C}/\Lambda)_\delta$ is a critical point of the function J, then $\varphi(\cdot, \zeta_c)$ is a solution of* (6.1).

Proof. There is a general first variation formula for (Ω_1, Ω_2) deformed to $(\Omega_{\varepsilon,1}, \Omega_{\varepsilon,2})$:

$$\frac{\partial \mathcal{J}(\Omega_{\varepsilon,1}, \Omega_{\varepsilon,2})}{\partial \varepsilon}\Big|_{\varepsilon=0} = -\sum_{j=1}^{2} \int_{\partial \Omega_j} \left(\kappa_j + \gamma_{j1} I(\Omega_1) + \gamma_{j2} I(\Omega_2) \right) \mathbf{N}_j \cdot \mathbf{X}_j \, ds \tag{6.3}$$

where ds is the length element; see [24, Lemma 2.4] or [25, Lemma 2.4]. Let (Ω_1, Ω_2) be the pair of perturbed discs specified by $\varphi(\cdot, \zeta)$ and the boundary of Ω_j be parametrized by $\mathbf{R}_j$; namely

$$\mathbf{R}_j(\theta) = \xi_j + \sqrt{r_j^2 + 2\varphi_j(\theta, \zeta)} \, e^{i\theta}, \ j = 1, 2. \tag{6.4}$$

The unit tangent and normal vectors of $\mathbf{R}_j$ are

$$\mathbf{T}_j(\theta) = \frac{\frac{\partial \mathbf{R}_j(\theta)}{\partial \theta}}{\left| \frac{\partial \mathbf{R}_j(\theta)}{\partial \theta} \right|}, \ \text{and } \mathbf{N}_j(\theta) = i \mathbf{T}_j(\theta), \tag{6.5}$$

respectively. Since $ds = \left|\frac{\partial \mathbf{R}_j(\theta)}{\partial\theta}\right| d\theta$,

$$\mathbf{T}_j(\theta)\frac{ds}{d\theta} = \frac{\partial \mathbf{R}_j(\theta)}{\partial\theta} = \frac{\frac{\partial\varphi_j}{\partial\theta}}{\sqrt{r_j^2+2\varphi_j}} e^{i\theta} + \sqrt{r_j^2+2\varphi_j}\ \mathrm{i}\, e^{\mathrm{i}\theta} \tag{6.6}$$

$$\mathbf{N}_j(\theta)\frac{ds}{d\theta} = \frac{\frac{\partial\varphi_j}{\partial\theta}}{\sqrt{r_j^2+2\varphi_j}}\ \mathrm{i}\, e^{\mathrm{i}\theta} - \sqrt{r_j^2+2\varphi_j}\ e^{\mathrm{i}\theta} \tag{6.7}$$

In (6.3), κ_j is the curvature of $\mathbf{R}_j$, and $\mathbf{N}_j$ points inwards.

In this proof we generate four deformations by varying $\xi_j = (\xi_j^1, \xi_j^2)$, $j = 1, 2$. They supply $\mathbf{X}_j$ in (6.3). First take ξ_1^1 to be a variable and keep the other ξ_j^k's fixed. This amounts to moving Ω_1 horizontally while changing the shapes of Ω_1 and Ω_2 slightly. Then, with $\zeta = \xi_2 - \xi_1$,

$$\mathbf{X}_1 = \frac{\partial \mathbf{R}_1}{\partial \xi_1^1} = (1,0) + \frac{\frac{\partial\varphi_1}{\partial\zeta^1}\frac{\partial\zeta^1}{\partial\xi_1^1}}{\sqrt{r_1^2+2\varphi_1}} e^{i\theta} = (1,0) - \frac{\frac{\partial\varphi_1}{\partial\zeta^1}}{\sqrt{r_1^2+2\varphi_1}} e^{\mathrm{i}\theta} \tag{6.8}$$

$$\mathbf{N}_1\cdot\mathbf{X}_1\frac{ds}{d\theta} = -\frac{\frac{\partial\varphi_1}{\partial\theta}}{\sqrt{r_1^2+2\varphi_1}}\sin\theta - \sqrt{r_1^2+2\varphi_1}\cos\theta + \frac{\partial\varphi_1}{\partial\zeta^1}, \tag{6.9}$$

$$\mathbf{X}_2 = \frac{\partial \mathbf{R}_2}{\partial \xi_1^1} = \frac{\frac{\partial\varphi_2}{\partial\zeta^1}\frac{\partial\zeta^1}{\partial\xi_1^1}}{\sqrt{r_2^2+2\varphi_2}} e^{\mathrm{i}\theta} = -\frac{\frac{\partial\varphi_2}{\partial\zeta^1}}{\sqrt{r_2^2+2\varphi_2}} e^{\mathrm{i}\theta}, \tag{6.10}$$

$$\mathbf{N}_2\cdot\mathbf{X}_2\frac{ds}{d\theta} = \frac{\partial\varphi_2}{\partial\zeta^1} \tag{6.11}$$

Since $\varphi \in \mathcal{X}_\flat$,

$$\int_0^{2\pi} \varphi_j\, d\theta = \int_0^{2\pi} \varphi_j \cos\theta\, d\theta = \int_0^{2\pi} \varphi_j \sin\theta\, d\theta = 0,\ \ j = 1, 2. \tag{6.12}$$

It follows that

$$\int_0^{2\pi} \frac{\partial\varphi_j}{\partial\zeta^k}\, d\theta = \int_0^{2\pi} \frac{\partial\varphi_j}{\partial\zeta^k}\cos\theta\, d\theta = \int_0^{2\pi} \frac{\partial\varphi_j}{\partial\zeta^k}\sin\theta\, d\theta = 0,\ \ j = 1, 2,\ \ k = 1, 2. \tag{6.13}$$

Because

$$\int_{\partial\Omega_1} \mathbf{N}_1\cdot\mathbf{X}_1\, ds = \int_0^{2\pi} \left[-\frac{d}{d\theta}\left(\sqrt{r_1^2+2\varphi_1}\ \sin\theta\right) + \frac{\partial\varphi_1}{\partial\zeta^1}\right] d\theta = 0, \tag{6.14}$$

$$\int_{\partial\Omega_1} \mathbf{N}_2\cdot\mathbf{X}_2\, ds = \int_0^{2\pi} \frac{\partial\varphi_2}{\partial\zeta^2}\, d\theta = 0, \tag{6.15}$$

by (6.13), one deduces from (3.22), (3.25), (3.26), and (6.3)

$$\frac{\partial \mathcal{J}(\Omega_1,\Omega_2)}{\partial\xi_1^1} = -\sum_{j=1}^{2}\int_{\partial\Omega_j} \mathcal{S}_j(\varphi,\zeta)\,\mathbf{N}_j\cdot\mathbf{X}_j\, ds. \tag{6.16}$$

Since $\Pi\mathcal{S}(\varphi(\cdot,\zeta),\zeta) = 0$, there exist constants $A_j(\zeta)$ and $B_j(\zeta)$ depending on ζ such that

$$\mathcal{S}_j(\varphi(\cdot,\zeta),\zeta) = A_j(\zeta)\cos\theta + B_j(\zeta)\sin\theta,\ \ j = 1, 2. \tag{6.17}$$

When $\xi_2 - \xi_1 = \zeta_c$, the left side of (6.16) vanishes and, with the help of (6.13),

$$
\begin{aligned}
0 &= -\int_0^{2\pi} (A_1(\zeta_c)\cos\theta + B_1(\zeta_c)\sin\theta)\left(-\frac{\frac{\partial\varphi_1}{\partial\theta}}{\sqrt{r_1^2+2\varphi_1}}\sin\theta - \sqrt{r_1^2+2\varphi_1}\cos\theta + \frac{\partial\varphi_1}{\partial\zeta^1}\right)d\theta \\
&\quad - \int_0^{2\pi} (A_2(\zeta_c)\cos\theta + B_2(\zeta_c)\sin\theta)\,\frac{\partial\varphi_2}{\partial\zeta^1}\,d\theta \\
&= \int_0^{2\pi} (A_1(\zeta_c)\cos\theta + B_1(\zeta_c)\sin\theta)\,\frac{d}{d\theta}\left(\sqrt{r_1^2+2\varphi_1}\,\sin\theta\right)d\theta \\
&= A_1(\zeta_c)\int_0^{2\pi}\sqrt{r_1^2+2\varphi_1}\,\sin^2\theta\,d\theta - B_1(\zeta_c)\int_0^{2\pi}\sqrt{r_1^2+2\varphi_1}\,\sin\theta\cos\theta\,d\theta.
\end{aligned}
\tag{6.18}
$$

