An efficient backtracking method for solving a system of linear equations over a finite set with application for construction of magic squares

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Melencolia I by Albrecht Dürer, 1514, features a magic square with the magic sum 34.

Melencolia I, 1514
Normal magic square is an $n \times n$ table filled with integers 1, 2, \ldots, $n^2$ such that the sum in each row, column, and diagonal equals the same number, called the *magic sum*.

\[
\begin{array}{cccc}
7 & 12 & 1 & 14 \\
2 & 13 & 8 & 11 \\
16 & 3 & 10 & 5 \\
9 & 6 & 15 & 4 \\
\end{array}
\]

\[\rightarrow 34\]

\[\rightarrow 34\]

\[\rightarrow 34\]

\[\rightarrow 34\]

34

34

34

34

34

34

34

34

34

34

34

34

34

34

34

34

In *non-normal* magic squares, elements can be any distinct numbers with some specified properties (e.g., prime numbers, squares, Smith numbers, etc.).
In 1987, Martin Gardner challenged his readers with the construction of a $3 \times 3$ magic square composed of 9 consecutive prime numbers and even offered $100 for such a square.
In 1987, Martin Gardner challenged his readers with the construction of a $3 \times 3$ magic square composed of 9 consecutive prime numbers and even offered $100 for such a square.

Such square was first constructed by H. L. Nelson, who collected the prize.

<table>
<thead>
<tr>
<th>1480028201</th>
<th>1480028129</th>
<th>1480028183</th>
</tr>
</thead>
<tbody>
<tr>
<td>1480028153</td>
<td>1480028171</td>
<td>1480028189</td>
</tr>
<tr>
<td>1480028159</td>
<td>1480028213</td>
<td>1480028141</td>
</tr>
</tbody>
</table>

This is the smallest such square and its magic sum is 4440084513.
Magic Squares Composed of Any Primes

Magic squares composed of *distinct* prime numbers with the smallest possible magic sum $S$:

<table>
<thead>
<tr>
<th>3 × 3</th>
<th>4 × 4</th>
<th>5 × 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>17</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>89</td>
<td>37</td>
<td>19</td>
</tr>
<tr>
<td>71</td>
<td>61</td>
<td>61</td>
</tr>
<tr>
<td>113</td>
<td>19</td>
<td>19</td>
</tr>
<tr>
<td>59</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>47</td>
<td>43</td>
<td>43</td>
</tr>
<tr>
<td>29</td>
<td>41</td>
<td>41</td>
</tr>
<tr>
<td>101</td>
<td>23</td>
<td>23</td>
</tr>
</tbody>
</table>

$S = 177$ | $S = 120$ | $S = 233$

Minimal magic sums of $n \times n$ magic squares form sequence A164843 in the Online Encyclopedia of Integer Sequences: [http://oeis.org/A164843](http://oeis.org/A164843)

How one would construct such magic squares?

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Solving a system of linear equations over a finite set
Suppose we want to construct a $4 \times 4$ magic square composed of distinct primes. Let us first fix a set of elements of our square. The first 16 odd primes do not work as their sum is not a multiple of 4. Instead let us take primes from $p_2 = 3$ to $p_{18} = 61$ but skipping the prime $p_{17} = 59$:

\[ \{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 61\}. \]

Their sum is 440, so the magic sum of the corresponding square would be $440/4 = 110$. 

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Solving a system of linear equations over a finite set
Now, we start filling out the cells of a 4 × 4 square: 

\[
\begin{array}{cccc}
3 & 5 & 7 & 11 \\
13 & 17 & 19 & 23 \\
29 & 31 & 37 & 41 \\
43 & 47 & 53 & 61 \\
\end{array}
\] → 

\[
\begin{array}{cccc}
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } \\
\end{array}
\] 

\[S = 110\]
Now, we start filling out the cells of a $4 \times 4$ square:

