On enumeration of seating arrangements of couples around a circular table

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Famous *ménage problem* asks for the number of seating arrangements of $n$ married couples of opposite sex such that

1. no spouses seat next to each other;
2. females and males alternate.

(We assume the labeled case: couples and seat are all labeled.)

The problem was formulated by Edouard Lucas in 1891. A complete solution was obtained by Touchard in 1934.
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The *ternary ménage problem*:

1. no spouses seat next to each other;
2. no TRIPLE of adjacent people are of the same sex.

Posed by Hugo Pfoertner in 2006.
Solution #1: Ladies First

A straightforward approach to the ménage problem is first to seat all ladies (in $2 \cdot n!$ ways) and then to seat all gentlemen, obeying the couple restriction. This way the problem reduces to enumerating placements of non-attacking rooks on a board like

Using the rook theory, this leads to the Touchard formula:

$$M_n = 2 \cdot n! \cdot \sum_{k=0}^{n} (-1)^k \frac{2n}{2n-k} \binom{2n-k}{k} (n-k)!.$$
The seating arrangements satisfying the ménage problem correspond to directed Hamiltonian cycles in the crown graph (obtained from the complete bipartite graph $K_{n,n}$ with removal of a perfect matching):

Crown graphs represent a particular case of circulant graphs. There is a general formula for the number of Hamiltonian cycles in circulant graphs (Golin and Leung, 2004).
Solution #3: Non-Sexist Inclusion-Exclusion

Following Bogart and Doyle (1986), let us count the number of alternating male-female seating arrangements of \( n \) couples with no \emph{close couples} (i.e., spouses seating next to each other). By inclusion-exclusion,

\[
M_n = 2 \cdot \sum_{j=0}^{n} (-1)^n \cdot \binom{n}{j} \cdot (n-j)!^2 \cdot \frac{2n}{2n-j} \binom{2n-j}{j} \cdot j!
\]

\[
= 2 \cdot n! \cdot \sum_{j=0}^{n} (-1)^n \cdot (n-j)! \cdot \frac{2n}{2n-j} \binom{2n-j}{j}.
\]

- factor 2 accounts for two ways to reserve seats for males/females;
- \( j \) stands for the number of close couples;
- \( \binom{n}{j} \) is the number of ways to select \( j \) out of \( n \) couples (to be close);
- \( \frac{2n}{2n-j} \binom{2n-j}{j} \) is the number of ways to select \( 2j \) seats for \( j \) close couples;
- \( j! \) is the number of seating arrangements of the \( j \) close couples at \( 2j \) selected seats;
- \( (n-j)!^2 = (n-j)! \cdot (n-j)! \) is the number of ways to seat females and males from the \( n-j \) non-selected couples.
Solution #3: Selecting Seats for Close Couples

\[ \frac{2n}{2n-j} \binom{2n-j}{j} \] is the number of ways to select \(2j\) seats for \(j\) close couples.

**Proof.** View seats as points on a circle so that adjacent seats connected by arcs. Then we need to select \(j\) disjoint arcs out of total \(2n\) arcs.

We proceed with the “stars and bars” approach. Fix any arc \(a\). There are two cases:

1. **Arc \(a\) is not selected.** Imagine that we contracted the \(j\) arcs (to be selected), resulting in a circle with \(2n - j\) points with one arc being \(a\). There are \(\binom{2n-j}{j}\) ways to select \(j\) out of \(2n - j\) points for uncontraction.

2. **Arc \(a\) is selected.** Similarly, we get \(\binom{2n-j-1}{j-1}\) ways to select the remaining \(j-1\) out of \(2n - j - 1\) points for uncontraction.

Thus, there are

\[ \binom{2n-j}{j} + \binom{2n-j-1}{j-1} = \binom{2n-j}{j} + \frac{j}{2n-j} \binom{2n-j}{j} = \frac{2n}{2n-j} \binom{2n-j}{j} \]

total ways to to select \(2j\) seats for \(j\) close couples.
Ternary Ménage Problem

Our goal is to find the number of seating arrangements of $n$ couples such that

1. there are no close couples;
2. no triple of adjacent people are of the same sex.

The described approaches do not seem to extend to this ternary ménage problem, since there is no nice male-female alternating structure anymore. In particular:

1. the ladies-first approach does not reduce the problem to an uniform board;
2. there is no obvious reduction to a Hamiltonian cycle problem;
3. the inclusion-exclusion approach is most prominent, but it is unclear what should be in place of $\frac{2n}{2n-j} \binom{2n-j}{j}$. 

