Zeitschr. f. math. Logik und Grundlagen d. Math.
Bd. 32, S. 467–472 (1986)

RELATIONS INTRINSICALLY RECURSIVE IN LINEAR ORDERS

by MICHAEL MOSES in Macomb, Illinois (U.S.A.)[1])

A recursively enumerable (r.e.) relation R on a recursive structure $\mathfrak{A}$ is said to be *intrinsically r.e.* on $\mathfrak{A}$ if every isomorphism of $\mathfrak{A}$ with a recursive structure carries R to an r.e. relation. This concept was introduced in [1] by C. J. ASH and A. NERODE. It was shown in that paper that R is intrinsically r.e. if and only if it is formally r.e., where a relation is defined to be *formally r.e.* on $\mathfrak{A}$ if it is equivalent in $\mathfrak{A}$ to an r.e. disjunction $\bigvee_{n<\omega} \varphi_n(\bar{a}, \bar{x})$ of existential formulae with finitely many parameters. The result is proved under an "extra decidability assumption" that there is a recursive procedure for determining, given an existential formula $\varphi_n(\bar{y}, \bar{x})$ and elements $\bar{a}$ of $\mathfrak{A}$, whether the implication $\varphi(\bar{a}, \bar{x}) \to R(\bar{x})$ is true in $\mathfrak{A}$. It has been shown (GONCHAROV [2], MANASSE [4]) that without this assumption formally r.e. is strictly stronger than intrinsically r.e.

A relation is *intrinsically recursive* if both it and its complement are intrinsically r.e. Relations that are intrinsically recursive have been considered in particular cases. In [1], ASH and NERODE showed that the *successivity relation* $S(x, y)$ and the *block relation* $B(x, y)$ are not intrinsically recursive on ω and $\omega + \omega^*$ respectively. ($S(x, y)$ is satisfied by elements with no other elements between them and $B(x, y)$ by elements with at most finitely many elements between them.) [1] also discusses the relations of linear dependence in recursive vector spaces, algebraic dependence in recursive algebraically closed fields and atoms in recursive Boolean algebras.

In this paper we characterize relations that are intrinsically recursive on a recursive linear order $\mathfrak{A}$ as precisely those that are equivalent in $\mathfrak{A}$ to a quantifier-free formula with finitely many parameters (Theorem 2). In proving Theorem 2 we use a result presented in MOSES [5]. This result, Theorem 1, gives a condition sufficient for a relation to be non-intrinsically r.e. on a recursive structure. The proof of Theorem 2 is complicated by the generality of the relation R. For this reason we present it as an inductive argument on the number of arguments of R. The first step of the induction, when R is a one-place relation, is proved in Lemma 1. The inductive step is proved in Theorem 2. Both proofs use Theorem 1.

We consider a recursive structure $\mathfrak{A}$ with recursive universe $A = \{a_0, a_1, \ldots\}$. A_s denotes the set $\{a_0, a_1, \ldots, a_s\}$ and $\mathfrak{A}_s$ is the restriction of $\mathfrak{A}$ to A_s. A general element of N^2 is represented by $\langle m, s\rangle$. Theorem 1 is proved in MOSES [5].

Theorem 1. *If $\mathfrak{A}$ is a recursive structure with language consisting solely of a finite number of predicate symbols, and R is a recursive relation on $\mathfrak{A}$, then* (1) *implies* (2):

[1]) These results were presented at the Conference on Logic and Computation in 1984, held at Monash University, Melbourne, Australia.

