Cotensor products of modules

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Abstract

Let C be a coalgebra over a field k and A its dual algebra. The category of C-comodules is equivalent to a category of A-modules. We use this to interpret the cotensor product $M\Box N$ of two comodules in terms of the appropriate Hochschild cohomology of the A-bimodule $M \otimes N$, when A is finite-dimensional, profinite, graded or differential-graded. The main applications are to Galois cohomology, comodules over the Steenrod algebra, and the homology of induced fibrations.

1 Introduction

Cotensor product and its derived functors Cotor^i were originally introduced by Eilenberg-Moore [EM] to calculate the homology of an induced fibration. These functors have been discussed in the literature, primarily in homotopy theory, but they have developed a reputation of being inaccessible. To counter this, we describe cotensor product and Cotor in terms of Hochschild cohomology, and provide a new class of examples to which they are relevant.

Let C be a coalgebra over a field k and A its dual algebra. It is classical [Car] that one can identify comodules over C with a subcategory of modules over A. Thus comodule constructions may be translated into module constructions. In this paper, we interpret the cotensor product $M\Box N$ of two comodules as the vector space $\mathrm{Hom}_{A^e}(A, M \otimes N)$ of bimodule maps from A to $M \otimes N$. We also interpret the right derived functors $\mathrm{Cotor}^*_C(M, N)$ in terms of the Hochschild cohomology of the A-bimodule $M \otimes N$. Our results tie together several apparently disparate areas of mathematics.

When C and A are finite-dimensional, we do indeed have $\mathrm{Cotor}^n_C(M, N) = H^n(A, M \otimes N)$. This straightforward result is presented in Theorem 3.4.

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When C is infinite-dimensional, it is important to think of A as a topological algebra, as first suggested by Radford [Rad]. In this case, we prove in Theorem 4.11 that $\mathrm{Cotor}^n_C(M, N)$ is the *continuous* Hochschild cohomology $H^n_{\mathrm{ctn}}(A, M \otimes N)$.

This result has an interesting application to the Galois cohomology of a profinite group G. The profinite group ring $k[G]$ is dual to the coalgebra C of locally constant functions from G to k, and the Galois cohomology $H^*_{\mathrm{ctn}}(G, M)$ of G with coefficients in a discrete G-module M is the same as $\mathrm{Cotor}^*_C(M, k)$; see Section 4 for details.

When C is graded and each C_n is finite-dimensional, we prove that $\mathrm{Cotor}^*_C(M, N)$ is the *graded* Hochschild cohomology $H^*_{gr}(M, N)$, at least when the graded modules M and N are bounded above (Theorem 5.6). Here there is extra bookkeeping for indices, but no topology is needed.

For example, consider the Steenrod algebra A^* over $\mathbb{F}_p$, and let X, Y denote topological spaces. Their homology are comodules over the dual A_*, and we have $\mathrm{Cotor}^n_{A_*}(H_*(X), H_*(Y)) = H^n(A^*, H_*(X \times Y))$; see Section 5.

In Section 6 we extend our results to the differential graded case, and tie them in with the original formulation and application of Eilenberg and Moore [EM].

The special case when A is a Frobenius algebra (and therefore finite-dimensional) was previously considered by the first author in [Abr]. The construction there makes use of the special duality characterizing Frobenius algebras to produce a coalgebra structure on the dual of A and to translate the comodule constructions to module constructions. Proposition 3.3 here fills in a gap in the proof of [Abr, 4.6.1].

Notation

The symbol $\otimes$ denotes tensor product over k, and the term "algebra" always refers to a k-algebra. For any vector space V, write DV for the dual space $\mathrm{Hom}_k(V, k)$. When V is finite-dimensional there are natural isomorphisms $D(V_1 \otimes V_2) \cong (DV_1) \otimes (DV_2)$ and $V \cong D(DV)$.