The two integrals in the last line are estimated by the bound on φ from Lemma 5.3 and one finds

$$A_1(\zeta_c)\left(\pi r_1 + O(|\gamma|\rho^6)\right) - B_1(\zeta_c)\left(0 + O(|\gamma|\rho^6)\right) = 0. \tag{6.19}$$

Next vary ξ_1^2 while keeping the other ξ_j^k's fixed to derive analogously

$$-A_1(\zeta_c)\left(0 + O(|\gamma|\rho^6)\right) + B_1(\zeta_c)\left(\pi r_1 + O(|\gamma|\rho^6)\right) = 0.. \tag{6.20}$$

Equations (6.19) and (6.20) form a homogeneous linear system for $A_1(\zeta_c)$ and $B_1(\zeta_c)$, and since the system is nonsingular when ρ is small,

$$A_1(\zeta_c) = B_1(\zeta_c) = 0. \tag{6.21}$$

Taking variations with respect to ξ_2^1 and ξ_2^2 shows in a similar way that

$$A_2(\zeta_c) = B_2(\zeta_c) = 0. \tag{6.22}$$

By (6.21) and (6.22), (6.17) implies

$$\mathcal{S}_j(\varphi(\cdot,\zeta_c),\zeta_c) = 0,\ j = 1,2, \tag{6.23}$$

and this proves the lemma. □

Proof of Theorem 1.1. Let ζ_ρ be a minimum of J on $\overline{(\mathbb{C}/\Lambda)_\delta}$. It suffices to show that $\zeta_\rho \in (\mathbb{C}/\Lambda)_\delta$, so ζ_ρ is a critical point of J and Lemma 6.1 applies. Let $\zeta_\rho \to \zeta_0$ as $\rho \to 0$, possibly along a sequence. By Lemma 5.5 and conditions 2 and 3 of Theorem 1.1,

$$\frac{1}{\gamma_{12}\pi^2 r_1^2 r_2^2}\left[J(\zeta) - \sum_{j=1}^{2}\left(2\pi r_j + \frac{\gamma_{jj}}{2}\left(-\frac{\pi r_j^4}{2}\log\frac{2\pi r_j}{\sqrt{|\Lambda|}} + \frac{\pi r_j^4}{8} + \pi^2 r_j^4 H(0)\right)\right)\right] \to G(\zeta),\ \text{as } \rho \to 0, \tag{6.24}$$

uniformly with respect to $\zeta \in \overline{(\mathbb{C}/\Lambda)_\delta}$.

If ζ_0 were not a minimum of G in $\overline{(\mathbb{C}/\Lambda)_\delta}$, let ζ_* be a minimum of G in $\overline{(\mathbb{C}/\Lambda)_\delta}$. Then $G(\zeta_*) < G(\zeta_0)$, and, by (6.24), $J(\zeta_*) < J(\zeta_\rho)$ when ρ is small. This contradicts our assumption that ζ_ρ is a minimum of J on $\overline{(\mathbb{C}/\Lambda)_\delta}$. Now that ζ_0 is a minimum of G in $\overline{(\mathbb{C}/\Lambda)_\delta}$, we deduce $\zeta_0 \in (\mathbb{C}/\Lambda)_\delta$ by (2.10). Then $\zeta_\rho \in (\mathbb{C}/\Lambda)_\delta$ when ρ is sufficiently small.

Because the minimum ζ_ρ of J is in the interior of $\overline{(\mathbb{C}/\Lambda)_\delta}$, ζ_ρ is a critical point of J and $\varphi(\cdot,\zeta_\rho)$ is a stationary point of $\mathcal{J}$ by Lemma 6.1. Since ζ_0 is also a global minimum of G in $\mathbb{C}/\Lambda$ by (2.10), any limit of the relative displacement ζ_ρ as $\rho \to 0$ along a sequence is a global minimum of G. □

The stationary point $\varphi(\cdot,\zeta_\rho)$ is in a sense stable. In the first step of our Lyapunov-Schmidt reduction procedure, the pseudo-solution $\varphi(\cdot,\zeta)$ is constructed as a fixed point and its second variation $\Pi\mathcal{S}'(\varphi)$ is positive definite, Lemma 5.4. This means that $\varphi(\cdot,\zeta)$ locally minimizes $\mathcal{J}$ among the pairs whose relative displacement equals ζ described by the members in $\mathcal{X}_\flat$. In the second step of the Lyapunov-Schmidt reduction, $\varphi(\cdot,\zeta_\rho)$ is found as a minimum of $\mathcal{J}$ among $\varphi(\cdot,\zeta)$ with respect to the relative displacement $\zeta \in \overline{(\mathbb{C}/\Lambda)_\delta}$. Therefore as a minimum of minimum, we claim a sense of stability for $\varphi(\cdot,\zeta_\rho)$.

Proof of Theorem 1.2. Let ζ_* be a non-degenerate critical point of Green's function G. Make δ smaller if necessary, so that $\zeta_* \in (\mathbb{C}/\Lambda)_\delta$. One proceeds to show that for sufficiently small ρ, there exists a critical point ζ_ρ of the function J such that as $\rho \to 0$, $\zeta_\rho \to \zeta_*$. The theorem then follows from Lemma 6.1.

Suppose that this assertion is false. Then there exist $\rho_n \to 0$ and $\epsilon > 0$, such that J has no critical point in the closed disc $\overline{B}(\zeta_*, \epsilon) \subset (\mathbb{C}/\Lambda)_\delta$ if $\rho = \rho_n$, and ζ_* is the only critical point of G in $\overline{B}(\zeta_*, \epsilon)$.

Note that $J(\zeta) = \mathcal{J}(\varphi(\cdot, \zeta), \zeta)$ depends on ζ in two ways. First $\varphi(\cdot, \zeta)$ depends on ζ; second the functional $\mathcal{J}$ depends on G which contains ζ in its variable as in (3.17). Then the derivative of J with respect to ζ^1, the first cordinate of ζ, is

$$\begin{aligned} \frac{\partial J}{\partial \zeta^1} &= \frac{D\mathcal{J}}{D\varphi} \, \frac{\partial \varphi}{\partial \zeta^1} + \frac{D\mathcal{J}}{D\zeta^1} \\ &= \langle \mathcal{S}(\varphi), \frac{\partial \varphi}{\partial \zeta^1} \rangle + \frac{D\mathcal{J}}{D\zeta^1} \\ &= \frac{D\mathcal{J}}{D\zeta^1}. \end{aligned} \tag{6.25}$$

Here the expression $\frac{D\mathcal{J}}{D\varphi}$ is just the first variation of $\mathcal{J}$ at φ and $\frac{D\mathcal{J}}{D\varphi}\frac{\partial \varphi}{\partial \zeta^1} = \langle \mathcal{S}(\varphi), \frac{\partial \varphi}{\partial \zeta^1}\rangle$. Since $\Pi \mathcal{S}(\varphi) = 0$, $\mathcal{S}(\varphi) \perp \mathcal{Z}_\flat$. But $\varphi \in \mathcal{X}_\flat$ implies that $\frac{\partial \varphi}{\partial \zeta^1} \in \mathcal{X}_\flat \subset \mathcal{Z}_\flat$. Hence $\langle \mathcal{S}(\varphi), \frac{\partial \varphi}{\partial \zeta^1}\rangle = 0$. For $\frac{D\mathcal{J}}{D\zeta^1}$, by (3.17) and G being even,

$$\begin{aligned} \frac{\partial J}{\partial \zeta^1} &= \frac{D\mathcal{J}}{D\zeta^1} \\ &= \gamma_{12} \int_0^{2\pi} \int_0^{\sqrt{r_1^2 + 2\varphi_1(\theta)}} \int_0^{2\pi} \int_0^{\sqrt{r_2^2 + 2\varphi_2(\eta)}} \frac{\partial}{\partial \zeta^1} G(-\zeta + te^{\mathrm{i}\theta} - qe^{\mathrm{i}\eta}) tq \, dq d\eta dt d\theta \\ &= \gamma_{12} \int_0^{2\pi} \int_0^{\sqrt{r_1^2 + 2\varphi_1(\theta)}} \int_0^{2\pi} \int_0^{\sqrt{r_2^2 + 2\varphi_2(\eta)}} \Big(\frac{\partial G(\zeta)}{\partial z^1} + O(\rho) \Big) tq \, dq d\eta \, dt d\theta \\ &= \gamma_{12} \pi^2 r_1^2 r_2^2 \frac{\partial G(\zeta)}{\partial z^1} + O(\gamma_{12} \rho^5). \end{aligned}$$

