The quantum Euler class and the quantum cohomology of the Grassmannians

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Abstract

The Poincaré duality of classical cohomology and the extension of this duality to quantum cohomology endows these rings with the structure of a Frobenius algebra. Any such algebra possesses a canonical "characteristic element;" in the classical case this is the Euler class, and in the quantum case this is a deformation of the classical Euler class which we call the "quantum Euler class." We prove that the characteristic element of a Frobenius algebra A is a unit if and only if A is semisimple, and then apply this result to the cases of the quantum cohomology of the finite complex Grassmannians, and to the quantum cohomology of hypersurfaces. In addition we show that, in the case of the Grassmannians, the [quantum] Euler class equals, as [quantum] cohomology element and up to sign, the determinant of the Hessian of the [quantum] Landau-Ginzbug potential.

1 Introduction

In [21], Witten's study of instantons in the context of supersymmetry of systems with deformed Hamiltonians gave rise to the notion of a deformed cohomology ring. This "quantum cohomology ring" has since then been formulated precisely in terms of Gromov-Witten invariants of symplectic manifolds (see [12] for details). Necessarily, much of the attention paid to quantum cohomology has been from the point of view of symplectic geometry, *e.g.* [15, 12]. There has also been a great deal of natural interest in the realm of algebraic geometry, *e.g.* [10, 7].

Nevertheless, there is strong motivation to pursue an approach which emphasizes and investigates the parallels between classical and quantum cohomology. The quantum cohomology ring of a manifold M is additively essentially the same as the classical cohomology ring of M, but possesses a multiplication which is a deformation of the classical cup product (see section 5). The strong analogy between the algebraic structures of these two rings is responsible for the fact that the Euler class has a quantum analogue which we refer to as the "quantum Euler class," defined in §1 below. We show here that this element of the quantum cohomology ring carries with it information about the semisimplicity, or lack therof, of the quantum cohomology ring.

The issue of semisimplicity of quantum cohomology rings has already been under investigation from other points of view, as in [9, 10]. In [9]), Dubrovin defines a **Frobenius manifold** M to be a manifold such that each fiber of the tangent bundle TM has a Frobenius algebra (FA) structure, which varies "nicely" from fiber to fiber. This context allows for a close investigation of the nature of the quantum deformations of classical cohomology, which is generally realized as T_0M, the tangent plane at "the origin" in M. Moreover, the fact that M is a Frobenius manifold is equivalent to the existence of a "Gromov-Witten potential" on M satisfying various differential equations, including the "WDVV" equations [*ibid*, p. 133]. Special manifolds, which Dubrovin calls **massive Frobenius manifolds**, have the additional property that for a generic point $t \in M$, the FA T_tM is semisimple. In this case, a variety of additional results relating to the classification of Frobenius manifolds hold [*ibid*, Lecture 3].

Kontsevich and Manin discuss aspects of Frobenius manifolds in [10], but deal with a different notion of semisimplicity. Working with a manifold M which is essentially the cohomology ring of some space, they define a particular section $K: M \to TM$ and, at each point $\gamma \in M$, the linear operator $B(\gamma): T_\gamma M \to T_\gamma M$ which is "multiplication by $K(\gamma)$." They also define a particular extension $\tilde{T}M$ of TM and show that if, over a subdomain of M, the operator $B(\gamma)$ is semisimple (*i.e.* has distinct eigenvalues), then $\tilde{T}M$ exhibits some special properties. The notion of semisimplicity of $B(\gamma)$ is referred to as "semisimplicity in the sense of Dubrovin" in [19] and other locations.

The quantum Euler class defined here also provides a section of the tangent bundle of a Frobenius manifold, although its exact connection with semisimplicity in the sense of Dubrovin is not yet clear.

The general structure and content of this article are as follows: The expository presentation of classical cohomology in §2 highlights the algebraic structures which are generalized and deformed in §§3 - 7. In particular, we offer a new canonical description of the Euler class e. The approach of §2 is extended to the general case of Frobenius algebras in §3, where the generalized analogue of the Euler class – "the characteristic element" – is shown to satisfy the following:

Theorem 3.4 *The characteristic element of a Frobenius algebra A is a unit if and only if A is semisimple.*

Strictly speaking, quantum cohomology should be viewed as a ring extension, and not an algebra. Section 4 provides the algebraic framework necessary to generalize the material of §3 to the case of a Frobenius extension (FE), *i.e.* when the base ring is not a field. This having been done, §5 sketches the elements of the definition of quantum cohomology, emphasizing its structure as a deformation of classical cohomology, and in particular as a FE. The "quantum Euler class" e_q, which is a deformation of e, is defined here to be the characteristic element of the FE structure of the quantum cohomology ring. Utilizing the material in §4, the semisimplicity test 3.4 can be applied to quantum cohomology rings.

In the classical and quantum cohomology rings of the complex Grassmannians, the Euler class and quantum Euler class take on additional significance. Section 6 outlines how these rings can be described as Jacobian algebras, where the ideal of relations is generated by the partial derivatives of the appropriate Landau-Ginzburg potential W (W_q in the quantum case). In this context we prove the following result:

Theorem 6.1 *The classical and quantum Euler classes are equal, up to sign, to the determinants of the Hessians of W and W_q, respectively.*

This connects the classical and quantum Euler classes to Morse-theoretic considerations regarding the functions W and W_q. In a sense, it brings them back to the roots of quantum cohomology in [21], which utilizes a Morse-theoretic approach. In addition, this result leads to a new proof of proposition 6.5 which, modulo technicalities, states that the quantum cohomology of any finite complex Grassmannian manifold is semisimple.

Finally, §7 applies the semisimplicity test 3.4 to the quantum cohomology of hyperplanes, providing good contrast to the situation for the Grassmannians.

2 Classical Cohomology and the Euler Class

Let X denote a connected K-oriented n-dimensional compact manifold, where n is even. Throughout this article, except where noted otherwise, homology and cohomology groups will use coefficients in a field K of characteristic 0. Denote by $[X] \in H_n(X)$ the fundamental orientation class of X, and let $\langle -,- \rangle: H^*(X) \otimes H_*(X) \to K$ denote the Kronecker index. The kernel of the linear form $\mu^*: H^*(X) \to K$, where μ denotes the generator of $H^n(X)$ satisfying $\langle \mu, [X] \rangle = 1$, contains no nontrivial ideals. This form can be used to define the "intersection form" $H^*(X) \otimes H^*(X) \to K$, by $a \otimes b \mapsto \mu * (a \cup b)$. The intersection form is nondegenerate.

