A GENUS BOUND FOR DIGITAL IMAGE BOUNDARIES

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Abstract. Shattuck and Leahy [4] conjectured—and Abrams, Fishkind, and Priebe [1],[2] proved—that the boundary of a digital image is topologically equivalent to a sphere if and only if certain related foreground and background graphs are both trees. In this manuscript we extend this result by proving upper and lower bounds on digital image boundary genus in terms of the foreground and background graphs, and we show that these bounds are best possible. Our results have current application to topology correction in medical imaging.

Key words. digital image, digital topology, combinatorial topology, surface.

AMS subject classifications. 05C10, 57M15.

1. Overview. Digital topology is an area of great theoretical interest having the additional bonus of significant application in imaging science and related areas. Our results are mathematical—the notation and setting are detailed in Section 2—but we begin with a brief description of a current application.

The human cerebral cortex, when viewed as closed at the brain stem, is topologically like a sphere. Magnetic resonance imaging (MRI) can differentiate between tissue that is interior to the cerebral cortex and tissue that is exterior to the cerebral cortex. Because of the finiteness of resolution, what is generated by MRI is a 3-dimensional array of cubes, each cube classified by MRI as “foreground” (tissue interior to the cerebral cortex) or “background” (tissue exterior to the cerebral cortex), and the boundary between the foreground and background is an approximation of the cerebral cortex itself.

Although topologically spherical, the cerebral cortex is densely folded, and the finite resolution, as well as noise, may lead to topological “handles” that don’t actually exist. The physiological and neurological function of regions of the cerebral cortex, as well as the relationship between the regions, is dictated by the spherical topology rather than just spatial proximity. It is therefore important to “correct” the topology, and a number of different strategies are currently used [3], [4].

The strategy of Shattuck and Leahy [4] is fundamentally based on the construction of certain foreground and background graphs related to the MRI data; they conjectured that the image boundary is topologically spherical if and only if both foreground and background graphs are trees. In situations where one or both of the graphs are not trees, the edges are weighted to reflect corresponding junctional thickness, and a maximum weight spanning tree is found. Edges not on the spanning tree are removed by adjusting the image at corresponding locations, and the resulting image is then, by their conjecture, topologically spherical.

The Shattuck and Leahy conjecture was proven and generalized by Abrams, Fishkind, and Priebe in [1] and [2], and the main result in this manuscript, Theorem 2.2, represents a further generalization. Theorem 2.2—articulated in Section 2 and proven in Section 3—gives bounds for the genus of the boundary of a digital image in terms of the foreground and background graphs, and these bounds are shown to be best possible. The truth of Shattuck and Leahy’s conjecture is, in fact, a special

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Theoretic trees. Will dictate whether we are viewing endpoints are connected component of tal image or as discrete graph-theoretic objects. In Figures 2.2 and 3.2 we show examples of $I$. Let $N$ be a fixed positive integer. For any triplet of integers $(i, j, k) \in \{1, 2, \ldots, N\}^3$, define the voxel $v_{i,j,k}$ to be the closed Euclidean unit-cube $[i-\frac{1}{2}, i+\frac{1}{2}] \times [j-\frac{1}{2}, j+\frac{1}{2}] \times [k-\frac{1}{2}, k+\frac{1}{2}]$. For any $A \subseteq \{1, 2, \ldots, N\}^3$, define the digital image $I_A := \bigcup_{(i,j,k) \in A} v_{i,j,k}$. Digital image $I_A$ is called surrounded if none of $i, j, k$ equals 1 or $N$, and $I_A$ is called standard if $I_A$ is surrounded and its boundary $\partial I_A$ is a surface. The complementary digital image $I^c_A$ is defined as $I_A^c$, where $A^c := \{1, 2, \ldots, N\}^3 \setminus A$. When there is no confusion we write $I$ in place of $I_A$.

For surrounded digital image $I$, when is $\partial I$ a surface? Of course, $\partial I$ is always compact. It is shown in [2] that $\partial I$ is locally homeomorphic to a disk if and only if it doesn’t have any of the three “forbidden” voxel configurations illustrated in Figure 2.1. When $\partial I$ is locally homeomorphic to a disk it is not difficult to show that $\partial I$ is connected, hence a surface, if and only if both $I$ and $I^c$ are connected.