Similarly,

$$\frac{\partial J}{\partial \zeta^2} = \gamma_{12} \pi^2 r_1^2 r_2^2 \frac{\partial G(\zeta)}{\partial z^2} + O(\gamma_{12} \rho^5).$$

Hence,

$$\frac{1}{\gamma_{12} \pi^2 r_1^2 r_2^2} \nabla J(\zeta) = \nabla G(\zeta) + O(\rho) \tag{6.26}$$

by conditions 2 and 3 of Theorem 1.2, and

$$\frac{1}{\gamma_{12} \pi^2 r_1^2 r_2^2} \nabla J \to \nabla G \text{ uniformly in } \overline{(\mathbb{C}/\Lambda)_\delta}, \text{ as } n \to \infty. \tag{6.27}$$

Since ζ_* is non-degenerate and the only critical point of G in $\overline{B}(\zeta_*, \epsilon)$, and J has no critical point in $\overline{B}(\zeta_*, \epsilon)$, by the topological degree theory (see [14, Chapter 1]),

$$\deg\big(\nabla G, \overline{B}(\zeta_*, \epsilon), (0,0)\big) = -1, \text{ or } 1, \tag{6.28}$$

$$\deg\left(\frac{1}{\gamma_{12} \pi^2 r_1^2 r_2^2} \nabla J, \overline{B}(\zeta_*, \epsilon), (0,0) \right) = 0. \tag{6.29}$$

However (6.27) implies that

$$\deg\big(\nabla G, \overline{B}(\zeta_*, \epsilon), (0,0)\big) = \deg\left(\frac{1}{\gamma_{12} \pi^2 r_1^2 r_2^2} \nabla J, \overline{B}(\zeta_*, \epsilon), (0,0) \right), \tag{6.30}$$

when ρ_n is sufficiently small. A contradiction. $\square$

7 Shape versus size

In the last two sections we study the impact of the underlying lattice Λ on the disc-disc structure. The subscript Λ is now restored in notations like $\mathcal{J}_\Lambda$, I_Λ, G_Λ, and H_Λ. When we compare lattices, it is more appropriate to consider the energy per area instead of the energy; namely

$$\widetilde{\mathcal{J}}_\Lambda(\Omega_1,\Omega_2) = \frac{1}{|\Lambda|}\mathcal{J}_\Lambda(\Omega_1,\Omega_2). \tag{7.1}$$

Since there are stationary points with relative displacements close to half periods (provided the half periods are non-degenerate as critical points of G_Λ) by Theorem 1.2 and according to Lemma 5.3, any stationary point $\varphi(\cdot,\zeta_c)$ is very well approximated by a disc-disc configuration $\phi = (0,0)$ of the same relative displacement, we shall be content with an analysis of the energy per area of disc-disc configurations, instead of disc-disc stationary points, with their relative displacements equal to half periods.

The role played by the size of a lattice can be separated from the role played by the shape of the lattice. Write a lattice as $t\Lambda$ with $t > 0$ and $|\Lambda| = 1$, then t measures its size and Λ its shape. Also write an admissible pair as $t\Omega = (t\Omega_1, t\Omega_2)$, so its energy per area becomes

$$\begin{aligned}\widetilde{\mathcal{J}}_{t\Lambda}(t\Omega) &= \Big(\frac{1}{t}\Big)^2 \mathcal{J}_{t\Lambda}(t\Omega)\\ &= \frac{1}{t}\Big(\frac{1}{2}\sum_{j=1}^{3}\mathcal{P}_{\mathbb{C}/\Lambda}(\Omega_j)\Big) + t^2\sum_{j,k=1}^{2}\frac{\gamma_{jk}}{2}\int_{\mathbb{C}/\Lambda}\nabla I_\Lambda(\Omega_j)(z)\cdot\nabla I_\Lambda(\Omega_k)(z)\,dz.\end{aligned} \tag{7.2}$$

With respect to t, (7.2) is minimized at

$$t = t_\Omega = \left(\frac{\frac{1}{2}\sum_{j=1}^{3}\mathcal{P}_{\mathbb{C}/\Lambda}(\Omega_j)}{\sum_{j,k=1}^{2}\gamma_{jk}\int_{\mathbb{C}/\Lambda}\nabla I_\Lambda(\Omega_j)(z)\cdot\nabla I_\Lambda(\Omega_k)(z)\,dz}\right)^{1/3} \tag{7.3}$$

and the minimum value is

$$\begin{aligned}\widetilde{\mathcal{J}}_{t_\Omega\Lambda}(t_\Omega\Omega) &= \frac{3}{2}\Big(\frac{1}{2}\sum_{j=1}^{3}\mathcal{P}_{\mathbb{C}/\Lambda}(\Omega_j)\Big)^{2/3}\Big(\sum_{j,k=1}^{2}\gamma_{jk}\int_{\mathbb{C}/\Lambda}\nabla I_\Lambda(\Omega_j)(z)\cdot\nabla I_\Lambda(\Omega_k)(z)\,dz\Big)^{1/3}\\ &= \frac{3}{2}\Big(\mathcal{J}_{s,\Lambda}(\Omega_1,\Omega_2)\Big)^{2/3}\Big(2\mathcal{J}_{l,\Lambda}(\Omega_1,\Omega_2)\Big)^{1/3}.\end{aligned} \tag{7.4}$$

As explained earlier, we set

$$(\Omega_1,\Omega_2) = (B(0,r_1), B(h,r_2)) := \mathbf{B}, \tag{7.5}$$

where h is a half period; namely

$$h \notin \Lambda, \text{ but } 2h \in \Lambda. \tag{7.6}$$

There are three Λ-inequivalent half periods. If (α_1,α_2) is a basis of Λ, then they are

$$\frac{\alpha_1}{2},\ \frac{\alpha_2}{2},\ \frac{\alpha_1+\alpha_2}{2}. \tag{7.7}$$

To ensure that the discs do not overlap, we require that $(\Lambda, h) \in \mathcal{A}_{r_1,r_2}$ and the radii r_1 and r_2 are sufficiently small as in the conditions 1 and 2 before (1.24).

Note that

$$\mathcal{J}_{s,\Lambda}(\mathbf{B}) = 2\pi r_1 + 2\pi r_2, \tag{7.8}$$

is independent of Λ and h, so to minimize

$$\widetilde{\mathcal{J}}_{t_\mathbf{B}\Lambda}(t_\mathbf{B}\mathbf{B}) = \frac{3}{2}\Big(\mathcal{J}_{s,\Lambda}(\mathbf{B})\Big)^{2/3}\Big(2\mathcal{J}_{l,\Lambda}(\mathbf{B})\Big)^{1/3} \tag{7.9}$$

with respect to Λ and h, it suffices to minimize $\mathcal{J}_{l,\Lambda}(\mathbf{B})$ with respect to (Λ, h) with $|\Lambda| = 1$ and h being a half period of Λ. This is also equivalent to minimizing $\mathcal{J}_\Lambda(\mathbf{B})$ with respect to (Λ, h), as in the statement of Theorem 1.3.