Notice that we may view $H_*(X)$ as a (left) $H^*(X)$-module via the cap product $\cap: H^*(X) \otimes H_*(X) \to H_*(X)$. Viewing $H^*(X)$ as the regular (left) module over itself, we see that the Poincaré duality map

$$D: H^*(X) \to H_*(X), \quad \zeta \mapsto \langle -, \zeta \cap [X] \rangle$$

is an $H^*(X)$-module isomorphism.

Let $\Delta: X \to X \times X$ denote the diagonal map. The transfer map $\Delta^!: H^*(X) \to H^*(X) \otimes H^*(X)$ is defined to be the map which makes the following diagram commutative:

$$\begin{tikzcd} H^*(X) \arrow[r, "\Delta^!"] \arrow[d, "D"'] & H^*(X) \otimes H^*(X) \\ H_*(X) \arrow[r, "\Delta_*"] & H_*(X) \otimes H_*(X) \arrow[u, "D^{-1} \otimes D^{-1}"'] \end{tikzcd}$$

Here, we implicitly use the isomorphism $H_*(X \times X) \cong H_*(X) \otimes H_*(X)$, and the corresponding isomorphism for cohomology. Modulo this latter isomorphism, the cup-product in $H^*(X)$ is given by $\Delta^*: H^*(X) \otimes H^*(X) \to H^*(X)$.

Let $j: (X \times X, \emptyset) \to (X \times X, X \times X \setminus \Delta(X))$ denote inclusion of pairs. Consider the element $\tau := \Delta_!(1) = (D^{-1} \otimes D^{-1}) \circ \Delta_*([X])$. By the canonical isomorphism of the tangent bundle TX to the normal bundle of $\Delta(X)$ in $X \times X$ [14], this is just the image under j^* of the Thom class of TX. It follows that $\Delta^* \circ \Delta^!(1) \in H^*(X)$ is in fact the Euler class $e(X)$.

We recall the well known formula [14]

$$e(X) \;=\; \sum_i e_i e_i^{\#},$$

where e_i ranges over a basis for $H^*(X)$, and $e_j^{\#}$ ranges over the corresponding dual basis relative to the intersection form, *i.e.* $\mu^*(e_i \cup e_j^{\#}) = \delta_{ij}$.

3 Frobenius Algebras and the Characteristic Element

Let K be a field of characteristic 0 and let A be a finite-dimensional (as a vector space) commutative algebra over K, with unity 1_A. Let $\beta\colon A \otimes A \to A$ denote multiplication in A, and let $\bar{\beta}\colon A \to \mathrm{End}(A)$ denote the regular representation of A, *i.e.* $\bar{\beta}(a)$ is "multiplication by a." View A as the regular module over itself, and view the vector space dual A^* as an A-module via the action $A \otimes A^* \to A^*$ given by $a \otimes \zeta \mapsto a \cdot \zeta := \zeta \circ \bar{\beta}(a)$.

A is referred to as a **Frobenius algebra** (FA) if there exists an A-module isomorphism $\lambda\colon A \to A^*$, *i.e.* a nondegenerate pairing. In [8, pages 414-418] this is shown to be equivalent to the existence of a linear form $f\colon A \to K$ whose kernel contains no nontrivial ideals, and to the existence of a nondegenerate linear form $\eta\colon A \otimes A \to K$ which is associative, *i.e.* $\eta(ab \otimes c) = \eta(a \otimes bc)$. In fact, we may take $f := \lambda(1_A)$ and $\eta := f \circ \beta$, and we will henceforth presume that λ, f and η are related in this way. When it is useful to emphasize the FA structure of A endowed by particular f, η, and λ, the algebra A will be denoted by (A, f).

For the next result, view $A \otimes A$ as an A-module via the usual module action $\beta \otimes I\colon A \otimes A \otimes A \to A \otimes A$.

Theorem 3.1 *A finite dimensional commutative algebra A with 1_A is a FA if and only if it has a cocommutative comultiplication $\alpha\colon A \to A \otimes A$, with a counit, which is a map of A-modules.*

Proof. A complete proof appears in [1]. Here, we simply note that if A is a Frobenius algebra with pairing λ, then the comultiplication α is defined to be the map $(\lambda^{-1} \otimes \lambda^{-1}) \circ \beta^* \circ \lambda$:

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\lambda"'] & A \otimes A \\ A^* \arrow[r, "\beta^*"] & A^* \otimes A^* \arrow[u, "\lambda^{-1}\otimes\lambda^{-1}"'] \end{tikzcd}$$

■

Define the **characteristic element of** (A, f) to be the element $\omega_{A,f} := \beta \circ \alpha(1_A) \in A$. This is a canonical element which is shown in [1] to be of the form

$$\omega_{A,f} = \sum_i e_i e_i^{\#} ,$$

where e_i ranges over a basis for A and $e_j^{\#}$ ranges over the corresponding dual basis relative to η.

It is easy to show that theorem 3.1 still holds if "commutative" is replaced by "skew-commutative," as would be the case for $H^*(X)$. We see that in that case $f, \lambda, \alpha, \omega$ correspond to $\mu^*, D, \Delta^!, e(X)$, respectively.

Given FA's (A, f) and (B, g), we can form the **direct sum** $(A \oplus B, f \oplus g)$, where $A \oplus B$ denotes the "orthogonal direct sum" of algebras, and $f \oplus g$ acts by $f \oplus g(a \oplus b) := f(a) + g(b) \in K$. $(A \oplus B, f \oplus g)$ is in fact a FA [1].