2 Modules and Comodules

The results in this section are well known; they appear in [Car] and later on in [Sw]. Let k be a field and A a finite-dimensional algebra. It is elementary (see [Sw, 1.1.2]) that the dual of the structure maps $k \to A$ and $A \otimes A \to A$ make the dual $DA = \mathrm{Hom}_k(A, k)$ into a co-associative coalgebra.

If M is a left A-module, the formula $\Delta_M(m)(a) = am$ defines a map

$$\Delta_M\colon M \to \mathrm{Hom}_k(A, M) \cong M \otimes DA\,.$$

By [Car, 4-03] or [Sw, 2.1.2], this makes M into a right DA-comodule. Conversely, if M is a right DA-comodule the map

$$A \otimes M \xrightarrow{1\otimes\Delta} A \otimes (M \otimes DA) \cong (A \otimes DA) \otimes M \xrightarrow{ev\otimes 1} M$$

makes M into a left A-module by [Sw, 2.1.1]. As observed in *loc. cit.*, these processes are inverse to each other, and module maps correspond to comodule maps by [Car] or [Sw, 2.1.3(e)]. We summarize this as follows.

Theorem 2.1 *Let A be a finite-dimensional algebra. Then there is an equivalence between the category of left A-modules and the category of right DA-comodules.*

$$A\text{-}\mathbf{mod} \cong \mathbf{comod}\text{-}DA$$

Note in particular that the comodule structure map

$$\Delta_A\colon A \to \mathrm{Hom}_k(A, A) \cong \mathrm{End}_k(A)$$

is the right regular representation: $\Delta_A(b)(a) = ab$. In particular, the composition with $k \to A$ yields the usual trace map $\tau_A\colon k \to A \otimes DA \cong \mathrm{End}_k(A)$.

When M is finite-dimensional, we can use duality to give another interpretation of this equivalence. If M is any left A-module, $\mathrm{Hom}_k(M, k)$ is naturally a right A-module by the rule $(fa)(m) = f(am)$. If M is finite-dimensional, the dual of the module structure map $\rho\colon DM \otimes A \to DM$ is a map $D\rho\colon M \to M \otimes DA$; it is straightforward that $D\rho$ makes M into a right DA-comodule. Elementary considerations show that this duality $M \mapsto DM$ gives an equivalence between the category of finite-dimensional left A-modules and the category of finite-dimensional right DA-comodules.

Lemma 2.2 *When M is finite-dimensional, the structure map Δ_M agrees with the duality map $D\rho\colon M \to M \otimes DA$.*

Proof. Let $\{e_i\}$ be a basis for A and $\{e^i\}$ the dual basis of DA. By inspection, when $M = A$ we have

$$(D\rho)(1)(e^i \otimes e_j) = \rho(e^i \otimes e_j)(1) = \delta_{ij},$$

so $(D\rho)(1) = \sum e_i \otimes e^i = \Delta_A(1)$. Now given $m \in M$, let $f_m\colon A \to M$ be the module homomorphism with $f_m(1) = m$. By naturality of ρ and Δ,

$$\Delta_M(m) = (f_m \otimes DA)\Delta_A(1) = (f_m \otimes DA)(D\rho)(1) = D\rho(m). \qquad \blacksquare$$

Note that even when M is not finite-dimensional, and hence $D\rho$ might not be a comodule structure map, the map Δ_M can still be expressed by $\Delta_M(m) = (f_m \otimes DA)(D\rho)(1)$, where $f_m\colon A \to M$ is as in the proof of the lemma.

Any construction on A-modules translates into a corresponding construction on DA-comodules. The contragradient representation is one example.

Definition 2.3 Let C be a finite-dimensional coalgebra. If M is a right C-comodule, $DM = \mathrm{Hom}_k(M, k)$ is the left C-comodule with structure map

$$\Delta = \Delta_{DM}\colon DM \to DM \otimes C \cong \mathrm{Hom}(M, C)$$

defined by the formula $(\Delta f)(m) = (f \otimes 1)(\Delta m)$.