For each $k \in \{1, 2, \ldots, N\}$, the $k$’th level is $L_k := \bigcup_{i,j \in \{1,2,\ldots,N\}} v_{i,j,k}$, and the $k,k+1$’th sheet is $S_{k,k+1} := L_k \cap L_{k+1}$. Associated to a digital image $I$ is the (multi)-graph $G_I$ with vertex set $V_I$ and edge set $E_I$ defined as follows: For each $k \in \{1, 2, \ldots, N\}$, we declare each connected component of $I \cap L_k$ to be a vertex in $V_I$. For any two vertices $u$ and $v$ on adjacent levels, say $L_k$ and $L_{k+1}$, we declare each connected component of $u \cap v \subseteq S_{k,k+1}$ to be an edge in $E_I$ whose graph-theoretic endpoints are $u$ and $v$. When referring to a vertex $u \in V_I$ or an edge $e \in E_I$, context will dictate whether we are viewing $u$ or $e$ as Euclidean subsets, i.e. subsets of $R^3$, or as discrete graph-theoretic objects. In Figures 2.2 and 3.2 we show examples of $I$, $G_I$, and $G_{I^c}$.

The following result was conjectured by Shattuck and Leahy [4] and proved by Abrams, Fishkind, and Pribe [1].

**Theorem 2.1 (Spherical Homeomorphism Theorem).** For any standard digital image $I$, $\partial I$ is a topological sphere if and only if both $G_I$ and $G_{I^c}$ are graph-theoretic trees.

For digital image $I$, define $r_I := |E_I| - |V_I| + 1$. This value is called the corank or cycle rank of $G_I$. If $I$ is connected then $G_I$ is connected as well, so $r_I \geq 0$. In

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1In [2] we discuss a corrective strategy — involving slightly altering the digital image — for medical imaging applications in which $\partial I$ is not locally homeomorphic to a disk.
particular, \( r_T = 0 \) if and only if \( G_T \) is a tree. The cycle ranks \( r_T \) and \( r_{T^c} \) are the first Betti numbers of \( G_T \) and \( G_{T^c} \), respectively. The following is our main result; \( g(\partial I) \) denotes the genus of the surface \( \partial I \), which is the first Betti number of \( I \).

**Theorem 2.2.** For any standard digital image \( I \),

\[
\max \{ r_T, r_{T^c} \} \leq g(\partial I) \leq r_T + r_{T^c}.
\]

Moreover, this is best possible in the sense that, for any nonnegative integers \( a, b, \) and \( c \) such that \( \max \{ a, b \} \leq c \leq a + b \), there exists a standard digital image \( I \) such that \( r_T = a, r_{T^c} = b \), and \( g(\partial I) = c \).

It is not hard to see that Theorem 2.2 implies Theorem 2.1: For standard digital image \( I \), if \( \partial I \) is topologically spherical then \( \max \{ r_T, r_{T^c} \} \leq 0 \) implies both \( G_T \) and \( G_{T^c} \) are trees and, conversely, if both \( G_T \) and \( G_{T^c} \) are trees then \( g(\partial I) \leq 0 + 0 \) implies that \( \partial I \) is topologically spherical. We may therefore think of Theorem 2.2 as a generalization of the Spherical Homeomorphism Theorem.

3. **Proof of the main result, Theorem 2.2.** We begin the proof of Theorem 2.2 with the following slightly weakened version.

**Lemma 3.1.** For any standard digital image \( I \), \( r_T \leq g(\partial I) \leq r_T + r_{T^c} \).

The proof of Lemma 3.1 extends and refines the development and strategies in [1].

If a surface \( S \) has a 2-cell embedding of some graph \( H \) with \( n \) vertices, \( e \) edges, and \( f \) faces, then Euler’s classical result states that \( n - e + f = 2 - 2g(S) \). The value \( \chi(S) \overset{\text{def}}{=} n - e + f \) is called the Euler characteristic of \( S \). Our assumption that \( \partial I \) is a surface implies that, for every \( v \in V_T \), \( \partial v \) is a surface; it is useful to view the voxel vertices, voxel edges, and voxel faces on \( \partial I \) or \( \partial v \), respectively, as a 2-cell embedding of a graph on \( \partial I \) or \( \partial v \).

Suppose \( \epsilon \in E_T \) is a subset of sheet \( S \); the assumption that \( \partial I \) is a surface implies that the boundary of \( \epsilon \) in \( S \), denoted \( \partial S \epsilon \), consists of a disjoint union of simple, closed curves. Let \( h_\epsilon \) denote the number of “punctures” in \( \epsilon \), i.e. \( h_\epsilon \) is one less than the number of connected components of \( \partial S \epsilon \). Even though \( \epsilon \) and \( \partial S \epsilon \) are not surfaces (they have boundaries, and \( \partial S \epsilon \) may not be connected) the Euler characteristics \( \chi(\epsilon) \) and \( \chi(\partial S \epsilon) \) are well defined; in fact, \( \chi(\partial S \epsilon) = 0 \) (since it has equal numbers of voxel edges and voxel vertices, and no faces), and

\[
\chi(\epsilon) = 2 - (h_\epsilon + 1) = 1 - h_\epsilon.
\]