\[
\begin{array}{cccc}
\emptyset & 5 & 7 & 11 \\
13 & 17 & 19 & 23 \\
29 & 31 & 37 & 41 \\
43 & 47 & 53 & 61 \\
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc}
3 & & & \\
& & & \\
& & & \\
& & & \\
\end{array}
\]

\[S = 110\]
Filling Cells with Primes

Now, we start filling out the cells of a $4 \times 4$ square:

\[
\begin{array}{cccc}
\empty & \empty & 7 & 11 \\
13 & 17 & 19 & 23 \\
29 & 31 & 37 & 41 \\
43 & 47 & 53 & 61 \\
\end{array}
\]

\[
\begin{array}{cc}
3 & 5 \\
\end{array}
\]

\[S = 110\]
Now, we start filling out the cells of a $4 \times 4$ square:

\[
\begin{array}{cccc}
\varnothing & \varnothing & \checkmark & 11 \\
13 & 17 & 19 & 23 \\
29 & 31 & 37 & 41 \\
43 & 47 & 53 & 61 \\
\end{array}
\rightarrow
\begin{array}{ccc}
3 & 5 & 7 \\
\end{array}
\]

\[S = 110\]
Now, we start filling out the cells of a $4 \times 4$ square:

\[
\begin{array}{cccc}
\alpha & \alpha & \alpha & 11 \\
13 & 17 & 19 & 23 \\
29 & 31 & 37 & 41 \\
43 & 47 & 53 & 61 \\
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc}
3 & 5 & 7 & 95 \\
\hline
\hline
\end{array}
\]

\[S = 110\]

It is time to use the known magic sum $S$ and compute the next element as

\[110 - 3 - 5 - 7 = 95\]

but this number is not in our set, meaning a deadend. So we need to change elements in the previous cells.
Now, we start filling out the cells of a $4 \times 4$ square:

\[
\begin{array}{cccc}
\emptyset & \emptyset & 7 & \emptyset \\
13 & 17 & 19 & 23 \\
29 & 31 & 37 & 41 \\
43 & 47 & 53 & 61 \\
\end{array}
\]

\[
\begin{array}{cccc}
3 & 5 & 11 & 91 \\
S = 110 \\
\end{array}
\]

\[110 - 3 - 5 - 11 = 91\]

Another deadend!
Now, we start filling out the cells of a $4 \times 4$ square:

\[
\begin{array}{cccc}
3 & 5 & 41 & 61 \\
3 & 5 & 41 & 61 \\
13 & 17 & 19 & 23 \\
29 & 31 & 37 & 41 \\
43 & 47 & 53 & 61 \\
\end{array}
\]

After some more trials, we get

\[110 - 3 - 5 - 41 = 61\]

and it is an available element from our set!
Now, we start filling out the cells of a $4 \times 4$ square:

\[
\begin{array}{cccc}
8 & 5 & 7 & 11 \\
17 & 19 & 23 & \\
29 & 31 & 37 & 41 \\
43 & 47 & 53 & 61 \\
\end{array}
\quad \rightarrow \quad
\begin{array}{cccc}
3 & 5 & 41 & \\
7 & 11 & 13 & 79 \\
\end{array}
\]

\[S = 110\]

\[110 - 7 - 11 - 13 = 79\]
Now, we start filling out the cells of a $4 \times 4$ square:

\[
\begin{array}{cccc}
3 & 5 & 41 & 61 \\
7 & 13 & 37 & 53 \\
\end{array}
\]

\[
S = 110
\]

\[
110 - 7 - 13 - 37 = 53
\]
Now, we start filling out the cells of a $4 \times 4$ square:

\[
\begin{array}{cccc}
3 & 5 & 41 & 61 \\
7 & 13 & 37 & 53 \\
11 & \text{ } & \text{ } & \text{ } \\
89 & \text{ } & \text{ } & \text{ } \\
\end{array}
\]

\[S = 110\]

\[110 - 3 - 7 - 11 = 89\]

and so on...
In fact, our algorithm would not be able to construct a $4 \times 4$ magic square from the set of primes

$$\{3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 61\}.$$  

However when completed, our algorithm would \textit{prove} that no such magic square exists!