We leave it to the reader to check that this is the comodule structure obtained by regarding M as a left module over $A = DC$, and translating the right A-module structure on DM into a left DA-comodule structure.

Remark 2.4 As an exercise, the reader might enjoy verifying that Δ_{DM} is the dual of the map $\mu_M\colon A \otimes M \to M$.

3 Cotensor product and Cotor

In this section, A will be a finite-dimensional k-algebra. If M is a left A-module, and N a right A-module, then the results of the last section allow us to view M as a right DA-comodule, and N as a left DA-comodule. As such, we can form the cotensor product $M \square N = M \square_{DA} N$. Recall from [EM] that $M \square N$ is defined to be the kernel of the map

$$\phi = \phi_{M,N}\colon M \otimes N \to M \otimes DA \otimes N, \ \phi(m \otimes n) = \Delta_1(m) \otimes n - m \otimes \Delta_2(n).$$

Here $\Delta_1\colon M \to M \otimes DA$ and $\Delta_2\colon N \to DA \otimes N$ are the comodule structure maps.

Proposition 3.1 *If M and N are left and right A-modules, respectively, then there is a natural isomorphism*

$$M \Box_{DA} N \;\cong\; \mathrm{Hom}_{A^e}(A, M \otimes N)$$

Proof. Note first that every bimodule map $f\colon A \to M \otimes N$ is determined by $f(1) \in M \otimes N$, so the map $\mathrm{Hom}_{A^e}(A, M \otimes N) \to M \otimes N$ defined by $f \mapsto f(1)$ is injective. To identify the image, consider $\omega = \Sigma\, m_i \otimes n_i$. The assertion that $\omega = f(1)$ for some bimodule map f is equivalent to the condition

$$\Sigma\, a \cdot m_i \otimes n_i \;=\; \Sigma\, m_i \otimes n_i \cdot a, \text{ for all } a \in A. \qquad (*)$$

Now, $(*)$ holds if and only if, for all $a \in A$,

$$(1_M \otimes a \otimes 1_N)(\Sigma\, \Delta_1(m_i) \otimes n_i) \;=\; (1_M \otimes a \otimes 1_N)(\Sigma\, m_i \otimes \Delta_2(n_i)).$$

or, equivalently, if and only if

$$\Sigma\, \Delta_1(m_i) \otimes n_i \;=\; \Sigma\, m_i \otimes \Delta_2(n_i),$$

i.e., if and only if $\omega \in M \Box N$. Thus $f(1) \in M \Box N$ for all bimodule maps f, and each $\omega \in M \Box N$ defines a bimodule map f with $f(1) = \omega$. ∎

Remark 3.2 Because Hochschild cohomology is defined as a functor on bimodules (see §4 and 5), it is useful to generalize the $\Box$ construction to A-bimodules. If B is an A-bimodule, the left and right A-module structures yield maps $\Delta_1\colon B \to B \otimes DA$ and $\Delta_2\colon B \to DA \otimes B$. Writing T for the twisting operator, we set

$$\phi_B = \Delta_1 - T\Delta_2\colon B \to B \otimes DA,$$

and define $\Box(B) := \ker(\phi_B)$. If $B = M \otimes N$, where M and N are left and right A-modules respectively, then this ϕ_B agrees with the previous $\phi_{M,N}$ modulo the natural twist isomorphism $M \otimes DA \otimes N \cong M \otimes N \otimes DA$. By definition, $\Box(M \otimes N) = M \Box N$.

Choose a k-basis $\{e_i\}$ for A and dual basis $\{e^i\}$ for DA. In the particular case $B = A \otimes A$, the structure maps Δ_1, Δ_2 are essentially the right and left regular representations, respectively. In terms of the bases we have, for any $b \in B = A \otimes A$,

$$\phi_B(b) \;=\; \sum_{i,j} \left(e_i b \otimes e^i \;-\; b e_j \otimes e^j\right) \;=\; \sum_i (e_i b - b e_i) \otimes e^i$$

By naturality of ϕ we can use a bimodule map $A \otimes A \to B$ to show that this formula holds for any bimodule B.