Lemma 7.1. *Let (α_1, α_2) be a basis of Λ, $|\Lambda| = 1$, and $\tau = \frac{\alpha_2}{\alpha_1} \in \mathbb{H}$. Then*

$$\mathcal{J}_{l,\Lambda}(B(0,r_1), B(h,r_2)) = \left(\frac{\gamma_{11}\pi^2 r_1^4 + \gamma_{22}\pi^2 r_2^4}{2}\right)\left(g_*(\tau) - \frac{1-b}{4\pi}\log 2\right) + \sum_{j=1}^{2}\frac{\gamma_{jj}}{2}\left(-\frac{\pi r_j^4}{2}\log(2\pi r_j) + \frac{\pi r_j^4}{8} + \frac{\pi^2 r_j^6}{4}\right) + \gamma_{12}\left(\frac{\pi^2(r_1^2 r_2^4 + r_1^4 r_2^2)}{8}\right).$$

Here

1. $g_*(\tau) = b\,g(\tau) + (1-b)\,g(2\tau)$ *if* $h = \dfrac{\alpha_1}{2}$,
2. $g_*(\tau) = b\,g(\tau) + (1-b)\,g\left(\dfrac{\tau}{2}\right)$ *if* $h = \dfrac{\alpha_2}{2}$,
3. $g_*(\tau) = b\,g(\tau) + (1-b)\,g\left(\dfrac{\tau+1}{2}\right)$ *if* $h = \dfrac{\alpha_1+\alpha_2}{2}$,

where b is given in (1.27),

$$g(\tau) = -\frac{1}{4\pi}\log\left|\operatorname{Im}(\tau)\eta^4(\tau)\right|, \tag{7.10}$$

and η is Dedekind's eta function:

$$\eta(\tau) = e^{\frac{\pi i\tau}{12}}\prod_{n=1}^{\infty}\left(1 - e^{2\pi i n\tau}\right). \tag{7.11}$$

Proof. By Lemma 2.2,

$$\begin{aligned}
\mathcal{J}_{l,\Lambda}(\mathbf{B}) &= \sum_{j=1}^{2}\frac{\gamma_{jj}}{2}\left(-\frac{\pi r_j^4}{2}\log(2\pi r_j) + \frac{\pi r_j^4}{8} + \pi^2 r_j^4 H_\Lambda(0) + \frac{\pi^2 r_j^6}{4}\right) \\
&\quad + \gamma_{12}\left(\pi^2 r_1^2 r_2^2 G_\Lambda(h) + \frac{\pi^2(r_1^2 r_2^4 + r_2^2 r_1^4)}{8}\right) \\
&= \frac{\left(\gamma_{11}\pi^2 r_1^4 + \gamma_{22}\pi^2 r_2^4\right)}{2} H_\Lambda(0) + \gamma_{12}\pi^2 r_1^2 r_2^2 G_\Lambda(h) \\
&\quad + \frac{\gamma_{11}}{2}\left(-\frac{\pi r_1^4}{2}\log(2\pi r_1) + \frac{\pi r_1^4}{8} + \frac{\pi^2 r_1^6}{4}\right) \quad + \frac{\gamma_{22}}{2}\left(-\frac{\pi r_2^4}{2}\log(2\pi r_2) + \frac{\pi r_2^4}{8} + \frac{\pi^2 r_2^6}{4}\right) \\
&\quad + \gamma_{12}\left(\frac{\pi^2(r_1^2 r_2^4 + r_1^4 r_2^2)}{8}\right).
\end{aligned}$$

Only $H_\Lambda(0)$ and $G_\Lambda(h)$ depend on Λ and only $G_\Lambda(h)$ depends on h, so we focus on

$$\begin{aligned}
&\frac{\left(\gamma_{11}\pi^2 r_1^4 + \gamma_{22}\pi^2 r_2^4\right)}{2} H_\Lambda(0) + \gamma_{12}\pi^2 r_1^2 r_2^2 G_\Lambda(h) \\
&= \frac{\left(\gamma_{11}\pi^2 r_1^4 + \gamma_{22}\pi^2 r_2^4 - 2\gamma_{12}\pi^2 r_1^2 r_2^2\right)}{2} H_\Lambda(0) + \gamma_{12}\pi^2 r_1^2 r_2^2\big(H_\Lambda(0) + G_\Lambda(h)\big) \\
&= \left(\frac{\gamma_{11}\pi^2 r_1^4 + \gamma_{22}\pi^2 r_2^4}{2}\right)\Big(b H_\Lambda(0) + (1-b)\big(H_\Lambda(0) + G_\Lambda(h)\big)\Big)
\end{aligned}$$

where

$$b = \frac{\gamma_{11} r_1^4 + \gamma_{22} r_2^4 - 2\gamma_{12} r_1^2 r_2^2}{\gamma_{11} r_1^4 + \gamma_{22} r_2^4}, \quad 1 - b = \frac{2\gamma_{12} r_1^2 r_2^2}{\gamma_{11} r_1^4 + \gamma_{22} r_2^4}.$$

The lemma will follow after one deduces the following identities.

$$H_\Lambda(0) = -\frac{1}{4\pi}\log\left|\operatorname{Im}(\tau)\eta^4(\tau)\right| \tag{7.12}$$

$$H_\Lambda(0) + G_\Lambda\left(\frac{\alpha_1}{2}\right) = -\frac{1}{4\pi}\log\left|\operatorname{Im}(2\tau)\eta^4(2\tau)\right| - \frac{1}{4\pi}\log 2 \tag{7.13}$$

$$H_\Lambda(0) + G_\Lambda\left(\frac{\alpha_2}{2}\right) = -\frac{1}{4\pi}\log\left|\operatorname{Im}\left(\frac{\tau}{2}\right)\eta^4\left(\frac{\tau}{2}\right)\right| - \frac{1}{4\pi}\log 2 \tag{7.14}$$

$$H_\Lambda(0) + G_\Lambda\left(\frac{\alpha_1+\alpha_2}{2}\right) = -\frac{1}{4\pi}\log\left|\operatorname{Im}\left(\frac{\tau+1}{2}\right)\eta^4\left(\frac{\tau+1}{2}\right)\right| - \frac{1}{4\pi}\log 2 \tag{7.15}$$

To show (7.12), let $z \to 0$ in (2.9) to find

$$\begin{aligned} H_\Lambda(0) &= -\frac{1}{2\pi}\log\left|\mathrm{e}\left(\frac{\tau}{12}\right)\left(-\frac{\mathrm{i}}{\alpha_1}\right)\prod_{n=1}^{\infty}(1-\mathrm{e}(n\tau))^2\right| \\ &= -\frac{1}{4\pi}\log\left|\operatorname{Im}(\tau)\,\mathrm{e}\left(\frac{\tau}{6}\right)\prod_{n=1}^{\infty}(1-\mathrm{e}(n\tau))^4\right| \\ &= -\frac{1}{4\pi}\log\left|\operatorname{Im}(\tau)\eta^4(\tau)\right| \end{aligned}$$

since

$$\left|-\frac{\mathrm{i}}{\alpha_1}\right|^2 = \frac{1}{|\alpha_1|^2} = \operatorname{Im}\frac{\overline{\alpha_1}\alpha_2}{|\alpha_1|^2} = \operatorname{Im}\left(\frac{\alpha_2}{\alpha_1}\right) = \operatorname{Im}(\tau)$$

by $\operatorname{Im}(\overline{\alpha_1}\alpha_2) = |\Lambda| = 1$.

By (2.6) and (7.12),

$$\begin{aligned} G_\Lambda\left(\frac{\alpha_1}{2}\right) &= -\frac{1}{2\pi}\log\left|\mathrm{e}\left(\frac{\tau}{12}\right)(2)\prod_{n=1}^{\infty}(1+\mathrm{e}(n\tau))^2\right|, \\ H_\Lambda(0) + G_\Lambda\left(\frac{\alpha_1}{2}\right) &= -\frac{1}{4\pi}\log\left|2\operatorname{Im}(2\tau)\,\mathrm{e}\left(\frac{\tau}{3}\right)\prod_{n=1}^{\infty}(1-\mathrm{e}(2n\tau))^4\right| \\ &= -\frac{1}{4\pi}\log\left|\operatorname{Im}(2\tau)\eta^4(2\tau)\right| - \frac{1}{4\pi}\log 2, \end{aligned}$$

which is (7.13).

To see (7.14), note that, since $\operatorname{Im}(\overline{\alpha_1}\alpha_2) = 1$,

$$\frac{\alpha_2^2\overline{\alpha_1}}{16\mathrm{i}\alpha_1} = \frac{\alpha_2}{16\mathrm{i}\alpha_1}(\overline{\alpha_1}\alpha_2) = \frac{\alpha_2}{16\mathrm{i}\alpha_1}(\alpha_1\overline{\alpha_2} + 2\mathrm{i}) = \frac{|\alpha_2|^2}{16\mathrm{i}} + \frac{\tau}{8},$$

and consequently by (2.6),

$$\begin{aligned} G_\Lambda\left(\frac{\alpha_2}{2}\right) &= -\frac{1}{2\pi}\log\left|\mathrm{e}\left(-\frac{\tau}{24}\right)\left(1-\mathrm{e}\left(\frac{\tau}{2}\right)\right)\prod_{n=1}^{\infty}\left(1-\mathrm{e}\left(n\tau+\frac{\tau}{2}\right)\right)\left(1-\mathrm{e}\left(n\tau-\frac{\tau}{2}\right)\right)\right| \\ &= -\frac{1}{4\pi}\log\left|\mathrm{e}\left(-\frac{\tau}{12}\right)\left(1-\mathrm{e}\left(\frac{\tau}{2}\right)\right)^2\prod_{n=1}^{\infty}\left(1-\mathrm{e}\left((2n+1)\frac{\tau}{2}\right)\right)^2\left(1-\mathrm{e}\left((2n-1)\frac{\tau}{2}\right)\right)^2\right| \\ &= -\frac{1}{4\pi}\log\left|\mathrm{e}\left(-\frac{\tau}{12}\right)\prod_{n=1}^{\infty}\left(1-\mathrm{e}\left((2n-1)\frac{\tau}{2}\right)\right)^4\right|. \end{aligned}$$