**Proof of Lemma 3.1.** Since the relative interior of each \( \epsilon \in E_T \) is a subset of two vertex boundaries not contained in \( \partial I \), and the relative boundary of each \( \epsilon \) has Euler
characteristic 0, a simple inclusion-exclusion argument gives

\[ \chi(\partial \mathcal{I}) = \sum_{v \in V_\mathcal{I}} \chi(\partial v) - 2 \sum_{e \in E_\mathcal{I}} \chi(e). \]  

(3.2)

Next, note that there is a natural one-to-one correspondence between the genus holes\(^2\) of the vertices of \(V_\mathcal{I}\) and the vertices of \(V_{\mathcal{I}c}\) other than the \(N\) “outermost” vertices of \(V_{\mathcal{I}c}\), one per level. Thus

\[ \sum_{v \in V_\mathcal{I}} \chi(\partial v) = \sum_{v \in V_\mathcal{I}} [2 - 2g(\partial v)] = 2|V_\mathcal{I}| - 2(|V_{\mathcal{I}c}| - N). \]

(3.3)

Combining (3.1), (3.2), and (3.3), we obtain

\[ 2 - 2g(\partial \mathcal{I}) = \chi(\partial \mathcal{I}) = \sum_{v \in V_\mathcal{I}} \chi(\partial v) - 2 \sum_{e \in E_\mathcal{I}} \chi(e) \\
= 2|V_\mathcal{I}| - 2(|V_{\mathcal{I}c}| - N) - 2 \sum_{e \in E_\mathcal{I}} (1 - h_e) \\
= 2 - 2(|E_\mathcal{I}| - |V_\mathcal{I}| + 1) - 2 \left(|V_{\mathcal{I}c}| - N - \sum_{e \in E_\mathcal{I}} h_e \right), \]

and it follows that

\[ g(\partial \mathcal{I}) = r_\mathcal{I} + \left(|V_{\mathcal{I}c}| - N - \sum_{e \in E_\mathcal{I}} h_e \right). \]

(3.4)

For \(k = 1, 2, \ldots, N-1\), let \(b_{k,k+1}\) denote the number of connected components in the subgraph of \(G_{\mathcal{I}c}\) induced by the vertices of \(V_{\mathcal{I}c}\) in the \(k\)’th and \(k+1\)’th levels, and denote by \(B_{k,k+1}\) the Euclidean set \(S_{k,k+1} \setminus \bigcup_{e \in E_\mathcal{I}: e \subseteq S_{k,k+1}} e\). Observe that the Euclidean set \(B_{k,k+1}\) has \(b_{k,k+1}\) components.

It now follows that

\[ 1 + \sum_{e \in E_\mathcal{I}: e \subseteq S_{k,k+1}} h_e = b_{k,k+1}. \]

Summing this equation over \(k\) yields

\[ N - 1 + \sum_{e \in E_\mathcal{I}} h_e = \sum_{k=1}^{N-1} b_{k,k+1}. \]

(3.5)

Substituting (3.5) into (3.4) and simplifying, we find that Lemma 3.1 is now equivalent to the assertion that, for any standard digital image \(\mathcal{I}\),

\[ 2 (|V_{\mathcal{I}c}| - 1) - |E_{\mathcal{I}c}| \leq \sum_{k=1}^{N-1} b_{k,k+1} \leq |V_{\mathcal{I}c}| - 1. \]

(3.6)

To show the right-hand side of (3.6), suppose first that we remove all edges of \(G_{\mathcal{I}c}\).

Without any edges, \(\sum_{k=1}^{N-1} b_{k,k+1} = 2|V_{\mathcal{I}c}| - 2\), since each vertex in \(V_{\mathcal{I}c}\), with the

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\(^2\) Suppose \(v \in V_\mathcal{I}\) is in level \(L\). We use the term “genus hole” of \(v\) to refer to each component of the complement of \(v\) in \(L\) which is bounded horizontally in \(\mathbb{R}^3\) by \(v\). For each genus hole of \(v\), there is a unique \(w \in V_{\mathcal{I}c}\) such that this genus hole of \(v\) is precisely the union of \(w\) and \(w\)’s genus holes.
exception of the single vertices in $L_1$ and $L_N$, respectively, is counted as a distinct component in the tabulation of $b_{i-1,k}$ as well as $b_{k,k+1}$ for some $k$ (the single vertices in $L_1$ and $L_N$ are tabulated only once, in $b_{1,2}$ and $b_{N-1,N}$, respectively). Now consider a spanning tree $T$ of the original graph $G_{T^c}$. Each of the $|V_{T^c}| - 1$ edges of $T$, when returned to $G_{T^c}$, reduces the total number of components by 1, so $\sum_{k=1}^{N-1} b_{k,k+1} = |V_{T^c}| - 1$ after all the edges of $T$ have been restored to $G_{T^c}$. Returning the remaining edges to the original graph $G_{T^c}$ can only further reduce the sum, and thus the right-hand side of (3.6) holds. Since reduction of the sum occurs only through this returning of edges, the largest possible reduction equals $|E_{T^c}|$, confirming the left-hand side of (3.6) and completing the proof of Lemma 3.1. \[\square\]