Proposition 3.2 [1] *The characteristic element respects direct sum structure. Specifically,*

$$\omega_{A' \oplus A'', f' \oplus f''} = \omega_{A', f'} \oplus \omega_{A'', f''} \in A' \oplus A''$$

The minimal essential ideal $\mathcal{S} = \mathcal{S}(A)$ of a ring A is called the **socle**. When A is indecomposable, the socle is $\mathrm{ann}(\mathcal{N})$, where $\mathcal{N} = \mathcal{N}(A) \subset A$ is the ideal of nilpotents. See [3, §9] for details.

Proposition 3.3 *In a FA A, the ideal ωA is the socle of A.*

This result is independent of the choice of FA structure.

The construction in the following proof is essentially taken from Sawin [16], although this result does not explicitly appear there.

Proof. Because the socle of a finite-dimensional commutative algebra is the direct sum of the socles of its indecomposable constituents [3, §9], it suffices to prove this proposition for the indecomposable cases. Furthermore, we showed in [1] that the socle $\mathcal{S}$ of a FA is a principal ideal, any of whose elements is a generator, so it suffices to show that ω lies in the socle. Notice that ω is not 0; we have $f(\omega) = (A : K) \in K$, and this is not 0 in K, since K has characteristic 0.

If A is a field extension then $\mathcal{N}(A) = \{0\}$, so the socle $\mathcal{S} = \mathrm{ann}(\mathcal{N}) = A$. But ω is not zero, so it is a unit, and thus $\omega A = A = \mathcal{S}$.

If A is not a field extension, define a chain of ideals $\mathcal{S} = S_1 \subset S_2 \subset \cdots \subset S_n = A$, where each S_k is the preimage in A of the socle of A/S_{k-1}. Choose a basis for S_1. Now, starting with $i = 1$, iteratively take the basis for S_i and extend it to a basis for S_{i+1}. Denote the elements of the basis for $S_n = A$ by $e_1, \ldots, e_n$, and let $e_1^\#, \ldots, e_n^\#$ denote the corresponding dual basis elements. Suppose $e_i \in S_k \setminus S_{k-1}$ and that $a \in A$ is any nilpotent element. Then $ae_i \in S_{k-1}$, and therefore can be expressed as a linear combination of basis elements other than e_i. It follows that $f(ae_ie_i^\#) = 0$, so $e_ie_i^\#\mathcal{N}(A) \subset \operatorname{Ker} f$. But $\operatorname{Ker} f$ can contain no nontrivial ideals, as mentioned above, so we must have $e_ie_i^\#\mathcal{N}(A) = 0$, ie. $e_ie_i^\# \in \mathcal{S}$. This follows for each i, so $\omega = \sum_i e_ie_i^\# \in \mathcal{S}$. ■

Theorem 3.4 *The characteristic element ω of a FA A is a unit if and only if A is semisimple.*

Proof. First, recall from the proof of 3.3 that ω is not 0. Because A is commutative, it is semisimple if and only if it is a direct sum of fields. In such a case, the component of ω in each component of A is nonzero (each component is a FA [1]), and hence a unit. Since a direct sum of units is a unit, ω is a unit.

If some component A' of A is not a field, then it contains nontrivial nilpotents. In this case, $\mathcal{S}(A') = \operatorname{ann}(\mathcal{N}(A'))$ is nilpotent, so ω has a nilpotent component, and cannot be a unit. ■

In a skew-commutative context, such as $H^*(X)$, the characteristic element is not necessarily nonzero. For instance, if X is an odd-dimensional compact oriented manifold then the characteristic element, *i.e.* the Euler class, is 0. However, if the characteristic element is in fact nonzero, then 3.4 still holds.

4 Frobenius Extensions

Suppose A/R is a finite-dimensional (as a module) commutative ring extension with identity. By analogy with FA's, if there exists a module isomorphism $\lambda\colon A \to A^*$, we call A a **Frobenius extension** (FE). As in the case of FA's, this is equivalent to the existence of maps η and α. There is also a "FE form" $f := \lambda(1_A)\colon A \to R$, but in this context it is not sufficient for the kernel of f to contain no nontrivial ideals. The characteristic element $\omega_{A,f}$ may be defined as for FA's, but note that theorem 3.4 no longer applies. This section provides an approach for dealing with this circumstance.

Suppose $\theta\colon R \to S$ is a surjective homomorphism of rings (sending $1_R \mapsto 1_S$). Let (A, f) denote a FE, and define $B = \theta_*(A)$ to be $A \otimes_R S$. In this ring, we have $ra \otimes s = a \otimes \theta(r)s$ for all $r \in R$, $s \in S$, and $a \in A$. Let $\tilde{\theta}\colon A \to B$ denote the ring homomorphism $a \mapsto a \otimes 1_S$. Define the linear form $\bar{f}\colon B \to S$ by

$$\bar{f}(a \otimes s) := \theta \circ f(a)s.$$

The form $\bar{f}$ is well-defined, since

$$\bar{f}(ra \otimes s) = \theta \circ f(ra)s = \theta(rf(a))s = \theta(r)(\theta \circ f(a))s = \bar{f}(a \otimes \theta(r)s),$$

and $\bar{f}$ satisfies the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\tilde{\theta}"] \arrow[d, "f"] & B \arrow[d, "\bar{f}"] \\ R \arrow[r, "\theta"] & S \end{tikzcd}$$

Let $e_1, \ldots, e_n$ denote a basis for A, and let $e_i^\#, \ldots, e_n^\#$ denote the corresponding dual basis relative to η_A.

Proposition 4.1 *The form $\bar{f}$ endows $B = \theta_*(A)$ with a FE structure, and*

$$\omega_{B,\bar{f}} = \tilde{\theta}(\omega_{A,f}).$$

Proof. It suffices to show that the set $\{\tilde{\theta}(e_1), \ldots, \tilde{\theta}(e_n)\}$ is a basis for B, and that its dual basis relative to the form $B \otimes B \to S$, $a \otimes b \mapsto \bar{f}(ab)$ is $\{\tilde{\theta}(e_1^\#), \ldots, \tilde{\theta}(e_n^\#)\}$. The existence of a dual basis will show B is a FE. The particular form of the basis and dual basis, together with the fact that $\tilde{\theta}$ is a homomorphism, will prove the claim about $\omega_{B,\bar{f}}$.