Trying different sets of primes, we may have better luck.

(Hint: From an earlier slide, one may recall that the smallest possible magic sum for a $4 \times 4$ prime magic square is $S = 120$.)
Let us analyze the construction algorithm. It clearly relies on the magic sums appearing in the square and tries to compute the next element (rather than brute force it) whenever possible. The order in which the algorithm determines elements of the square is following:

<table>
<thead>
<tr>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_5$</td>
<td>$x_6$</td>
<td>$x_7$</td>
<td>$x_8$</td>
</tr>
<tr>
<td>$x_9$</td>
<td>$x_{11}$</td>
<td>$x_{13}$</td>
<td>$x_{14}$</td>
</tr>
<tr>
<td>$x_{10}$</td>
<td>$x_{12}$</td>
<td>$x_{15}$</td>
<td>$x_{16}$</td>
</tr>
</tbody>
</table>

The elements that are computed are colored green.

This looks like a rather efficient algorithm. In fact, this is not quite so.
First issue with our algorithm is that we did not take all magic sums into account. It can be seen that in any $4 \times 4$ magic square the sums in the cells labeled with the same letter below must also be equal to $S$:

\[
\begin{array}{cccc}
C & A & A & C \\
B & I & I & B \\
B & I & I & B \\
C & A & A & C \\
\end{array}
\]

and thus the order of cells in our algorithm can be improved to:

\[
\begin{array}{cccc}
\times_1 & \times_2 & \times_3 & \times_4 \\
\times_5 & \times_6 & \times_7 & \times_8 \\
\times_9 & \times_{14} & \times_{13} & \times_{11} \\
\times_{10} & \times_{16} & \times_{16} & \times_{12} \\
\end{array}
\]

The number of dependent cells here is 9 and this is the maximum possible (can be easily verified with linear algebra).
Issue 2: Order of Cells

Let us take a close look at the cells order in which the improved algorithm determines their values:

\[
\begin{array}{cccccccccccccc}
& x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\
\end{array}
\]
Let us take a close look at the cells order in which the improved algorithm determines their values:

Now consider a different order:

that is

Is this new order any better?
Comparing the two orders, we notice that the latter has dependent (green) cells more closely packed towards the beginning. Since each green cell means a potential deadend (which we earlier colored red) for the currently selected values, the earlier we encounter green cells, the better!

But even if we ignore possible deadends, the first order would iterate over

$$16 \cdot 15 \cdot 14 \cdot 12 \cdot 11 \cdot 10 \cdot 8 = 35481600$$

values of the independent cells, while the second one would iterate over

$$16 \cdot 15 \cdot 14 \cdot 12 \cdot 10 \cdot 8 \cdot 4 = 12902400,$$

which is almost a 3-fold speed up.
The dependent cells are computed as follows:

\[
\begin{align*}
    x_4 &= S - x_1 - x_2 - x_3, \\
    x_6 &= x_2 + x_3 - x_5, \\
    x_8 &= S - x_1 - x_2 - x_3 + x_5 - x_7, \\
    x_{10} &= x_1 + x_2 + x_3 - x_5 - x_9, \\
    x_{11} &= S - x_1 - 2 \cdot x_2 - x_3 + x_5 - x_7 + x_9, \\
    x_{12} &= x_1 + x_2 - x_5 + x_7 - x_9, \\
    x_{14} &= S - x_7 - x_9 - x_{13}, \\
    x_{15} &= S - x_1 - x_5 - x_{13}, \\
    x_{16} &= -S + x_1 + x_5 + x_7 + x_9 + x_{13}.
\end{align*}
\]
Elements of an $n \times n$ magic square satisfy a homogeneous matrix equation (in other words, a system of linear equations) for certain matrix $M$:

$$M \cdot y = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \text{ where } y = \begin{bmatrix} y_1 \\ \vdots \\ y_{n^2} \end{bmatrix}.$$  