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So far, a seating arrangement was viewed as a cyclic sequence of people:

\[ f_{i_1} \rightarrow m_{j_1} \rightarrow f_{i_2} \rightarrow m_{j_2} \rightarrow \cdots \rightarrow f_{i_n} \rightarrow m_{j_n} \rightarrow f_{i_1} \cdot \]

However, it can also be viewed as a cyclic sequence of *pairs* of people seating next to each other:

\[(f_{i_1}, m_{j_1}) \rightarrow (m_{j_1}, f_{i_2}) \rightarrow (f_{i_2}, m_{j_2}) \rightarrow \cdots \rightarrow (f_{i_n}, m_{j_n}) \rightarrow (m_{j_n}, f_{i_1}) \rightarrow (f_{i_1}, m_{j_1}) \cdot \]
De Bruijn Graph Perspective

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\[(f_{i_1}, m_{j_1}) \to (m_{j_1}, f_{i_2}) \to (f_{i_2}, m_{j_2}) \to \cdots \to (f_{i_n}, m_{j_n}) \to (m_{j_n}, f_{i_1}) \to (f_{i_1}, m_{j_1})\].

The same idea was used by Nicolaas de Bruijn to construct a sequence, which contains every subsequence of a fixed length $k$. He introduced graphs, now named after him, whose nodes represent $(k-1)$-mers and edges represent $k$-mers of letters (an edge corresponding to a $k$-mer $s$ connects prefix of $s$ with suffix of $s$).

We will employ de Bruijn graphs for $k = 3$ for solving the ménage problem. However, in contrast to conventional unweighted de Bruijn graphs, we will use algebraic weights to account for

- the balance between females and males; and
- the number of close couples.
The starred nodes $fm^*$ and $mf^*$ stand for close couples.
The indeterminate $y$ accounts for the males-females balance.
The indeterminate $z$ accounts for the number of close couples.
Any seating arrangement corresponds to a cyclic sequence of type (nodes) \( fm \) and \( mf \), some of which may be starred to indicate close couples.

Such sequence with \( j \) close couples corresponds to a cycle (starting/ending at the same node) of length \( 2n \) and algebraic weight \( y^0z^j \).

The number of such cycles equals by the coefficient of \( y^0z^j \) in \( \text{tr}(A^{2n}) \).

\[
A = \begin{bmatrix}
0 & y^{-1} & y^{-1}z & 0 \\
y & 0 & 0 & yz \\
y & 0 & 0 & 0 \\
0 & y^{-1} & 0 & 0 
\end{bmatrix}
\]
So, we get a matrix inclusion-exclusion formula for $M_n$:

$$M_n = n! \cdot \sum_{j=0}^{n} (-1)^n \cdot (n-j)! \cdot [y^0 z^j] \text{tr}(A^{2n}).$$

It can be verified that

$$[y^0 z^j] \text{tr}(A^{2n}) = 2 \cdot \frac{2n}{2n-j} \binom{2n-j}{j}$$

as expected.

This matrix formula can be generalized for the ternary ménage problem.
De Bruijn Graph for Ternary Ménage Problem

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Let $B$ be the adjacency matrix for the de Bruijn graph:

$$B = \begin{bmatrix}
0 & y^{-1} & 0 & y & 0 & 0 \\
yz & 0 & y^{-1} & 0 & y & 0 \\
yz & 0 & 0 & 0 & y & 0 \\
0 & y^{-1} & 0 & 0 & 0 & y^{-1}z \\
0 & y^{-1} & 0 & y & 0 & y^{-1}z \\
0 & 0 & y^{-1} & 0 & y & 0
\end{bmatrix}$$

The number $T_n$ of seating arrangements for the ternary ménage problem is

$$T_n = n! \cdot \sum_{j=0}^{n} (-1)^n \cdot (n-j)! \cdot [y^0 z^j] \operatorname{tr}(B^{2n}).$$
The ménage problem is represented by the following sequences in the Online Encyclopedia of Integer Sequences (http://oeis.org):

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Terms for $n = 1, 2, \ldots$</th>
<th>OEIS index</th>
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<td>$T_n$</td>
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<td>A258338</td>
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</tbody>
</table>
Acknowledgements

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Online Encyclopedia of Integer Sequences (OEIS)
Website: http://oeis.org

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