We first prove the orthogonality relations:

$$\bar{f}\left(\tilde{\theta}(e_i)\tilde{\theta}(e_j^\#)\right) = \bar{f}\left(\tilde{\theta}(e_i e_i^\#)\right) = \theta \circ f(e_i e_i^\#) = \theta(\delta_{ij}) = \delta_{ij} \in S.$$

To prove that we have a basis as claimed, note that the elements $\tilde{\theta}(e_1), \ldots, \tilde{\theta}(e_n)$ clearly span B, since $\tilde{\theta}$ is surjective. Suppose that for some $\{s_i\} \in S$ we have $\sum_i s_i \tilde{\theta}(e_i) = 0$. Then, for all j,

$$0 = \bar{f}(0) = \bar{f}\left(\sum_i s_i \tilde{\theta}(e_i)\tilde{\theta}(e_j^\#)\right) = s_j.$$

It follows that $\tilde{\theta}(e_1), \ldots, \tilde{\theta}(e_n)$ are independent, and thus form a basis. The orthogonality relations show that $\{\tilde{\theta}(e_1^{\#}), \ldots, \tilde{\theta}(e_n^{\#})\}$ is a basis as well. ■

In the next result, let $\theta\colon R \to K$ be any surjective K-linear ring homomorphism, where K is a field.

Proposition 4.2 *The element $\omega_{A,f}$ is either a unit in A or a zero divisor.*

(i) *If $\omega_{A,f}$ is a unit in A then $B = \theta_*(A)$ is semisimple.*

(ii) *If $\omega_{A,f}$ is a zero divisor and $\operatorname{ann}(\omega_{A,f}) \not\subseteq \operatorname{Ker}\tilde{\theta}$, then $\theta_*(A)$ is not semisimple.*

Proof. If $\omega_{A,f}$ is a unit, then there exists a $u \in A$ such that $\omega_{A,f}u = 1_A$. But then, by 4.1

$$\omega_{B,\bar{f}}\tilde{\theta}(u) = \tilde{\theta}(\omega_{A,f})\tilde{\theta}(u) = \tilde{\theta}(1_A) = 1_B,$$

so $\omega_{B,f}$ is a unit as well.

All FE structures on A are given by $(A, f \circ \bar{\beta}(u))$, for some unit $u \in A$ [1, Proposition 2, *mutatis mutandis*]. Thus, if ω is not a unit in A, then the map $\omega_{A,f} \cdot f$ is not a FE form. This implies that there exists an $a \in A$ such that $f(\omega_{A,f}aA) = \omega_{A,f} \cdot f(aA) = \{0\}$. But f is a FE form, so it must be that $\omega_{A,f}a = 0$. If follows that $\tilde{\theta}(\omega_{A,f})\tilde{\theta}(a) = 0$. Since, by assumption, there exists some $a \in \operatorname{ann}(\omega_{A,f})$ such that $a \notin \operatorname{Ker}\tilde{\theta}$, we see that $\tilde{\theta}(\omega_{A,f}) = \omega_{B,\bar{f}}$ is a zero divisor as well. Both statements (i) and (ii) now follow from theorem 3.4. ■

5 Quantum Cohomology and the Quantum Euler Class

Let X be a $2n$-dimensional compact oriented manifold which, in addition, is symplectic, and let $H_2'(X)$ denote the free part of $H_2(X, \mathbb{Z})$. Taking $B_1, \ldots, B_n$ to denote a basis of $H_2'(X)$, the group algebra $\Lambda := K[H_2'(X)]$ may be expressed as $K[q^{B_1}, \ldots, q^{B_n}]$, where q is a formal variable and the addition of exponents is the group operation of $H_2'(X)$. This is essentially an algebraic version of the Novikov ring (see [12, §9.2]). As an additive group, the **quantum cohomology** ring $\mathbf{QH^*(X)}$ has the same structure as $H^*(X) \otimes \Lambda$, but has a "deformed" multiplication, which we describe briefly:

The classical cup product of two elements $a, b \in H^*(X)$ is given by

$$a \cup b = \sum_i (\alpha \cdot \beta \cdot \gamma_i) c_i \,,$$

where c_i runs over a basis for $H^*(X)$ and α, β, γ_i are the Poincaré duals of a, b, c_i, respectively, and "$\cdot$" denotes the homology intersection index. The quantum multiplication

$$*\colon QH^*(X) \otimes QH^*(X) \to QH^*(X)$$

is defined on elements $a, b \in H^*(X) \hookrightarrow QH^*(X)$ by

$$a * b := \sum_{i,B} \Phi_B(\alpha, \beta, \gamma_i) q^B c_i \,,$$

and extended by linearity to all of $QH^*(X)$. Here, B ranges over $H_2'(X)$, and $\Phi_B(\alpha, \beta, \gamma_i)$ denotes the Gromov (Gromov-Witten) invariants. Intuitively, these count intersections (subject to dimension requirements!) of the cells α, β, γ_i not with themselves, but with the fourth cell B. When $B = 0$, the Gromov invariant is the classical intersection index. Thus,

$$a * b \;=\; a \cup b + \text{other terms.}$$

For details regarding the definition of quantum cohomology, and in particular proofs of the associativity of $*$, see [12, 15].

Extend $\mu^*\colon H^*(X) \to K$ (defined in section 2) by linearity over Λ to a form $\mu^*\colon QH^*(X) \to \Lambda$.

Proposition 5.1 *The form μ^* endows $QH^*(X)$ with a FE structure.*

Proof. See [2] for a rigorous proof. ■

Let $\iota\colon H^*(X) \hookrightarrow QH^*(X)$ denote the obvious inclusion map. Note that although $H^*(X)$ and $QH^*(X)$ share essentially the same basis $\{e_i\}$ and the same FA form, the respective dual bases are not necessarily equal. In other words, the fact that the element $e_i^\#$ is the dual in $H^*(X)$ to e_i does not necessarily imply that $\iota(e_i^\#)$ is dual to $\iota(e_i)$ in $QH^*(X)$. However, it does hold that the q^0 term of $\iota(e_i)^\#$ is in fact $\iota(e_i^\#)$. It follows that the q^0 term of the characteristic element ω_q of $(QH^*(X), \mu^*)$ is $e(X)$. In other words, we have:

The characteristic element ω_q is a deformation of the classical Euler class.