By (7.12),

$$H_\Lambda(0) + G_\Lambda\left(\frac{\alpha_2}{2}\right) = -\frac{1}{4\pi}\log\Big|\operatorname{Im}(\tau)\,\mathrm{e}\left(\frac{\tau}{12}\right)\left(\prod_{n=1}^{\infty}(1-\mathrm{e}(n\tau))^4\right)\left(\prod_{n=1}^{\infty}\left(1-\mathrm{e}\left((2n-1)\frac{\tau}{2}\right)\right)^4\right)\Big|$$
$$= -\frac{1}{4\pi}\log\Big|\operatorname{Im}(\tau)\,\mathrm{e}\left(\frac{\tau}{12}\right)\prod_{n=1}^{\infty}\left(1-\mathrm{e}\left(n\frac{\tau}{2}\right)\right)^4\Big|$$
$$= -\frac{1}{4\pi}\log\Big|\operatorname{Im}\left(\frac{\tau}{2}\right)\eta^4\left(\frac{\tau}{2}\right)\Big| - \frac{1}{4\pi}\log 2.$$

Finally for (7.15), start with

$$\begin{aligned}
\frac{(\alpha_1+\alpha_2)^2\overline{\alpha_1}}{16\mathrm{i}\alpha_1} &= \frac{|\alpha_1+\alpha_2|^2(\alpha_1+\alpha_2)\overline{\alpha_1}}{16\mathrm{i}\overline{(\alpha_1+\alpha_2)}\alpha_1}\\
&= \frac{|\alpha_1+\alpha_2|^2\left(\overline{(\alpha_1+\alpha_2)}\alpha_1+2\mathrm{i}\right)}{16\mathrm{i}\overline{(\alpha_1+\alpha_2)}\alpha_1}\\
&= \frac{|\alpha_1+\alpha_2|^2}{16\mathrm{i}} + \frac{\alpha_1+\alpha_2}{8\alpha_1}\\
&= \frac{|\alpha_1+\alpha_2|^2}{16\mathrm{i}} + \frac{\tau+1}{8},
\end{aligned}$$

so that

$$G_\Lambda\left(\frac{\alpha_1+\alpha_2}{2}\right) = -\frac{1}{2\pi}\log\Big|\mathrm{e}\left(-\frac{\tau}{24}\right)\left(1+\mathrm{e}\left(\frac{\tau}{2}\right)\right)\prod_{n=1}^{\infty}\left(1+\mathrm{e}\left(\left(n+\frac{1}{2}\right)\tau\right)\right)\left(1+\mathrm{e}\left(\left(n-\frac{1}{2}\right)\tau\right)\right)\Big|$$
$$= -\frac{1}{4\pi}\log\Big|\mathrm{e}\left(-\frac{\tau}{12}\right)\prod_{n=1}^{\infty}\left(1+\mathrm{e}\left(\left(n-\frac{1}{2}\right)\tau\right)\right)^4\Big|,$$

and

$$H_\Lambda(0) + G_\Lambda\left(\frac{\alpha_1+\alpha_2}{2}\right) = -\frac{1}{4\pi}\log\Big|\operatorname{Im}(\tau)\,\mathrm{e}\left(\frac{\tau}{12}\right)\prod_{n=1}^{\infty}(1-\mathrm{e}(n\tau))^4\prod_{n=1}^{\infty}\left(1+\mathrm{e}\left(\left(n-\frac{1}{2}\right)\tau\right)\right)^4\Big|$$
$$= -\frac{1}{4\pi}\log\Big|\operatorname{Im}\left(\frac{\tau+1}{2}\right)\mathrm{e}\left(\frac{\tau+1}{12}\right)\prod_{n=1}^{\infty}\left(1-\mathrm{e}\left(n\frac{\tau+1}{2}\right)\right)^4\Big| - \frac{1}{4\pi}\log 2$$
$$= -\frac{1}{4\pi}\log\Big|\operatorname{Im}\left(\frac{\tau+1}{2}\right)\eta^4\left(\frac{\tau+1}{2}\right)\Big| - \frac{1}{4\pi}\log 2.$$

Here, to simplify the two infinite products, one uses the identity

$$\prod_{n=1}^{\infty}(1-\mathrm{e}(n\tau))\prod_{n=1}^{\infty}\left(1+\mathrm{e}\left(\left(n-\frac{1}{2}\right)\tau\right)\right) = \prod_{n=1}^{\infty}\left(1-\mathrm{e}\left(2n\frac{\tau+1}{2}\right)\right)\prod_{n=1}^{\infty}\left(1-\mathrm{e}\left((2n-1)\frac{\tau+1}{2}\right)\right)$$
$$= \prod_{n=1}^{\infty}\left(1-\mathrm{e}\left(n\frac{\tau+1}{2}\right)\right).$$

This completes the proof of the lemma. □

Remark 7.2. *Although $\mathcal{J}_{l,\Lambda}(B(0,r_1),B(h,r_2))$ is defined only for $(\Lambda,h)\in\mathcal{A}_{r_1,r_2}$, the three g_*'s found in Lemma 7.1 are meaningful for all (Λ,h) pairs as long as $|\Lambda|=1$ and h is a half period of Λ, even if (Λ,h) is not in $\mathcal{A}_{r_1,r_2}$.*

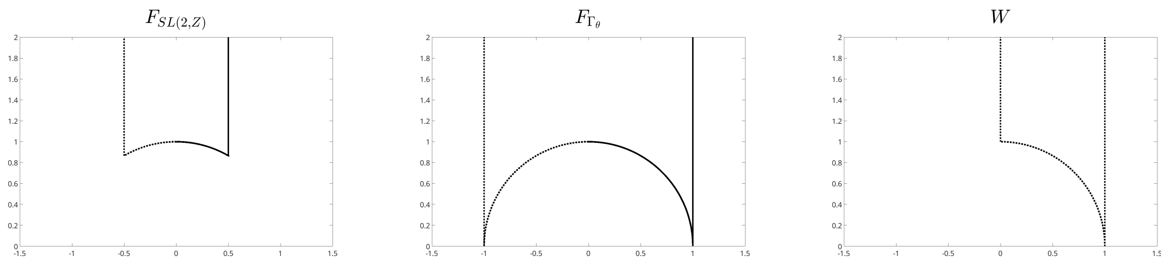

Figure 2: The fundamental domain $F_{SL(2,\mathbb{Z})}$, the fundamental domain F_{Γ_θ}, and the set W.

8 Optimal lattice

The modular group $SL(2,\mathbb{Z})$ is the set of all two by two matrices of integer entries with determinants equal to 1,

$$SL(2,\mathbb{Z}) = \left\{ \begin{bmatrix} d_1 & d_2 \\ d_4 & d_3 \end{bmatrix} : \ d_j \in \mathbb{Z}, \ j = 1,2,3,4, \ d_1 d_3 - d_2 d_4 = 1 \right\}. \tag{8.1}$$

This group is generated by

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{8.2}$$

It acts on $\mathbb{H}$ by the linear fractional transform

$$\tau \to M\tau = \frac{d_1\tau + d_2}{d_4\tau + d_3}, \ \tau \in \mathbb{H}, \ M = \begin{bmatrix} d_1 & d_2 \\ d_4 & d_3 \end{bmatrix} \in SL(2,\mathbb{Z}). \tag{8.3}$$

A fundamental domain of this action by $SL(2,\mathbb{Z})$ on $\mathbb{H}$ is

$$\begin{aligned} F_{SL(2,\mathbb{Z})} = & \left\{ \tau \in \mathbb{H} : \ |\tau| > 1, \ -\frac{1}{2} < \operatorname{Re}\tau < \frac{1}{2} \right\} \cup \left\{ \tau \in \mathbb{H} : \ \operatorname{Re}\tau = \frac{1}{2}, \ \operatorname{Im}\tau \geq \frac{\sqrt{3}}{2} \right\} \\ & \cup \left\{ \tau \in \mathbb{H} : \ |\tau| = 1, \ 0 \leq \operatorname{Re}\tau < \frac{1}{2} \right\}; \end{aligned} \tag{8.4}$$

see the left plot of Figure 2 and consult [6, VI.1] for a proof.