To describe the right derived functors of $\Box$, we require a preliminary result on injective modules over finite-dimensional algebras.

Proposition 3.3 *If E_1 is an injective left A-module, and E_2 is an injective right A-module, then $E_1 \otimes E_2$ is an injective left A^e-module.*

Proof. Because A is noetherian, every injective module is a direct sum of indecomposable modules [AF, 25.6]. Since direct sums of injective A^e-modules are injective, we may assume that E_1 and E_2 are indecomposable injectives, *i.e.*, $E_1 = E(A/I_1)$ and $E_2 = E(A/I_2)$ for some one-sided ideals I_1 and I_2 (see [AF, 18.12.3]). But there are surjections $A^n \to D(A/I_i)$, yielding injections $A/I_i \to (DA)^n$. Hence E_1 and E_2 are direct summands of some $(DA)^n$. A fortiori, $E_1 \otimes E_2$ is a direct summand of $(DA)^n \otimes (DA)^n$, which is a sum of copies of the injective A^e-module $DA \otimes DA$. ■

A hint to the following result appears in [Doi, 3.1], but the discussion there remains in the category of comodules.

Theorem 3.4 *Let A be a finite-dimensional algebra. If M is a left A-module, and N a right A-module, there is a natural isomorphism*

$$\mathrm{Cotor}^*_{DA}(M, N) \cong H^*(A, M \otimes N).$$

Proof. This is a formal consequence of 3.1 and 3.3. Choosing an injective resolution $M \to E_1^*$ in A-**mod**, and $N \to E_2^*$ in **mod**-A, the Cotor groups are the cohomology of the chain complex

$$E_1^* \Box E_2^* \cong \mathrm{Hom}_{A^e}(A, E_1^* \otimes_k E_2^*).$$

On the other hand, $M \otimes N \to E_1^* \otimes E_2^*$ is an injective resolution in A^e-**mod**, so by 3.3 the same complex computes the Hochschild cohomology groups $H^*(A, M \otimes N) = \mathrm{Ext}^*_{A^e}(A, M \otimes N)$. ■

4 Profinite algebras

We now generalize the results of §3 to comodules over an arbitrary coalgebra C. Given any right C-comodule M, we can make M into a left module over $A = DC$, via the composite [Car][Sw, 2.1.1]

$$A \otimes M \xrightarrow{1 \otimes \Delta} A \otimes (M \otimes C) \cong M \otimes (A \otimes C) \xrightarrow{1 \otimes ev} M.$$

Similarly, we can make any left C-comodule N into a right A-module. As before, we can consider $M \otimes N$ as an A-bimodule. The proof of Proposition 3.1 readily extends to the current context.

Proposition 4.1 *Let M and N be right and left C-comodules, respectively. There is a natural isomorphism*

$$M \Box_C N \cong \mathrm{Hom}_{A^e}(A, M \otimes N)$$

The analogue of Theorem 3.4 for the Cotor groups requires a more robust module context, which begins with the correspondence between arbitrary coalgebras and profinite algebras.

Any coalgebra C is the union of its finite-dimensional subcoalgebras C^α, so its dual $A = \mathrm{Hom}_k(C, k)$ is the inverse limit of the finite-dimensional algebras $A_\alpha = D(C^\alpha)$. We view $A = \{A_\alpha\}$ as a pro-object in the category of finite-dimensional algebras, a structure we will refer to as a *profinite algebra.* Each A_α is isomorphic to A/I_α, where I_α is the ideal of functions vanishing on C^α; these ideals define a topology on A.

Conversely, suppose that $A = \{A_\alpha\}$ is a profinite algebra. Then the union C of the filtered system of duals $C^\alpha = D(A_\alpha)$ is a coalgebra, and clearly A is the profinite algebra associated to C. This establishes an equivalence between the category of coalgebras and the opposite category of profinite algebras [Br][Wit].