(1) *There is a recursive function f from N^2 into N such that for every $m \in \mathsf{N}$ there is a sequence $\bar{a} \in R$ for which there are infinitely many $s \in \mathsf{N}$ with embeddings $\varphi : \mathfrak{A}_s \to \mathfrak{A}_{f\langle m,s\rangle}$ with φ the identity on A_m and $\varphi(\bar{a}) \notin R$.*

(2) *R is not intrinsically r.e. on $\mathfrak{A}$.*

We now use Theorem 1 to prove our main result. In Lemma 1 we treat the case when R is a one-place relation. Notice that a one-place relation R is equivalent in a linear order $\mathfrak{A}$ to a quantifier-free formula with finitely many parameters $\bar{a}$ if and only if $\mathfrak{A}$ is partitioned by $\bar{a}$ into intervals, each of which either has all its elements in R or all its elements in $\neg R$.

Lemma 1. *If $\mathfrak{A}$ is a recursive linear order with R a one-place relation recursive on $\mathfrak{A}$, the following are equivalent:*

(1) *There is a finite sequence $\bar{a} \subseteq A$ and a quantifier-free formula $\theta(\bar{a}, x)$ such that the following equivalence holds in $\mathfrak{A}$:*

$$R(x) \leftrightarrow \theta(\bar{a}, x).$$

(2) *Every isomorphism from $\mathfrak{A}$ to a recursive linear order carries R to a recursive relation.*

Proof. We prove that (2) implies (1) by means of Theorem 1. (The converse is obvious.) The function f from N^2 into N is defined as follows. For $\langle m, s\rangle \in \mathsf{N}^2$, if $m \geqq s$ define $f\langle m, s\rangle = s$. If $m < s$, let $c_0, \ldots, c_m$ be the ordering of A_m in $\mathfrak{A}$ and take c_{-1} and c_{m+1} to be $-\infty$ and ∞ respectively. For $i \in \{0, \ldots, m+1\}$ and $t \geqq s$ we say the interval (c_{i-1}, c_i) is *large in* $\mathfrak{A}_t$ if $(c_{i-1}, c_i) \cap A_t \cap R$ is non-empty. Define $f\langle m, s\rangle$ to be the least $t < s$ for which either

(i) there is an $i \in \{0, \ldots, m+1\}$ with (c_{i-1}, c_i) not large in $\mathfrak{A}_s$ but large in $\mathfrak{A}_t$ or

(ii) there is an $i \in \{0, \ldots, m+1\}$ with (c_{i-1}, c_i) large in $\mathfrak{A}_s$ and with $(c_{i-1}, c_i) \cap A_t$ containing at least s elements of $A - R$.

Under the assumption that statement (1) is false it is evident that for every $m \in \mathsf{N}$ there is an $i \in \{0, \ldots, m+1\}$ such that (c_{i-1}, c_i) contains an infinite number of elements of $A - R$. It follows from this that the function f defined above satisfies the hypothesis of Theorem 1, thus proving our result.

Before we prove the general case we need a few definitions.

By an *arrangement* of the set $x_1, \ldots, x_n$ we mean a finite conjunction

$$\psi = \theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_{n-1},$$

where $y_1, \ldots, y_n$ is some permutation of $x_1, \ldots, x_n$ and each formula θ_i is either $y_i = y_{i+1}$ or $y_i < y_{i+1}$. It is clear that there are only a finite number of distinct (in the theory of linear order) arrangements of the finite set $x_1, \ldots, x_n$. (These formulae are discussed more fully in Moses [5].)

Consider an n-ary relation $R(x_1, \ldots, x_n)$ on $\mathfrak{A}$. For $j \in \{1, \ldots, n\}$ and $a \in A$ we define a relation R_a^j as follows:

$$R_a^j(x_1, \ldots, x_n) \leftrightarrow R(x_1, \ldots, x_{j-1}, a, x_{j+1}, \ldots, x_n).$$

Thus R_a^j is an $(n - 1)$-ary relation. Notice also that if R is intrinsically recursive on $\mathfrak{A}$, so is R_a^j. (For any isomorphism f from $\mathfrak{A}$ to $\mathfrak{B}$, we may 'choose' the element $f(a)$.) This does not mean that the relations R_a^j for $a \in A$ are intrinsically uniformly recursive. It just means that each R_a^j (and therefore any finite number of them) is intrinsically recursive.