Remark 8.1. *We use the term fundamental domain in the strict sense: every orbit of the action $SL(2,\mathbb{Z})$ on $\mathbb{H}$ has one and only one element in $F_{SL(2,\mathbb{Z})}$.*

Also needed is a subgroup of $SL(2,\mathbb{Z})$, called the theta group Γ_θ,

$$\Gamma_\theta = \left\{ M \in SL(2,\mathbb{Z}) : \ M \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mod 2, \text{ or } M \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \mod 2 \right\}. \tag{8.5}$$

This group is generated by

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}; \tag{8.6}$$

see [6, Appendix to VI.5] for more on the theta group. A fundamental domain of the action by Γ_θ on $\mathbb{H}$ is

$$\begin{aligned} F_{\Gamma_\theta} = & \{\tau \in \mathbb{H} : \ |\tau| > 1, \ -1 < \operatorname{Re}\tau < 1\} \cup \{\tau \in \mathbb{H} : \ \operatorname{Re}\tau = 1\} \\ & \cup \{\tau \in \mathbb{H} : \ |\tau| = 1, \ 0 \leq \operatorname{Re}\tau < 1\}; \end{aligned} \tag{8.7}$$

see the center plot of Figure 2.

Denote g_* in the third case of Lemma 7.1 by g_b:

$$g_b(\tau) = b\, g(\tau) + (1-b)\, g\left(\frac{\tau+1}{2}\right), \ \tau \in \mathbb{H}, \ b \in [0,1]. \tag{8.8}$$

Some of the properties of g_b below hold even if $b \in \mathbb{R}$. We state so explicitly in those cases.

Lemma 8.2. *The following transformation rules hold for g and g_b.*

1. *For all $M \in SL(2,\mathbb{Z})$ and $\tau \in \mathbb{H}$,*
$$g(M\tau) = g(\tau).$$

2. *For all $M \in \Gamma_\theta$, $\tau \in \mathbb{H}$, and $b \in \mathbb{R}$,*
$$g\left(\frac{M\tau+1}{2}\right) = g\left(\frac{\tau+1}{2}\right), \quad \textit{and} \ \ g_b(M\tau) = g_b(\tau).$$

3. *For all $\tau \in \mathbb{H}$ and $b \in \mathbb{R}$,*
$$g(-\overline{\tau}) = g(\tau), \ g\left(\frac{-\overline{\tau}+1}{2}\right) = g\left(\frac{\tau+1}{2}\right) \quad \textit{and} \ \ g_b(-\overline{\tau}) = g_b(\tau).$$

4. *Under the transform $\tau \to \sigma = \frac{\tau-1}{\tau+1}$ of $\mathbb{H}$,*
$$g(\tau) = g\left(\frac{\sigma+1}{2}\right), \quad \textit{and} \ \ g\left(\frac{\tau+1}{2}\right) = g(\sigma), \quad \tau \in \mathbb{H} \quad \textit{and} \quad \sigma = \frac{\tau-1}{\tau+1} \in \mathbb{H};$$
consequently, for all $b \in \mathbb{R}$,
$$g_b(\sigma) = g_{1-b}(\tau), \quad \tau \in \mathbb{H} \quad \textit{and} \quad \sigma = \frac{\tau-1}{\tau+1} \in \mathbb{H}.$$

Proof. Dedekind's eta function has the transformation properties that for all $\tau \in \mathbb{H}$,

$$\eta(\tau+1) = \exp\left(\frac{\pi \mathrm{i}}{12}\right)\eta(\tau) \text{ and } \eta\Big(-\frac{1}{\tau}\Big) = \sqrt{-\mathrm{i}\tau}\,\eta(\tau) \tag{8.9}$$

where $\sqrt{\cdot}$ stands for the principal branch of square root; see [2, Chapter 2]. It follows that

$$g(\tau+1) = g(\tau) \text{ and } g\left(-\frac{1}{\tau}\right) = g(\tau), \ \tau \in \mathbb{H}. \tag{8.10}$$

Hence $g(\tau)$ is invariant under the generators (8.2), and consequently invariant under the modular group $SL(2,\mathbb{Z})$. This proves part 1.

To prove part 2, consider the transforms $\tau \to \tau+2$ and $\tau \to -\frac{1}{\tau}$ by the generators (8.6) of Γ_θ, and use the transformation properties of g in (8.10) to derive

$$g\left(\frac{(\tau+2)+1}{2}\right) = g\left(\frac{\tau+1}{2}+1\right) = g\left(\frac{\tau+1}{2}\right)$$
$$g\left(\frac{-\frac{1}{\tau}+1}{2}\right) = g\left(\frac{\tau-1}{2\tau}\right) = g\left(\frac{-\tau-1}{2\tau}\right) = g\left(\frac{2\tau}{\tau+1}\right) = g\left(\frac{-2}{\tau+1}\right) = g\left(\frac{\tau+1}{2}\right)$$

Part 3 follows from the definitions (7.10) and (7.11) of g and η.

For part 4, note that the transform $\sigma = \frac{\tau-1}{\tau+1}$ and its inverse $\tau = \frac{-\sigma-1}{\sigma-1}$ are both maps from $\mathbb{H}$ onto $\mathbb{H}$, but they are not actions by elements of $SL(2,\mathbb{Z})$. Nevertheless,

$$g\left(\frac{\sigma+1}{2}\right) = g\left(\frac{\frac{\tau-1}{\tau+1}+1}{2}\right) = g\left(\frac{\tau}{\tau+1}\right) = g(\tau) \tag{8.11}$$

$$g\left(\frac{\tau+1}{2}\right) = g\left(\frac{\frac{-\sigma-1}{\sigma-1}+1}{2}\right) = g\left(\frac{-1}{\sigma-1}\right) = g(\sigma) \tag{8.12}$$

since $\tau \to \frac{\tau}{\tau+1}$ and $\sigma \to \frac{-1}{\sigma-1}$ are both actions by elements in $SL(2,\mathbb{Z})$ under which g is invariant. $\square$

An important number B is defined in [10]. When $b = B$, the second derivative of $g_b(\tau)$ with respect to $\operatorname{Im}\tau$ vanishes at i; namely

$$\frac{\partial^2 g_b(\tau)}{\partial(\operatorname{Im}\tau)^2}\Big|_{b=B,\tau=\mathrm{i}} = 0. \tag{8.13}$$

One can solve (8.13) for B and write it as a quotient of series and find its numerical value:

$$\begin{aligned} B &= \frac{1+\sum_{n=1}^{\infty}\frac{4\pi^2n^2(-e^{-\pi})^n}{(1-(-e^{-\pi})^n)^2}}{\sum_{n=1}^{\infty}\frac{4\pi^2n^2(-e^{-\pi})^n}{(1-(-e^{-\pi})^n)^2} - \sum_{n=1}^{\infty}\frac{16\pi^2n^2e^{-2n\pi}}{(1-e^{-2n\pi})^2}} \\ &= \frac{-0.2982...}{-1.2982... - 0.2982...} \\ &= 0.1867... \end{aligned} \tag{8.14}$$

The next two lemmas were proved by Luo, Ren, and Wei in [10].

Lemma 8.3 ([10, Lemma 4.4]). *The following properties hold for $t \to g_b(t\mathrm{i})$, $t \in (0,\infty)$.*

1. *When $b \in [0,B)$, the function $t \to g_b(t\mathrm{i})$, $t > 0$, has exactly three critical points at $\frac{1}{q_b}$, 1, and q_b, where $q_b \in (1,\sqrt{3}]$. Moreover*
   (a) $\frac{\partial g_b(\tau)}{\partial \operatorname{Im}\tau}\Big|_{\tau=t\mathrm{i}} < 0$ *if* $t \in \left(0,\frac{1}{q_b}\right)$,
   (b) $\frac{\partial g_b(\tau)}{\partial \operatorname{Im}\tau}\Big|_{\tau=t\mathrm{i}} > 0$ *if* $t \in \left(\frac{1}{q_b},1\right)$,
   (c) $\frac{\partial g_b(\tau)}{\partial \operatorname{Im}\tau}\Big|_{\tau=t\mathrm{i}} < 0$ *if* $t \in (1,q_b)$,
   (d) $\frac{\partial g_b(\tau)}{\partial \operatorname{Im}\tau}\Big|_{\tau=t\mathrm{i}} > 0$ *if* $t \in (q_b,\infty)$.