Note that a profinite algebra carries more structure, (*e.g.* topology) than its inverse limit. This is clearly seen in the papers by Radford and Witkowski on reflexive coalgebras [Rad][Wit].

Definition 4.2 Let $A = \{A_\alpha\}$ be a profinite algebra. We say that a (left) A-module M is *rational* if each element of M generates a finite-dimensional A-submodule isomorphic to a quotient of some A_α.

The category A-$\mathbf{mod_{rat}}$ of rational modules and A-module maps is an abelian category, and the inclusion A-$\mathbf{mod_{rat}} \subset A$-$\mathbf{mod}$ is exact. The argument of [Wei, 6.11.10] shows that A-$\mathbf{mod_{rat}}$ has enough injectives.

Radford [Rad, 2.2] and Witkowski [Wit] have shown that our notion of rational module agrees with Sweedler's notion [Sw, p. 37]. That is, an A-module M is rational in our sense if and only if the map $M \xrightarrow{\rho} \mathrm{Hom}(A, M)$ defined by $\rho(m)(a) = a \cdot m$ lands in the subspace $M \otimes C$ of $\mathrm{Hom}(A, M)$.

If M is a rational module, we can regard it as a C-comodule as follows. Since $M = \bigcup Am$, it suffices to consider Am. Choose α so that Am is an A_α-module. By Theorem 2.1, Am is a comodule over C^α and hence over C. This proves Sweedler's theorem [Sw, 2.1.3], which we record here.

Theorem 4.3 *If C is a coalgebra and A the dual profinite algebra, then there is an equivalence between the category of right C-comodules and the category of left rational A-modules.*

$$A\text{-}\mathbf{mod}_{\mathbf{rat}} \cong \mathbf{comod}\text{-}C$$

Example 4.4 Let G be a profinite group, with finite quotients G_α. The group rings $k[G_\alpha]$ are dual to the Hopf algebras k^{G_α} of functions $G_\alpha \to k$. Hence the profinite group algebra $\{k[G_\alpha]\}$ corresponds to the coalgebra $C = \bigcup k^{G_\alpha}$ of locally constant functions $G \to k$. Regarding each G_α as the algebraic group $\mathrm{Spec}(k^{G_\alpha})$, we get a pro-algebraic group $\{G_\alpha\}$.

A rational $k[G]$-module is the same thing as a *discrete G-module* in the sense of Galois cohomology [Wei]. By Sweedler's theorem, it is also just a comodule for C. Now a *rational representation* of the pro-algebraic group $\{G_\alpha\}$ is just a union of rational representations of the G_α, each of which is just a k^{G_α}-comodule (see [Fog, 2.23]). Thus a discrete G-module may be thought of as a rational representation of $\{G_\alpha\}$.

Example 4.5 (Taft [T]) An A-module M is called *locally finite* if each element of M generates a finite-dimensional A-submodule. This notion does not involve the topology on A, and is weaker than the notion of rational module.

For example, let V be an infinite-dimensional k-vector space and form the coalgebra $C := k \oplus V$, where the elements of V are primitive. The dual algebra is $A = k \oplus V^*$. The A-module $M := V^{**}$ is locally finite (for each $m \in M$, the submodule Am is two-dimensional) but not rational (Am need not be a quotient of any A_α).

We now turn to A-bimodules. In order to define continuous Hochschild cohomology, we again need to consider the topology of A.

Definition 4.6 We say that an A-bimodule B is *rational* if each element of B generates a finite-dimensional sub-bimodule isomorphic to a quotient of some A_α. That is, B is a rational module over the profinite algebra $A\widehat{\otimes}A^{\mathrm{op}} = \{A_\alpha \otimes A_\alpha^{\mathrm{op}}\}$.