Theorem 2. *If $\mathfrak{A}$ is a recursive linear order and R a relation recursive on $\mathfrak{A}$, the following are equivalent:*

(1) *There is a finite sequence $\bar{a} \subseteqq A$ and a quantifier-free formula $\theta(\bar{a}, \bar{x})$ such that the following equivalence holds in $\mathfrak{A}$:*

$$R(\bar{x}) \leftrightarrow \theta(\bar{a}, \bar{x}).$$

(2) *Every isomorphism from $\mathfrak{A}$ to a recursive linear order carries R to a recursive relation.*

Proof. Once again it is obvious that (1) implies (2). We prove the converse by induction on the number of arguments of R. Lemma 1 is the first step of the induction. We assume that Theorem 2 is true for every relation with less than n arguments and that statement (2) of the theorem is true for a particular n-ary relation R. We argue from these assumptions that statement (1) of our theorem is also true for this R; thus proving the theorem.

The tactic of the proof is as follows: We define a function f from N^2 into N. We show that *if this function is defined on all of* N^2 it then satisfies the hypothesis of Theorem 1 for (at least) one of the relations R, $\neg R$. Under the assumption that R is intrinsically recursive this implies that f is not defined on all of N^2. We argue from this and the inductive hypothesis that statement (1) of our theorem is true.

The function f from N^2 into N is defined as follows. For $\langle m, s\rangle \in \mathsf{N}^2$, if $m \geqq s$ define $f\langle m, s\rangle = s$. If $m < s$, let $\bar{x}$ be a sequence of n variables $\psi_1, \ldots, \psi_r$ the (finite number of) distinct arrangements of the set $A_m \cup \bar{x}$. For $i \in \{1, \ldots, r\}$ and $t \geqq s$ we say the arrangement ψ_i is *large in* $\mathfrak{A}_t$ if there are sequences $\bar{a}$ and $\bar{b}$ in A_t such that

$$\mathfrak{A} \models R(\bar{a}) \wedge \neg R(\bar{b}) \wedge \psi_i(A_m, \bar{a}) \wedge \psi_i(A_m, \bar{b}).$$

Define $f\langle m, s\rangle$ to be the least $t > s$ for which either

(i) there is an $i \in \{1, \ldots, r\}$ with ψ_i not large in $\mathfrak{A}_s$ but large in $\mathfrak{A}_t$

or

(ii) there is an $i \in \{1, \ldots, r\}$ with ψ_i large in $\mathfrak{A}_s$ and there is in A_t a sequence $\bar{c}$ satisfying the following two properties:

(a) $\mathfrak{A} \models \psi_i(A_m, \bar{c})$.

(b) For every element c_j of $\bar{c}$ there are in $\mathfrak{A}_t$ elements d_1, d_2 on either side of c_j such that the intervals (d_1, c_j) and (c_j, d_2) each contain at least s elements of A_t and no elements of $A_m \cup \bar{c}$.

Consider the condition (ii). Let ψ_i be an arrangement of $A_m \cup \bar{x}$ satisfying (ii) and let $\bar{a}$ be a sequence in A_s such that $\mathfrak{A} \models \psi_i(A_m, \bar{a})$. Property (b) ensures that there is an embedding of $\mathfrak{A}_s$ into $\mathfrak{A}_t$ that is the identity on A_m and that maps the sequence $\bar{a}$ to the sequence $\bar{c}$. We shall use this fact in the following argument.