   As b increases from 0 to B, q_b decreases from $\sqrt{3}$ towards 1.
2. *When $b \in [B,1]$, the function $t \to g_b(t\mathrm{i})$, $t > 0$, has only one critical point at 1, and*
   (a) $\frac{\partial g_b(\tau)}{\partial \operatorname{Im}\tau}\Big|_{\tau=t\mathrm{i}} < 0$ *if* $t \in (0,1)$,
   (b) $\frac{\partial g_b(\tau)}{\partial \operatorname{Im}\tau}\Big|_{\tau=t\mathrm{i}} > 0$ *if* $t \in (1,\infty)$.

Define a subset W and its closure $\overline{W}_{\mathbb{H}}$ in $\mathbb{H}$ as follows.

$$W = \{\tau \in \mathbb{H} : 0 < \operatorname{Re}\tau < 1,\ |\tau| > 1\}, \tag{8.15}$$

$$\overline{W}_{\mathbb{H}} = \{\tau \in \mathbb{H} : 0 \le \operatorname{Re}\tau \le 1,\ |\tau| \ge 1\}. \tag{8.16}$$

See the right plot of Figure 2.

Lemma 8.4 ([10, Lemma 5.2]). *Let $b \in [0, 1-B]$ and W be given in* (8.15). *Then*

$$\frac{\partial g_b(\tau)}{\partial(\operatorname{Re}\tau)} > 0, \ \textit{for all}\ \tau \in W.$$

Remark 8.5. *The function g_b here is equal to $-\frac{1}{4\pi} f_b$ in [10].*

Now we are ready to prove the main theorem.

Proof of Theorem 1.3. We temporarily dispense with the requirement $(\Lambda, h) \in \mathcal{A}_{r_1,r_2}$ and allow discs $B(0, r_1)$ and $B(h, r_2)$ to overlap. According to Remark 7.2 the three g_*'s in Lemma 7.1 are defined for all (Λ, h), provided that $|\Lambda| = 1$ and h is a half period of Λ, including those (Λ, h)'s not in $\mathcal{A}_{r_1,r_2}$.

First assume $b \in [0, 1-B]$, and consider the third case of Lemma 7.1 where $g_* = g_b$. We show that g_b in $\mathbb{H}$ is minimized at τ_b, where

$$\tau_b = \begin{cases} q_b \mathrm{i} & \text{if } b \in [0, B) \\ \mathrm{i} & \text{if } b \in [B, 1-B] \end{cases}, \tag{8.17}$$

and the points in the orbit of τ_b under the group Γ_θ. Recall that $q_b \in (1, \sqrt{3})$ is given in Lemma 8.3.1.

Consider g_b restricted to $\overline{W}_{\mathbb{H}}$. Lemma 8.4 asserts that g_b is strictly increasing in the horizontal direction in W, so it can only attain a minimum in $\overline{W}_{\mathbb{H}}$ on the part of the unit circle in the first quadrant, i.e. $\{\tau \in \mathbb{H} : \ 0 < \operatorname{Re}\tau < 1, \ |\tau| = 1\}$, or on the part of the imaginary axis above i, i.e. $\{\tau \in \mathbb{H} : \ \operatorname{Re}\tau = 0, \ \operatorname{Im}\tau \geq 1\}$.

The unit circle can be ruled out. By Lemma 8.2.4

$$g_b(\sigma) = g_{1-b}(\tau), \ \tau \in \mathbb{H}, \ \sigma = \frac{\tau - 1}{\tau + 1} \in \mathbb{H}. \tag{8.18}$$

Take $\tau = t\mathrm{i}$, $t > 1$, to be on the imaginary axis. Then

$$\sigma = \frac{t^2 - 1}{t^2 + 1} + \frac{2t}{t^2 + 1}\mathrm{i}$$

is on the unit circle. As τ moves from i to ∞ upwards along the imaginary axis, σ moves from i to 1 clockwise along the unit circle. When $b \in [0, 1-B]$, $1 - b \in [B, 1]$. Since $t \to g_{1-b}(t\mathrm{i})$ is strictly increasing for $t \in (1, \infty)$ by Lemma 8.3.2, $g_b(\sigma)$ is strictly increasing when σ moves from i to 1 clockwise along the unit circle. Then g_b cannot attain a minimum on $\{\sigma \in \mathbb{H} : \ \ 0 < \operatorname{Re}\sigma < 1, \ |\sigma| = 1\}$.

Therefore in $\overline{W}_{\mathbb{H}}$, g_b can only achieve a minimum on $\{\tau \in \mathbb{H} : \ \operatorname{Re}\tau = 0, \ \operatorname{Im}\tau \geq 1\}$. By Lemma 8.3, this minimum is $q_b\mathrm{i}$ if $b \in [0, B)$ and is i if $b \in [B, 1-B]$; according to (8.17), it is denoted τ_b.

By the invariance $g_b(-\overline{\tau}) = g_b(\tau)$, $\tau \in \mathbb{H}$, in Lemma 8.2.3, when restricted to F_{Γ_θ}, g_b achieves a unique minimum at τ_b. Since F_{Γ_θ} is a fundamental domain of the theta group Γ_θ and g_b is invariant under Γ_θ, g_b in $\mathbb{H}$ is minimized at the orbit of τ_b, $\{T\tau_b : \ T \in \Gamma_\theta\}$, under Γ_θ when $b \in [0, 1-B]$.

Next consider the case $b \in (1-B, 1]$. Since $1 - b \in [0, B)$, the above result applies to g_{1-b} which in $\mathbb{H}$ is minimized at $\{T\tau_{1-b} : \ T \in \Gamma_\theta\}$. Lemma 8.2.4 asserts that

$$g_b(\sigma) = g_{1-b}(\tau), \quad \tau \in \mathbb{H} \ \text{ and } \ \sigma = \frac{\tau - 1}{\tau + 1} \in \mathbb{H}.$$

Hence g_b in $\mathbb{H}$ is minimized at $\{AT\tau_{1-b} : \ T \in \Gamma_\theta\}$ where

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \tag{8.19}$$

Define

$$\tau_b = A\tau_{1-b}, \quad b \in (1-B, 1], \tag{8.20}$$

where τ_{1-b} on the right side is given by (8.17); more explicitly,

$$\tau_b = \frac{q_{1-b}^2 - 1}{q_{1-b}^2 + 1} + \frac{2q_{1-b}}{q_{1-b}^2 + 1}\mathrm{i}, \ b \in (1-B, 1] \tag{8.21}$$

where $q_{1-b} \in (1, \sqrt{3}]$ is given in Lemma 8.3.1. Note that in this case, τ_b is on the arc $\{\tau \in \mathbb{H} : \ 0 < \operatorname{Re}\tau \leq \frac{1}{2}, \ |\tau| = 1\}$.

From the definition (8.5) of the theta group, it is easy to see that

$$A\Gamma_\theta A^{-1} = \Gamma_\theta. \tag{8.22}$$

One writes $AT\tau_{1-b} = ATA^{-1}\tau_b$ by (8.20). Hence, as in the $b \in [0, 1-B]$ case, when $b \in (1-B, 1]$, g_b in $\mathbb{H}$ is minimized at $\{T\tau_b : \ T \in \Gamma_\theta\}$.

In summary, for each $b \in [0, 1]$, g_b in $\mathbb{H}$ is minimized at

$$\{T\tau_b : \ T \in \Gamma_\theta\}, \tag{8.23}$$

where τ_b is given by (8.17) and (8.21). Moreover,

1. when $b = 0$, $\tau_0 = \sqrt{3}\mathrm{i}$;
2. when $b \in (0, B)$, τ_b is on the segment $\{\tau \in \mathbb{H} : \ \operatorname{Re}\tau = 0, \ 1 < \operatorname{Im}\tau < \sqrt{3}\}$ of the imaginary axis, and as b increases from 0 to B, τ_b moves from $\sqrt{3}\,\mathrm{i}$ to i;
3. when $b \in [B, 1-B]$, $\tau_b = \mathrm{i}$;
4. when $b \in (1-B, 1)$, τ_b is on the arc $\{\tau \in \mathbb{H} : \ 0 < \operatorname{Re}\tau < \frac{1}{2}, \ |\tau| = 1\}$ of the unit circle, and as b increases from $1-B$ to 1, τ_b moves from i clockwise to $e^{\pi \mathrm{i}/3}$;
5. when $b = 1$, $\tau_1 = e^{\pi \mathrm{i}/3}$.