**Lemma 3.2.** For any standard digital image $I$, $r_{T^c} \leq g(\partial I)$.

The proof of Lemma 3.2, together with Lemma 3.1, will complete the proof of the bound $\max\{r_T, r_{T^c}\} \leq g(\partial I) \leq r_T + r_{T^c}$ in Theorem 2.2. However, using the approach of the proof of Lemma 3.1 to show $r_{T^c} \leq g(\partial I)$ requires more than simply reversing the roles of $I$ and $T^c$ since $T^c$, unlike $I$, isn’t standard.

**Proof of Lemma 3.2.** Enlarge the ambient space for digital images to allow voxels with one or more coordinates equal to 0 or $N + 1$. Accordingly, use the adjective surrounded to indicate that a digital image contains no voxel with any coordinate equal to 0 or $N + 1$; note that any digital image which is standard in the previous sense remains standard when ‘surrounded’ is redefined in this way.

For the sake of simplifying notation, we let $I$ denote a standard digital image which is surrounded in the original, smaller, ambient space, and let $T^c$ denote its complement in that smaller space. Let $v_{i,j,k}$ denote a voxel in $I$ with minimum $k'$. Let $J$ denote the digital image consisting of the union of the following Euclidean sets: $I$, the voxels $v_{i,j,k}$ such that at least one of $i, j, k = 0$ or $N + 1$, and the voxels $v_{i,j,k}$ such that $0 \leq k < k'$. Note that $J$ and $J^c$ are connected, and $J^c$ is standard. Moreover, we have topological equivalence of $\partial I$ and $\partial J^c$, since the change in $\partial I$ amounts to cutting out an open disk, attaching one end of a tube to $\partial I$ along the boundary of the removed disk, and then capping off the tube. (We consider $\partial J^c$ rather than $\partial J$ because $\partial J^c$ contains the additional component $\partial(J \cup J^c)$.) Figure 3.1 illustrates this process.

For each $k$ in $1, \ldots, k' - 1$ there is exactly one vertex of $G_{T^c}$ in level $k$, and no vertices of $G_T$. Since the removal of the voxels $\{v_{i,j,k} \mid 0 \leq k < k\}$ does not disconnect any of the vertices of $G_{T^c}$, and no other changes are made to $I$ in the process of constructing $J^c$, we see that $G_{J^c} = G_{T^c}$, and thus $r_{J^c} = r_{T^c}$.

Applying Lemma 3.1 to the standard digital image $J^c$ yields

$$r_{T^c} = r_{J^c} \leq g(\partial J^c) = g(\partial I).$$

and Lemma 3.2 is shown. \[\square\]

It is interesting to note that there is also a close relationship between $G_T$ and $G_J$. The voxels $\{v_{i,j,k} \mid 0 \leq i, j, k \leq N + 1\}$ and $\{v_{i,j,k} \mid 0 \leq k < k'\}$ give rise to a path $P$ in graph $G_J$. In fact, $G_J$ can be obtained from $G_T$ by attaching $P$ at a single endpoint. Since the edges of $P$ lie in no cycles, we have $r_J = r_T$.

We now establish that the bounds in Theorem 2.2 are ‘best possible’:

**Lemma 3.3.** For any nonnegative integers $a$, $b$, and $c$ such that $\max\{a, b\} \leq c \leq a + b$, there is a standard digital image $I$ such that $r_T = a$, $r_{T^c} = b$, and $g(\partial I) = c$. 

Fig. 3.1. An illustration of the topological effect of modifying $I$ to obtain $J$: (a) $\partial I$ with a disk cut out, (b) with a tube attached, and (c) with the tube capped off.

Fig. 3.2. Three key examples — vertical $n$-torus, horizontal $n$-torus, and $n$-ladder — used in the proof of Theorem 2.2. Note that vertices of degree 1 corresponding to levels which contain no voxel in $I$ have been omitted from the graphs $G_{Ic}$. 
Proof of Lemma 3.3. Suppose $a$, $b$, and $c$ are nonnegative integers such that $\max\{a, b\} \leq c \leq a+b$. Construct a digital image $I$ by connecting, in any topologically trivial way, a vertical $(c-b)$-torus, a horizontal $(c-a)$-torus, and an $(a+b-c)$-ladder (see Figure 3.2). Note that $g(\partial I) = (c-b) + (c-a) + (a+b-c) = c$, $r_T = (c-b) + (a+b-c) = a$, and $r_{T_e} = (c-a) + (a+b-c) = b$. \hfill \Box

Theorem 2.2 now follows directly from Lemmas 3.1, 3.2, and 3.3. \hfill \Box

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