Notice that if $f\langle m, s\rangle$ exists, we can find it effectively. If f is defined on all of N^2 it is therefore a recursive function from N^2 into N. We show that it also satisfies the rest of the hypothesis of Theorem 1. Consider any $m \in \mathsf{N}$. Notice that if an arrangement ψ_i of $A_m \cup \bar{x}$ is large in $\mathfrak{A}_s$ it is also large in $\mathfrak{A}_{s+1}$. It follows that $f\langle m, s\rangle$ is defined via (i) for only finitely many s and therefore via (ii) for infinitely many s. This implies that there is a particular arrangement ψ_i of $A_m \cup \bar{x}$ that satisfies (ii) for infinitely many s (and $t = f\langle m, s\rangle$). Consider the sequence $\bar{c}$ defined on these occasions. For each s this sequence is either in R or in $\neg R$. It follows that there are infinitely many s for which ψ_i satisfies (ii) and $\bar{c}$ is in R, or that there are infinitely many s for which ψ_i satisfies (ii) and $\bar{c}$ is in $\neg R$. Without loss of generality, we assume the latter. Consider any sequence $\bar{a} \subseteq A$ such that

$$\mathfrak{A} \models \psi_i(A_m, \bar{a}) \wedge R(\bar{a}).$$

There is such a sequence, as ψ_i is large in $\mathfrak{A}$. It follows that for infinitely many s there is an embedding $\varphi\colon \mathfrak{A}_s \to \mathfrak{A}_{f\langle m,s\rangle}$ such that φ is the identity on A_m and $\varphi(\bar{a})$ is in $\neg R$. Thus f satisfies the hypothesis of Theorem 1 for the relation R. Since R is intrinsically recursive we have therefore proved that the function f is not defined on all of N^2. We argue from this and the inductive hypothesis that statement (1) of our theorem is also true.

Let $f\langle m, s\rangle$ be undefined. Consider the arrangements $\psi_1, \ldots, \psi_r$ of $A_m \cup \bar{x}$. Evidently the following equivalence holds in $\mathfrak{A}$:

$$R \leftrightarrow (R \wedge \psi_1) \vee \ldots \vee (R \wedge \psi_r).$$

We show that R is equivalent to a quantifier-free formula with a finite number of parameters by showing this to be true of each of the formulae $R \wedge \psi_i$. This is evident for those arrangements that are not large in $\mathfrak{A}$. Consider now an arrangement ψ_i that is large in $\mathfrak{A}$. Since $f\langle m, s\rangle$ is undefined, this means that ψ_i is already large in $\mathfrak{A}_s$ but that there is no integer t satisfying the rest of condition (ii). It follows that for some x_j in $\bar{x}$ there are only finitely many points $d_1, \ldots, d_l$ that are related to the elements of A_m in the same way that x_j is in the arrangement ψ_i. (If not, a sequence $\bar{c}$ satisfying (ii) would exist in A and therefore in some A_t.) Consider the $(n-1)$-ary relations $R^j_{d_1}, \ldots, R^j_{d_l}$. The following equivalence holds in $\mathfrak{A}$.

$$R(\bar{x}) \wedge \psi_i(\bar{x}) \leftrightarrow \psi_i(\bar{x}) \wedge \Big(\bigvee_{k=1}^{l} (x_j = d_k) \wedge R^j_{d_k}(\bar{x})\Big).$$

R is intrinsically recursive on $\mathfrak{A}$ and therefore so are $R^j_{d_1}, \ldots, R^j_{d_l}$. By the inductive hypothesis each of these relations is therefore equivalent to a quantifier-free formula with finitely many parameters. $\psi_i(\bar{x})$ is a quantifier-free formula with finitely many parameters. It follows that R satisfies statement (1) of Theorem 2.

Theorem 2 is easily applicable to particular relations. The successivity relation $S(x, y)$ is equivalent in $\mathfrak{A}$ to a quantifier-free formula with finitely many parameters if and only if $\mathfrak{A}$ has finitely many successivities. Thus Theorem 2 in this case reads:

Corollary 1. *If $\mathfrak{A}$ is a recursive linear order with the successivity relation $S(x, y)$ recursive, the following are equivalent:*

(1) *$\mathfrak{A}$ has finitely many successivities.*

(2) *$S(x, y)$ is intrinsically recursive on $\mathfrak{A}$.*

It is clear from this, that $S(x, y)$ is not intrinsically recursive on ω. Notice that since $S(x, y)$ is equivalent to the formula $\forall z \neg((x < z < y) \overset{*}{\vee} (y < z < x))$, every decidable linear order must have $S(x, y)$ recursive. Also, the standard elimination of quantifiers argument applied to dense linear orders (for example see LANGFORD [3]), shows that any recursive dense linear order and therefore any recursive linear order with finitely many successivities, is decidable. We may therefore add to Corollary 1 the following equivalent statement.