Let $\alpha_b = (\alpha_{b,1}, \alpha_{b,2})$ be a basis associated with τ_b, i.e. $\tau_b = \frac{\alpha_{b,2}}{\alpha_{b,1}}$. The corresponding half period is

$$h_b = \frac{\alpha_{b,1} + \alpha_{b,2}}{2}. \tag{8.24}$$

The fundamental parallelogram P_{α_b} is a non-square rectangle in cases 1 and 2; P_{α_b} is a square in case 3; P_{α_b} is a rhombus in cases 4 and 5.

The function g_b arises in case 3 of Lemma 7.1 where the half period h is the center point $\frac{\alpha_1+\alpha_2}{2}$. It remains to study the other two half periods, $\frac{\alpha_1}{2}$ and $\frac{\alpha_2}{2}$. Consider case 1 of Lemma 7.1 where $h = \frac{\alpha_1}{2}$ under a basis (α_1, α_2) of lattice Λ. Let

$$M_1 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \in SL(2, \mathbb{Z}), \tag{8.25}$$

and introduce a new basis (β_1, β_2) of the same lattice Λ by

$$\begin{bmatrix} \beta_2 \\ \beta_1 \end{bmatrix} = M_1 \begin{bmatrix} \alpha_2 \\ \alpha_1 \end{bmatrix}. \tag{8.26}$$

Then, with $\sigma = \frac{\beta_2}{\beta_1}$,

$$\sigma = \frac{\beta_2}{\beta_1} = \frac{\alpha_2 + \alpha_1}{-\alpha_2} = -1 - \frac{\alpha_1}{\alpha_2} = -1 - \frac{1}{\tau}, \tag{8.27}$$

and the half period h is

$$h = \frac{\alpha_1}{2} = \frac{\beta_1 + \beta_2}{2}, \tag{8.28}$$

which is the center of P_β, the fundamental parallelogram of the new basis. This reduces case 1 of Lemma 7.1 to case 3 under the basis (β_1, β_2), with $\sigma = \frac{\beta_2}{\beta_1}$. Consequently,

$$b\, g(\tau) + (1-b)\, g\,(2\tau) = g_b(\sigma). \tag{8.29}$$

For case 2 of Lemma 7.1 where $h = \frac{\alpha_2}{2}$ under a basis (α_1, α_2), let

$$M_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \in SL(2, \mathbb{Z}), \tag{8.30}$$

and (β_1, β_2) be a new basis of the same lattice by

$$\begin{bmatrix} \beta_2 \\ \beta_1 \end{bmatrix} = M_2 \begin{bmatrix} \alpha_2 \\ \alpha_1 \end{bmatrix}. \tag{8.31}$$

Then

$$\sigma = \frac{\beta_2}{\beta_1} = \frac{\alpha_2 - \alpha_1}{\alpha_1} = \frac{\alpha_2}{\alpha_1} - 1 = \tau - 1, \tag{8.32}$$

and the half period h is

$$h = \frac{\alpha_2}{2} = \frac{\beta_1 + \beta_2}{2}. \tag{8.33}$$

This again reduces to case 3 of Lemma 7.1 under the basis (β_1, β_2), with $\sigma = \frac{\beta_2}{\beta_1}$, and consequently

$$b\, g(\tau) + (1-b)\, g\left(\frac{\tau}{2}\right) = g_b(\sigma). \tag{8.34}$$

All three cases of half periods have been reduced to the minimization of g_b in $\mathbb{H}$. Each $b \in [0,1]$ determines τ_b by (8.17) and (8.21). Any lattice Λ_b of unit area with a basis $(\alpha_{b,1}, \alpha_{b,2})$ such that $\frac{\alpha_{b,2}}{\alpha_{b,1}} = \tau_b$ and with a half period $h_b = \frac{\alpha_{b,1}+\alpha_{b,2}}{2}$, minimizes $\mathcal{J}_{l,\Lambda}(B(0,r_1), B(h,r_2))$ of Lemma 7.1. The five assertions of Theorem 1.3 regarding the shape of the fundamental parallelogram P_{α_b} follow from the remark after (8.23).

Now we reinstate the requirement $(\Lambda, h) \in \mathcal{A}_{r_1,r_2}$. This removes some (Λ, h) pairs from consideration. However, for every $b \in [0,1]$, any lattice Λ_b specified by τ_b and the center point h_b of the associated fundamental parallelogram belongs to $\mathcal{A}_{r_1,r_2}$. Hence, this (Λ_b, h_b), associated with the minimum τ_b of g_b, is also a minimum of the problem (1.24).

Finally we show that the minimum of (1.24) is unique up to rotation. Let (Λ, h) and (Λ', h') both minimize problem (1.24). Choose a basis (α_1, α_2) for Λ, with $\tau = \frac{\alpha_2}{\alpha_1}$, so that $h = \frac{\alpha_1+\alpha_2}{2}$. Similarly choose (α_1', α_2'), $\tau' = \frac{\alpha_2'}{\alpha_1'}$, for Λ' with $h' = \frac{\alpha_1'+\alpha_2'}{2}$. As both τ and τ' minimize g_b in $\mathbb{H}$, by (8.23) there exists $M \in \Gamma_\theta$ such that $\tau' = M\tau$; namely

$$\frac{\alpha_2'}{\alpha_1'} = \frac{d_1\alpha_2 + d_2\alpha_1}{d_4\alpha_2 + d_3\alpha_1}, \quad \text{where } M = \begin{bmatrix} d_1 & d_2 \\ d_4 & d_3 \end{bmatrix} \in \Gamma_\theta. \tag{8.35}$$

Then there exists $\kappa \in \mathbb{C}\backslash\{0\}$ such that

$$\begin{bmatrix} \alpha_2' \\ \alpha_1' \end{bmatrix} = \kappa \begin{bmatrix} d_1 & d_2 \\ d_4 & d_3 \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \alpha_1 \end{bmatrix}, \tag{8.36}$$

which means

$$\Lambda' = \kappa\Lambda. \tag{8.37}$$

Since $|\Lambda| = |\Lambda'| = 1$, $|\kappa| = 1$ and Λ' is a rotation of Λ by κ.

Moreover,

$$\begin{aligned} h' &= \frac{\alpha_1' + \alpha_2'}{2} \\ &= \kappa \frac{(d_2+d_3)\alpha_1 + (d_1+d_4)\alpha_2}{2}. \end{aligned}$$

By (8.5),

$$d_2 + d_3 \equiv d_1 + d_4 \equiv 1 \mod 2,$$

and hence

$$\frac{(d_2+d_3)\alpha_1+(d_1+d_4)\alpha_2}{2} \equiv \frac{\alpha_1+\alpha_2}{2} \mod \Lambda$$
$$= h.$$

Therefore

$$h' = \kappa h, \tag{8.38}$$

and h' is also a rotation of h by κ. □

In the proof of Theorem 1.3, the half periods $\frac{\alpha_1}{2}$ and $\frac{\alpha_2}{2}$ are transformed to the center $\frac{\beta_1+\beta_2}{2}$ of a new basis (β_1, β_2) by M_1 and M_2 in $SL(2,\mathbb{Z})$ respectively, so the center is not preserved under the modular group $SL(2,\mathbb{Z})$. However if one transforms a basis by the generators of the theta group Γ_θ in (8.6) from (α_1, α_2) to (β_1, β_2), then

$$\frac{\beta_1+\beta_2}{2} = \frac{\alpha_1+\alpha_2+2\alpha_1}{2} \equiv \frac{\alpha_1+\alpha_2}{2} \mod \Lambda \tag{8.39}$$

$$\frac{\beta_1+\beta_2}{2} = \frac{\alpha_2-\alpha_1}{2} \equiv \frac{\alpha_1+\alpha_2}{2} \mod \Lambda \tag{8.40}$$

for the two generators respectively. Therefore the center is preserved under the action of the theta group.

The theta group is a non-normal subgroup of the modular group of index three. One can write down a right coset decomposition

$$SL(2,\mathbb{Z}) = \Gamma_\theta \cup \Gamma_\theta M_1 \cup \Gamma_\theta M_2. \tag{8.41}$$

Every element in $\Gamma_\theta M_1$ transforms $\frac{\alpha_1}{2}$ to the center of another fundamental parallelogram; the elements in $\Gamma_\theta M_2$ do the same to $\frac{\alpha_2}{2}$.

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