Because of this, we refer to ω_q as **the quantum Euler class**, and denote it by $e_q(X)$. Unlike $e(X)$, the quantum Euler class may very well be a unit. Strictly speaking, however, the semisimplicity result 3.4 does not apply to $QH^*(X)$, because it is infinite dimensional (as a vector space) over K and only a ring extension (not an algebra) over Λ. We may, however, utilize the approach of section 4. Define the homomorphism $\theta\colon \Lambda \to K$ as follows: For each generator B_i of $H_2'(X)$ choose any nonzero $r_i \in K$ and define $\theta(q^{B_i}) := r_i$. Extending θ by linearity over K gives a surjective ring homomorphism, often referred to as "specialization." Theorem 3.4 now applies to $\theta_*[QH^*(X)]$, which is a FA.

6 The Quantum Cohomology of the Grassmannians

Let $G_{k,n}$ denote the Grassmannian manifold of complex k-dimensional subspaces in $\mathbb{C}^n$. Define the Chern polynomial of $X = G_{k,n}$ to be

$$c_t(G_{k,n}) := \sum_{i=1}^{k} x_i t^i = \prod_{i=1}^{k}(1 + \lambda_i t),$$

where t is a formal variable and the x_i's are the Chern classes of the canonical bundle $\mathbf{S}_{k,n}$. The λ_i are referred to as the **Chern roots** of $G_{k,n}$ (but they are *not* roots of c_t!). Obviously, x_i is the i'th elementary symmetric polynomial $\sigma_i(\lambda_1, \ldots, \lambda_k)$ in the Chern roots. Define

$$W(\lambda_1, \ldots, \lambda_k) = \sum_{i=1}^{k} \frac{1}{n+1}\lambda_i^{n+1}$$

and

$$\begin{aligned} W_q(\lambda_1, \ldots, \lambda_k) &= \sum_{i=1}^{k} \frac{1}{n+1}\lambda_i^{n+1} + (-1)^k q \lambda_i \\ &= W(\lambda_1, \ldots, \lambda_k) + (-1)^k q x_1. \end{aligned}$$

The function W_q is called the Landau-Ginzburg potential of $G_{k,n}$. Because W and W_q are symmetric functions in the λ_i, they may also be viewed as functions of $x_1, \ldots, x_k$. Define dW to be the ideal $(\frac{\partial W}{\partial x_1}, \ldots, \frac{\partial W}{\partial x_k})$, and define dW_q similarly. Then

$$H^*(G_{k,n}) \cong K[x_1, \ldots, x_k]/dW$$

and

$$QH^*(G_{k,n}) \cong K[q, q^{-1}][x_1, \ldots, x_k]/dW_q.$$

Denote by H and H_q the determinants of the Hessians

$$\mathcal{H} = \left(\frac{\partial^2 W}{\partial x_i \partial x_j} \right) \ , \ \ \mathcal{H}_q = \left(\frac{\partial^2 W_q}{\partial x_i \partial x_j} \right)$$

of W and W_q, respectively.

In this section, we will prove the following:

Theorem 6.1 *$e(G_{k,n}) = (-1)^{\binom{n}{2}} H$ and $e_q(G_{k,n}) = (-1)^{\binom{n}{2}} H_q$.*

Suppose that an algebra A (not necessarily a FA) is finite dimensional as a vector space and is given by the presentation $A \cong K[x_1, \ldots, x_n]/R$, where $R = (f_1, \ldots, f_p)$ is some finitely-generated ideal in $K[x_1, \ldots, x_n]$. Note that we continue to assume that K has characteristic 0. Because A is finite dimensional, we must have $p \geq n$. The **Jacobian ideal** $J = J(R)$ of R is defined to be the ideal generated by the determinants of the $n \times n$ minors of the matrix

$$\left(\frac{\partial(f_1, \ldots, f_p)}{\partial(x_1, \ldots, x_n)} \right) \operatorname{mod} R.$$

The ideal J is well-defined since it is a Fitting ideal of the module $\Omega_{A/K}$ of Kähler differentials of A (see [20, §1.1, §10.3]).

The following result of Scheja and Storch [17] is reported in more generality in [20, *ibid*] (although for the definition of "complete intersection" we refer the reader to [11]):

Proposition 6.2 *$J \neq \{0\}$ if and only if A is a complete intersection and J generates the socle of A.*

Now assume that $A \cong K[x_1, \ldots, x_n]/R$ is a FA with characteristic element ω for some choice of FA structure.

Proposition 6.3 *$J \neq \{0\}$ if and only if $J = \omega A$. If $p = n$, then $J \neq \{0\}$ if and only if*

$$\det\left(\frac{\partial f_i}{\partial x_j}\right) \bmod R = u\omega$$

for some unit $u \in A$.

Proof. This proposition follows immediately from 3.3 and 6.2. ∎

Proposition 6.4 *For each $G_{k,n}$ there is a $\kappa \in K$ such that $H = \kappa e(G_{k,n})$ and $H_q = \kappa e_q(G_{k,n})$.*

Proof. Because H and $e(G_{k,n})$ are the q^0 terms of H_q and $e_q(G_{k,n})$, respectively, it suffices to prove the proposition for the quantum case.