(3) *Every recursive linear order isomorphic to $\mathfrak{A}$ is decidable.*

Theorem 2 applied to the block relation $B(x, y)$ produces:

Corollary 2. *If $\mathfrak{A}$ is a recursive linear order with the block relation $B(x, y)$ recursive, the following are equivalent:*

(1) *$\mathfrak{A}$ can be partitioned, by a finite number of points, into intervals each of which is of order type η, ω or ω^*.*

(2) *$B(x, y)$ is intrinsically recursive on $\mathfrak{A}$.*

The result that $B(x, y)$ is not intrinsically recursive on $\omega + \omega^*$ is clearly implied by this.

One-place relations are especially easy to analyse. As was mentioned previously, Lemma 1 says that a one-place relation R is intrinsically recursive on $\mathfrak{A}$ if and only if $\mathfrak{A}$ can be partitioned, by a finite number of points, into intervals each of which either has all its elements in R or all its elements in $\neg R$. It follows that *a one-place-relation R is intrinsically recursive on ω if and only if either R or $\neg R$ is finite.*

In [1] ASH and NERODE prove that with an extra decidability assumption "intrinsically recursive" is equivalent to "formally recursive". A relation R is defined to be formally recursive if both it and its complement are equivalent to r.e. disjunctions $\bigvee_{n<\omega} \varphi_n(\bar{a}, \bar{x})$ of existential formulae with finitely many parameters. It can be shown that in linear orders, a relation is formally recursive if and only if it is equivalent to a quantifier-free formula with finitely many parameters. The argument hinges on the fact (discussed in [5]) that in linear orders, an existential formula $\varphi(\bar{a}, \bar{x})$ is equivalent to a finite conjunction of formulae representing some of the following statements (for y_i, y_j elements of $\bar{a} \cup \bar{x}$):

(i) $y_i = y_j$.

(ii) $y_i < y_j$.

(iii) There are (at least) n elements less than y_i.

(iv) There are (at least) n elements greater than y_i.

(v) There are (at least) n elements between y_i and y_j.

The "at least" in the last three statements allows us to show that an infinite disjunction of existential formulae is equivalent to a single existential formula. A similar argument proves that if R and $\neg R$ are equivalent to existential formulae with finitely many parameters, then R is equivalent to a quantifier-free formula with finitely many parameters. Thus Theorem 2 shows that in this case, the ASH and NERODE theorem is true without the extra decidability assumption.

References

[1] Ash, C., and A. Nerode, Intrinsically recursive relations. In: Aspects of effective algebra (J. N. Crossley, ed.), Upside Down A Book Co., Steel's Creek, Australia 1981, pp. 26–41.
[2] Goncharov, S. S., On the number of nonautoequivalent constructivizations. Algebra and Logic **16** (1977), 169–185.
[3] Langford, C. H., Some theorems on deducibility. Ann. Math., Ser. 2, 28 (1927), 16–40.
[4] Manasse, M., Notes on a theorem of Ash and Nerode. Presented at the Math. Soc. Summer Research Institute on Recursion Theory, Cornell University 1982.
[5] Moses, M., Recursive linear orders with recursive successivities. Ann. Pure and Applied Logic **27** (1984), 253–264.

Michael Moses
Department of Mathematics
Western Illinois University
Macomb, Illinois 61455
U.S.A.

(Eingegangen am 8. August 1985)