We define the *continuous Hochschild cohomology* of B to be the right-derived functors $H^*_{\mathrm{ctn}}(A,B)$ of

$$B \mapsto \mathrm{Hom}_{A^e}(A,B) = \{b \in B : ab = ba \text{ for all } a \in A\}.$$

Recall [Wei, 6.5.1] that if B is an A-bimodule, its Hochschild cohomology $H^*(A,B)$ is defined to be the homology of the cochain complex $C^*(A,B)$, where $C^n(A,B)$ denotes the k-module of $(n+1)$-fold multilinear maps from A to B, *i.e.,* maps $f\colon A^{\otimes n} \to B$. When $n = 0$, $C^0(A,B) = B$.

When $A = \{A_\alpha\}$ is a profinite algebra and B is a rational bimodule, we define $C^n_{\mathrm{ctn}}(A,B)$ to be the subspace of $C^n(A,B)$ consisting of *continuous cochains*, *i.e.,* maps which factor through some quotient $A_\alpha^{\otimes n}$ of $A^{\otimes n}$.

For each finite quotient $A_\alpha = A/I_\alpha$ of A, set $B^\alpha := \{b \in B : I_\alpha b = bI_\alpha = 0\}$. Then B^α is an A_α-bimodule, and $B = \bigcup B^\alpha$ if B is rational.

Proposition 4.7 *We have* $C^*_{\mathrm{ctn}}(A,B) = \varinjlim C^*(A_\alpha, B^\alpha)$.

Proof. The composite of $A^{\otimes n} \to A_\alpha^{\otimes n}$ with a map $A_\alpha^{\otimes n} \to B^\alpha$ is a continuous cochain, so $C^*(A_\alpha, B^\alpha) \to C^*(A,B)$ is an injection. Conversely, given a continuous cochain $f\colon A^{\otimes n} \to B$, there is an α so that f factors through $A_\alpha^{\otimes n}$. Because $A_\alpha^{\otimes n}$ is finite-dimensional and B is rational, the image $f(A^{\otimes n})$ lies in some B^β. Choosing γ so that A_γ maps to A_α and A_β, we see that $f \in C^n(A_\gamma, B^\gamma)$. ∎

Remark 4.8 Our terminology comes from the fact that the profinite algebra $A\widehat{\otimes}\cdots\widehat{\otimes}A = \{A_\alpha^{\otimes n}\}$ may be regarded as a topological algebra. If B has the discrete topology, a continuous map $A\widehat{\otimes}\cdots\widehat{\otimes}A \to B$ must factor through some $A_\alpha^{\otimes n}$ and hence be a continuous cochain.

Theorem 4.9 *If B is a rational bimodule over a profinite algebra A, then $H^*_{\mathrm{ctn}}(A,B)$ is the cohomology of the complex $C^*_{\mathrm{ctn}}(A,B)$. In particular,*

$$H^*_{\mathrm{ctn}}(A,B) \cong \varinjlim H^*(A_\alpha, B^\alpha).$$

Proof. For simplicity, set

$$T^n(B) = H^n C^*(A,B) = \varinjlim H^n C^*(A_\alpha, B^\alpha).$$

Note that $C^0(A,B) \cong B$, and thus $T^0(B) = H^0_{\mathrm{ctn}}(A,B)$. We now argue as in the proof of [Wei, 6.11.13]. The set $\{T^n\}$ forms a δ-functor because

$H^*C^*(A_\alpha, B^\alpha)$ is the Hochschild cohomology $H^*(A_\alpha, B^\alpha)$ for each α. To see that this δ-functor is universal, note that if J is an injective object in $A\widehat{\otimes}A$-$\mathbf{mod_{rat}}$ then each J^α is an injective A_α-bimodule (because $B \mapsto B^\alpha$ is right adjoint to the forgetful functor). Hence if $n \neq 0$ then

$$T^n(J) = \varinjlim H^n(A_\alpha, J^\alpha) = 0. \qquad \blacksquare$$

Example 4.10 Let $G = \{G_\alpha\}$ be a profinite group and M a discrete left G-module. A continuous cochain $f\colon G^n \to M$ in the sense of Galois cohomology is one which factors through some quotient G^n_α of G^n. Its linear extension $k[G]^{\otimes n} \to M_\epsilon$ is a continuous cochain, where M_ϵ is the bimodule with trivial right G-action. Thus $C^*_{\text{ctn}}(k[G], M_\epsilon)$ is the chain complex used to compute the Galois cohomology of G. It follows that $H^*_{\text{ctn}}(k[G], M) \cong H^*_{\text{ctn}}(G, M)$.