The polynomial W_q is homogeneous of degree $2(n+1)$ [12, §8.4]. In other words, each summand of W_q has degree $2(n+1)$ in $QH^*(G_{k,n})$, where q is taken to have degree $2n$. Also, $|x_i| = 2i$ for each i. Thus, for fixed i, j we have

$$\left|\left(\frac{\partial^2 W_q}{\partial x_i \partial x_j}\right)\right| \;=\; |W_q| - |x_i| - |x_j| \;=\; |W_q| - 2i - 2j.$$

We now show by induction that H_q is homogeneous of degree $2k(n-k)$. Each $s \times s$ minor M of $\mathcal{H}$ is a matrix with entries $m_{ij} := \mathcal{H}_{ij}$ where i and j run over elements of some ordered subsets $I, J \subset \{1, \ldots, k\}$ respectively, and $\#I = \#J = s$. Define $M\widehat{(i,j)}$ to be the minor of M which does not include the entry m_{ij}. We have already shown that when $s = 1$, the single entry of each M is homogeneous of degree $|W_q| - 2i - 2j$. Assume that for all minors M of $\mathcal{H}$ of size less than $(s+1) \times (s+1)$, the determinant of M is homogeneous of degree

$$s|W_q| - 2\sum_{i \in I} i \;-\; 2\sum_{j \in J} j.$$

Now consider any $(s+1) \times (s+1)$ minor M of $\mathcal{H}$ with index sets I, J. Take any $i' \in I$. Then

$$\det M = \sum_{j' \in J} \operatorname{sgn}(i', j') m_{i'j'} \det M\widehat{(i', j')},$$

where sgn is the appropriate function $I \times J \to \{+1, -1\}$. By the induction hypothesis,

$$\begin{aligned}
\left| m_{i'j'} \det M\widehat{(i',j')} \right| &= |m_{i'j'}| + |M\widehat{(i',j')}| \\
&= |W_q| - 2i' - 2j' + s|W_q| - \left(2 \sum_{i \in I \setminus \{i'\}} i \right) - \left(2 \sum_{j \in J \setminus \{j'\}} j \right) \\
&= (s+1)|W_q| - \; 2\sum_{i \in I} i \; - \; 2\sum_{j \in J} j.
\end{aligned}$$

But this is independent of the choice of j', so $\det M$ is homogeneous of degree $(s+1)|W_q| - 2\sum_{i\in I} i \; - \; 2\sum_{j \in J} j$. In particular, we can take $M = \mathcal{H}$, and thus H is homogeneous of degree

$$k|W_q| \; - \; 2\sum_{i=1}^{k} i \; - \; 2s\sum_{j=1}^{k} j \; = \; k(2n+2) - 2k(k+1) \; = \; 2k(n-k).$$

Of course, e_q is also homogeneous of degree $2k(n-k)$ since $e_q = \sum_i e_i e_i^{\#}$, where e_i runs over a basis for $H^*(G_{k,n})$, and since $|e_i^{\#}| = 2k(n-k) - |e_i|$.

Consider the algebra $A := K[q][x_1, \ldots, x_k]/dW_q$. By the definitions of H_q and e_q, and the nature of the relations given by dW_q, both H_q and e_q may be viewed as elements of A. Now, the proof of 4.1 applies equally well to the algebra A, so it is a FE, and e_q is in fact the characteristic element of A. Proposition 6.3, shows that $H_q = v e_q(G_{k,n})$ for some unit $v \in A$. Of course, v may also be viewed as an element of $QH^*(G_{k,n})$ which simply has no q^i-terms with $i < 0$.

Write $v = v' + v''$ where v' is homogeneous of degree 0, and v'' contains no terms of degree 0. Since $H_q = v'e_q + v''e_q$ and both H_q and $v'e_q$ are homogeneous of degree $2k(n-k)$, we see that $v''e_q$ must also be homogeneous of this degree. By degree considerations, we must have $v''e_q = 0$, and thus $H_q = v'e_q$.

Write $v' = \sum_{j \geq 0} v_j q^j$. Since v' is homogeneous of degree 0 and $|q^j| = 2jn$, we see that $|v_j| = -2jn$. But v' is an element of A, so we must have $v_j = 0$ for $j \neq 0$. Thus v' may in fact be viewed as a degree 0 element in $H^*(G_{k,n})$. In other words, v' is an element $\kappa \in K$. ■

Take $K = \mathbb{R}$ or $\mathbb{C}$, and for any nonzero $r \in K$ let θ_r denote a specialization homomorphism $K[q, q^{-1}] \to K$, $q \mapsto r$ as above. In the following paragraph, any reference to $QH^*(G_{k,n})$ or any element a therein should be interpreted as referring to $(\theta_r)_*[QH^*(G_{k,n})]$ and $\tilde{\theta}_r(a)$, respectively.

In this context, the relationship between the distinguished element and the Hessian provides $e_q(G_{k,n})$ with a nontrivial geometric interpretation: Denote the critical points of W_q by $z_1, \ldots, z_j$, and note that H_q may be viewed as a function $K^k \to K$, as may all the elements of $QH^*(G_{k,n})$. It is well known that, for each j, $H_q(z_j) = 0$ if and only if the critical point z_j is degenerate [13]. Because the elements of $QH^*(G_{k,n})$, viewed as functions, are completely determined by their values on the critical points of W_q, we see that H (and hence $e_q(G_{k,n})$) is a unit in $QH^*(G_{k,n})$ if and only if the critical points of W_q are all nondegenerate.

This relationship between $e_q(G_{k,n})$ and H also yields a new approach to the following known result [18]:

Proposition 6.5 *For all $G_{k,n}$ and all nonzero $r \in \mathbb{R}$, the algebra $(\theta_r)_*[QH^*(G_{k,n})]$ is semisimple.*

The proof is based on calculations appearing in [5].

Proof. The Jacobian matrix $V = (\partial x_i / \partial \lambda_j)$ associated to the elementary symmetric functions x_i is a Vandermonde matrix, and has determinant $\prod_{i<j}(\lambda_i - \lambda_j) \neq 0$. Let ∇_x denote the gradient vector operator with respect to $x_1, \ldots, x_k$, and let ∇_λ denote the gradient operator with respect to $\lambda_1, \ldots, \lambda_k$. Viewing the gradient operators as row vectors, we have $\nabla_x(W_q)V = \nabla_\lambda(W_q)$. Let $\nabla_x(W_q)_i$ denote the i'th entry of $\nabla_x(W_q)$, and let V_i denote the i'th row of V. Then the Hessian of W_q with respect to the λ's is

$$\begin{aligned}
\nabla_\lambda^T \nabla_\lambda(W_q) &= \nabla_\lambda^T(\nabla_x(W_q)V) \\
&= V^T \nabla_x^T \nabla_x(W_q) V + \textstyle\sum_i \nabla_x(W_q)_i \, \nabla_\lambda^T V_i.
\end{aligned}$$