Its solution (over the integers) has the form $y = K \cdot z$, where matrix $K$ is formed by column-vectors of the basis of the integer kernel of $M$, and $z$ is any vector with integer components.
In order to effectively solve this matrix equation over a finite set (e.g., of prime numbers) with the backtracking algorithm, we want an ordering $y' = [y_{i_1}, y_{i_2}, \ldots, y_{i_{n^2}}]^T$ of $y$ that satisfies:

$$y' = B \cdot y', \quad \text{where } B \text{ is an } n^2 \times n^2 \text{ matrix such that:}$$

- some rows of matrix $B$ represent unit vectors (which correspond to independent components of $y'$);
- $B$ is a lower-triangular matrix (i.e., dependent components of $y'$ depend only on preceding components);
- ranks of $B_1, B_2, \ldots, B_{n^2}$, where $B_i$ is the submatrix of $B$ consisting of first $i$ rows, have the slowest possible growth.

I developed an algorithm for constructing such a solution (to be described elsewhere).
The finite set, in which we look for solutions, may have more than $n^2$ elements. In this case, the magic sum $S$ is not known in advance. For example, we can search for a $4 \times 4$ magic squares with elements in the set of all primes below 1,000.

We can search for magic squares with additional properties such as:

- **pandiagonal**, which have the sums along broken diagonals equal $S$ as well;
- **associative**, which have the same sum in each pair of elements symmetric about the square center;
- **ultramagic**, which are pandiagonal and associative at the same time;

Each of them is similarly reduced to a systems of linear equations, which we can solve over a chosen finite set.

Additionally we can take into account the square symmetry (e.g., its rotations and reflections) to speed up the computation.
In 1991, A. W. Johnson, Jr. constructed a smallest $6 \times 6$ pandiagonal magic square composed of 36 consecutive primes (from 67 to 251):

$$
\begin{array}{cccccc}
67 & 193 & 71 & 251 & 109 & 239 \\
139 & 233 & 113 & 181 & 157 & 107 \\
241 & 97 & 191 & 89 & 163 & 149 \\
73 & 167 & 131 & 229 & 151 & 179 \\
199 & 103 & 227 & 101 & 127 & 173 \\
211 & 137 & 197 & 79 & 223 & 83 \\
\end{array}
$$

*Journal of Recreational Mathematics, 23:3 (1991), 190-191*

However, existence of similar $4 \times 4$ and $5 \times 5$ pandiagonal magic squares was an open question until recently.

In 2013, J. Wroblewski found several $4 \times 4$ such squares, smallest of which has elements about $3.2 \cdot 10^{17}$. 
In late July 2014, I constructed a $4 \times 4$ pandiagonal magic square composed of 16 consecutive primes (starting with $170, 693, 941, 183, 817$) with the smallest possible magic sum:

```
<table>
<thead>
<tr>
<th>170693941183817</th>
<th>170693941183933</th>
<th>170693941183949</th>
<th>170693941183981</th>
</tr>
</thead>
<tbody>
<tr>
<td>170693941183979</td>
<td>170693941183951</td>
<td>170693941183847</td>
<td>170693941183903</td>
</tr>
<tr>
<td>170693941183891</td>
<td>170693941183859</td>
<td>170693941184023</td>
<td>170693941183907</td>
</tr>
<tr>
<td>170693941183993</td>
<td>170693941183937</td>
<td>170693941183861</td>
<td>170693941183889</td>
</tr>
</tbody>
</table>
```

A $5 \times 5$ pandiagonal magic squares composed of consecutive primes (smallest or not) is unknown but being searched for. Its elements should be greater than $10^{14}$. 
Acknowledgements

Natalia Makarova, Russia
Website: http://klassikpoez.narod.ru/glavnaja.htm
Wonderful World of Magic Squares (in Russian)

Online Encyclopedia of Integer Sequences (OEIS)
Website: http://oeis.org

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