We can now prove the main theorem.

Theorem 4.11 *If C is a coalgebra, A its profinite dual, and M, N are C-comodules, then*

$$\operatorname{Cotor}^*_C(M, N) = H^*_{\text{ctn}}(A, M \otimes N).$$

Proof. Let $\{C^\alpha\}$ and $\{A_\alpha\}$ be as in the beginning of this section. Define $M^\alpha := \{x \in M \mid I_\alpha x = 0\} = \{x \in M \mid \Delta_M x \in M \otimes C^\alpha\}$ and define N^α similarly. Note that $M^\alpha \otimes N^\alpha = (M \otimes N)^\alpha$. By Theorem 4.3 M, N, M^α, N^α may all be viewed as rational A-modules.

Note that $\operatorname{Cotor}_{C^\alpha}(M^\alpha, N^\alpha)$ can be calculated with the complex

$$M^\alpha \otimes N^\alpha \to M^\alpha \otimes C^\alpha \otimes N^\alpha \to M^\alpha \otimes C^\alpha \otimes C^\alpha \otimes N^\alpha \to \cdots$$

Taking direct limits, we get a complex for $\operatorname{Cotor}_C(M, N)$. Since homology commutes with this particular limit, we have

$$\operatorname{Cotor}^*_C(M, N) \;=\; \varinjlim \operatorname{Cotor}^*_{C^\alpha}(M^\alpha, N^\alpha)$$

Applying Proposition 4.7 and Theorem 3.4 gives

$$\varinjlim \operatorname{Cotor}^*_{C^\alpha}(M^\alpha, N^\alpha) \;=\; \varinjlim H^*(A_\alpha, M^\alpha \otimes N^\alpha) \;=\; H^*_{\text{ctn}}(A; M \otimes N),$$

which completes the proof. $\blacksquare$

5 Graded modules and comodules

Replacing the category of vector spaces with the category of graded vector spaces does not change things very much, as we now explain.

First, we need to fix our notation, for which we follow [Mac] and [EM]. The category of graded vector spaces has an internal Hom, constructed as follows. Given two graded vector spaces V and V', the degree p component $\mathrm{Hom}_k^p(V, V')$ of $\mathrm{Hom}_k^*(V, V')$ is the vector space of all maps $f\colon V \to V'$ of degree p, i.e., satisfying $f(V^i) \subset (V')^{i+p}$. The indexing is set up so that the evaluation map $\mathrm{Hom}_k^*(V, V') \otimes V \to V'$ is homogeneous of degree zero. The *graded dual* $DV = \mathrm{Hom}^*(V, k)$ of V arises as the special case $V' = k$. Thus $(DV)_n = (DV)^{-n} = \mathrm{Hom}_k(V^n, k)$. If each V^n is finite-dimensional we say that V is of *finite type*; this is the hypothesis needed to have $D(DV) \cong V$.

By a graded algebra A we will mean a positively graded algebra $A^0 \oplus A^1 \oplus \cdots$. We will assume throughout this section that A is of finite type.

By a graded coalgebra C we will mean a positively graded coalgebra $C_0 \oplus C_1 \oplus \cdots$. If A is a graded k-algebra of finite type, then DA is a co-associative graded coalgebra by [MM, 3.1(4)] or [Sw, 6.0.2]. This is because the tensor product $A \otimes A$ is of finite type.