Evaluating at the critical points of W_q (*i.e.* assuming $\nabla_x(W_q) = 0$), and expressing everything in terms of the λ_i's, we see that

$$H = \det\left(\nabla_\lambda^T \nabla_\lambda(W_q)\right) \det(V^{-2}) = \frac{n^k \prod_{i=1}^k \lambda_i^{n-1}}{(\prod_{i<j}(\lambda_i - \lambda_j))^2}.$$

Now, because V is invertible, the relation $\nabla_x(W_q) = 0$ is equivalent to $\nabla_\lambda(W_q) = 0$. In other words, for each i we have $\lambda_i^n = (-1)^{k+1} q$ at the critical points of W_q. This implies that

$$x_k \prod_{i=1}^{k} \lambda_i^{n-1} = \prod_{i=1}^{k} \lambda_i^n = (-1)^{k(k+1)} q^k.$$

Since $q \neq 0$, the numerator of H, and thus H itself, is nonzero at the critical point of W_q. It follows that H, as an element of $QH^*(G_{k,n})$, has an inverse, and therefore, by 6.4, so does $e_q(G_{k,n})$. By proposition 4.2, $\theta_r[QH^*(G_{k,n})]$ is semisimple. ∎

As discussed above, the Chern classes $x_1, \ldots, x_k$ arising from the bundle $\mathbf{S}_{k,n}$ are the elementary symmetric polynomials in the Chern roots $\lambda_1, \ldots, \lambda_k$. An analogous situation holds for the "normal" classes $y_1, \ldots, y_{n-k}$, which arise from the quotient bundle $\mathbf{Q}_{k,n}$. Define $\mu_1, \ldots, \mu_{n-k}$ to be the Chern roots corresponding to the formal polynomial

$$\sum_{i=1}^{n-k} y_i t^i.$$

Then for all i, we have $y_i = \sigma_i(\mu_1, \ldots, \mu_{n-k})$. In fact, the λ_i's and μ_i's are the first Chern classes of the line bundles in the splitting of $\mathbf{S}_{k,n}$ and $\mathbf{Q}_{k,n}$, respectively [6, §21]. Together with the well known bundle-isomorphism of the tangent bundle $\mathbf{T}_{k,n} \cong \mathbf{S}_{k,n}^* \otimes \mathbf{Q}_{k,n}$, this fact allows us to write the characteristic classes $c_i(\mathbf{T}_{k,n})$ in terms of the x_i's and y_i's: The Chern polynomial for $\mathbf{T}_{k,n}$ is

$$\sum_{i=1}^{k(n-k)} c_i(\mathbf{T}_{k,n}) t^i \;=\; \prod_{i,j} (1 + (\mu_j - \lambda_i) t),$$

where i and j in the product range over possible indices [6, *ibid*]. In other words, for each i we have $c_i(\mathbf{T}_{k,n}) = \sigma_i(\{\mu_j - \lambda_i\}_{i,j})$. This shows that each $c_i(\mathbf{T}_{k,n})$ is symmetric in the λ_i's and the μ_j's, and can therefore be written in terms of the x_i's and y_j's.

In particular, the Euler class $e(G_{k,n})$ can be lifted to a polynomial

$$P \in K[x_1, \ldots, x_k, y_1, \ldots, y_{n-k}]$$

or, using the relations between the x_i's and y_j's, to a polynomial

$$P' \in K[x_1, \ldots, x_k].$$

P' is referred to as the "Euler polynomial."

Bertram [5] has proven the following:

Proposition 6.6 *For each (k, n), the Euler polynomial P' is a lifting of $(-1)^{\binom{n}{2}} H_q \in QH^*(G_{k,n})$.*

We can now prove theorem 6.1.

Proof. (of theorem 6.1) Proposition 6.4 shows that $e_q := e_q(G_{k,n}) = \kappa H_q$ for some $\kappa \in K$. Let $\pi \colon QH^*(G_{k,n}) \to H^*(G_{k,n})$ denote the module homomorphism sending $q \mapsto 0$. By definition, P' is a lifting of $e := e(G_{k,n})$, so (by 6.6) we have

$$
\begin{tikzcd}
K[x_1, \ldots, x_k] \arrow[d] \arrow[dr] & \\
H^*(G_{k,n}) & QH^*(G_{k,n}) \arrow[l, "\pi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
P' \arrow[d] \arrow[dr] & \\
e(G_{k,n}) & (-1)^{\binom{n}{2}} H_q \arrow[l, "\pi"']
\end{tikzcd}
$$

where the unlabeled arrows are the canonical projection maps. Now $\pi(e_q) = e$, by definition of the quantum multiplication $*$, so proposition 6.4 shows that

$$(-1)^{\binom{n}{2}} e \;=\; \pi\left((-1)^{\binom{n}{2}} e_q\right) \;=\; \pi\left((-1)^{\binom{n}{2}} \kappa H_q\right) \;=\; \kappa e,$$

and thus $\kappa = (-1)^{\binom{n}{2}}$. ■

7 Quantum Cohomology of Hyperplanes

For the sake of contrast with the Grassmannians, this section provides another class of examples of a quantum cohomology ring, and determines which of these are semisimple. In [19], Tian and Xu discuss a more general class of examples along these lines from the point of view of semisimplicity in the sense of Dubrovin (as defined in the introduction).

Let $X \subset \mathbb{C}P^{n+r}$ be a smooth complete intersection of degree $(d_1, \ldots, d_r)$ and dimension $n \geq 2$ satisfying $n \geq \sum(d_i - 1) - 1$. Let Γ denote the hyperplane class generating $H^2(X, \mathbb{Z})$. By the "primitive cohomology $H^n(X)_0$ of X" we mean $H^n(X)$ if n is odd, and the subspace of $H^n(X)$ orthogonal to $\Gamma^{n/2}$ if n is even. Beauville shows in [4] (although he unnecessarily presumes $q = 1$), that $QH^*(X)$ is the algebra over $K[q, q^{-1}]$ generated by Γ and $H^n(X)_0$, subject to the relations

$$\Gamma^{n+1} \;=\; d_1^{d_1} \cdots d_r^{d_r} \Gamma^{d-1} q$$

and, for all $a, b \in H^n(X)_0$,

$$\Gamma a = 0 \quad \text{and} \quad ab = \langle a, b\rangle \frac{1}{d}\left(\Gamma^n - d_1^{d_1}\cdots d_r^{d_r}\Gamma^{d-2}q\right).$$

Here, $\langle\cdot,\cdot\rangle$ denotes the classical intersection form $a\otimes b \mapsto f(a\cup b)$, where $f := (\Gamma^n)^*$.

Proposition 7.1 *Let X denote a hyperplane of degree d. For any nonzero $r \in K$, if $d > 2$ then $(\theta_r)_*[QH^*(X)]$ is not semisimple. If $d = 2$ then $(\theta_r)_*[QH^*(X)]$ is semisimple.*

Proof. Denote $(H^n(X)_0 : K)$ by R, and choose a basis $e_1, \ldots, e_R$ for $H^n(X)_0$. Together with the elements $1, \Gamma, \Gamma^2, \ldots, \Gamma^n$, this provides a full vector-space basis for $QH^*(X)$. Thus the characteristic element of $(QH^*(X), f)$ is

$$\begin{aligned}\omega &= \textstyle\sum_{i=0}^n \Gamma^i\Gamma^{n-i} + \sum_{i=1}^R e_i e_i^\# \\ &= (n+1)\Gamma^n + \tfrac{R}{d}\left(\Gamma^n - d^d\Gamma^{d-2}q\right).\end{aligned}$$

Notice that if $d > 2$ then ω is divisible by Γ, so $\omega e_1 = 0$ (for example). Since $e_1 \notin \operatorname{Ker}\tilde{\theta}$ (it is a basis element for any choice of coefficients!), proposition 4.2 shows that $(\theta_r)_*[QH^*(X)]$ is not semisimple for any choice of $r \in K$.

If $d = 2$ then we have

$$\omega = (n+1+\frac{R}{2})\Gamma^n - 2Rq,$$

and thus $\Gamma\omega = 4(n+1)q\Gamma$. Order the basis for $QH^*(X)$ as follows: $1, \Gamma, \Gamma^2, \ldots, \Gamma^n$, $e_1, \ldots, e_R$. Then the matrix $[\bar{\beta}(\omega)]$ corresponding to ω under the regular representation $\bar{\beta}$ is

$$\left(\begin{array}{ccccc|ccc}
-2Rq & & & & n+1+\frac{R}{2} & & & \\
 & 4(n+1)q & & 0 & & & & \\
 & & 4(n+1)q & & & & & \\
 & 0 & & \ddots & & & & \\
 & & & & 4(n+1)q & & & \\
\hline
 & & & & & -2nq & & 0 \\
 & & & & & & \ddots & \\
 & & & & & 0 & & -2nq
\end{array}\right).$$

Since the determinant of this matrix is a unit in $K[q, q^{-1}]$, we see that ω is a unit in $QH^*(X)$; by 4.2 this shows that $(\theta_r)_*[QH^*(X)]$ is semisimple for any choice of r. ∎

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References

[1] Lowell Abrams. Two-dimensional topological quantum field theories and Frobenius algebras. *J. Knot Theory and its Ramifications*, 5:569–587, 1996.

[2] Lowell Abrams. *Frobenius Algebra Structures in Topological Quantum Field Theory and Quantum Cohomology.* Doctoral dissertation. The Johns Hopkins University, May 1997.

[3] Frank W. Anderson and Kent R. Fuller. *Rings and Categories of Modules.* Springer-Verlag, New York, second edition, 1992.

[4] Arnaud Beauville. Quantum cohomology of complete intersections. Preprint alg-geom/9501008, 1995.

[5] Aaron Bertram. Towards a Schubert calculus for maps from a Riemann surface to a Grassmannian. *Intern. J. Math.*, 5:811–825, 1994.

[6] Raoul Bott and Loring W. Tu. *Differential Forms in Algebraic Topology.* Springer-Verlag, New York, 1982.

[7] Bruce Crauder and Rick Miranda. Quantum cohomology of rational surfaces. In *The moduli space of curves* (Texel Island, 1994), 33–80, Progr. Math. 129, Birkhauser, Boston, 1995.

[8] Charles Curtis and Irving Reiner. *Representation Theory of Finite Groups and Associative Algebras.* Interscience Publishers, New York, 1962.

[9] Boris Dubrovin. Geometry of 2d topological field theories. In *Integrable Systems and Quantum Groups*, volume 1620 of *Lecture Notes*, pages 120–348. Springer-Verlag, 1996.

[10] M. Kontsevich and Yu. Manin. Gromov-Witten classes, quantum cohomology, and enumerative geometry. *Comm. Math. Phys.*, 164:525–562, 1994.

[11] Ernst Kunz. *Introduction to Commutative Algebra and Algebraic Geometry.* Birkhäuser, Boston, 1985.

[12] Dusa McDuff and Dietmar Salamon. *J-holomorphic Curves and Quantum Cohomology.* American Mathematical Society, Providence, 1994.

[13] John W. Milnor. *Morse Theory.* Princeton University Press, Princeton, 1963.

[14] John W. Milnor and James D. Stasheff. *Characteristic Classes.* Princeton University Press, Princeton, 1974.

[15] Yongbin Ruan and Gang Tian. A mathematical theory of quantum cohomology. *J. Diff. Geom.*, 42:259–367, 1995.

[16] Stephen Sawin. Direct sum decomposition and indecomposable tqft's. *J. Math. Phys.*, 36:6673–6680, 1995.

[17] Gunter Scheja and Uwe Storch. Uber Spurfunktionen bei vollstandigen Durchschnitten. *J. fur die Reine und Angew. Math.*, 278/279:174–189, 1975.

[18] Bernd Siebert and Gang Tian. On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator. Preprint alg-geom/9403010, 1994.

[19] Gang Tian and Geng Xu. On the semisimplicity of the quantum cohomology algebras of complete intersections. Preprint alg-geom/9611035, 1996.

[20] Wolmer V. Vasconcelos. *Arithmetic of Blowup Algebras.* Cambridge University Press, New York, 1994.

[21] Edward Witten. Supersymmetry and Morse Theory. *J. Diff. Geom.*, 17:661–692, 1982.

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