Note that a graded module M is rational if and only if it is locally finite (example 4.5); the topology on A plays no role when A is of finite type. As a typical example, if M is a *bounded-above* module (*i.e.,* $M^p = 0$ for $p \gg 0$) then clearly M is locally finite, because A has finite type.

Proposition 5.1 *If A is of finite type, then the category of graded right DA-comodules is equivalent to the category of locally finite, graded left A-modules. Under this equivalence, the category of bounded-below comodules ($M_p = 0$ for $p \ll 0$) is equivalent to the category of bounded-above modules.*

Proof. If M_* is a comodule, the classical formula for the A-module structure on M makes it a locally finite graded module (see [MM]). Conversely, if M^* is a graded module, the formula $\Delta_M(m)(a) = am$ is homogenous, and $\Delta_M(m)$ is in $M \otimes DA$ if M is locally finite. ∎

Suppose now that M is a graded right comodule and N a graded left comodule. Then we can consider M and N as left and right modules, respectively, and form the graded bimodule $M \otimes N$. Applying the graded $\mathrm{Hom}_{A^e}(A, -)$ (defined as in [Mac, p. 185]) allows us to construct the graded Hochschild cohomology $H^*_{gr}(A, M \otimes N)$, as in [Mac, p. 300].

The proof of Proposition 3.1 goes through to prove the following analogue.

Proposition 5.2 *Let A be a graded algebra of finite type, with dual coalgebra C. If M and N are locally finite left and right graded A-modules, respectively, then there is a natural isomorphism*

$$M \square_C N \;\cong\; \mathrm{Hom}_{A^e}(A, M \otimes N)$$

In order to prove the analogue of Theorem 3.4, we require some preliminary results.

Lemma 5.3 *If V is a bounded above graded vector space, then $DA \otimes V$ is an injective graded A-module (and hence also an injective graded DA-comodule), and $DA \otimes V \otimes DA$ is an injective graded A-bimodule.*

Proof. Because $\mathrm{Hom}_k(A, -)$ is right adjoint to the forgetful functor from graded A-modules to graded vector spaces, it preserves injectives [Wei, 2.3.10]. Hence $\mathrm{Hom}_k(A, V)$ is injective. In general, $\mathrm{Hom}_k^n(A, V) = \prod_{i+j=n} DA^i \otimes V^j$ is larger than $(DA \otimes V)^n = \bigoplus DA^i \otimes V^j$. But if V is bounded above, we have $DA \otimes V = \mathrm{Hom}(A, V)$, which is injective, and similarly

$$DA \otimes V \otimes DA = D(A \otimes A) \otimes V = \mathrm{Hom}_k(A^e, V)$$

is injective as a graded A^e-module, *i.e.,* a graded bimodule. ∎

Remark 5.4 If A is a Noetherian algebra, then direct sums of injective modules are injective. Since DA and $DA \otimes DA$ are injective (compare with the proof of Proposition 3.3), we may remove the hypothesis that V be bounded above.

Corollary 5.5 *If M and N are bounded-above graded A-modules, there are bounded-above graded vector spaces V^i and W^i and injective resolutions $M \to I^*$, $N \to J^*$ with $I^i \cong DA \otimes V^i$, $J^i \cong W^j \otimes DA$. Moreover, $M \otimes N \to I^* \otimes J^*$ is an injective bimodule resolution.*

Theorem 5.6 *Suppose A is a graded k-algebra of finite type, with dual coalgebra C, and that M, N are bounded-above graded A-modules. Then for each n there is a natural isomorphism of graded vector spaces:*

$$\mathrm{Cotor}^n_C(M, N) \cong H^n_{gr}(A, M \otimes N).$$

Proof. Take injective resolutions $M \to I^*$ and $N \to J^*$ as in Corollary 5.5. These lie in the category of graded DA-comodules. Because I^p and J^q are bounded below, the proof of Proposition 3.1 shows that $I^p \Box J^q = \mathrm{Hom}_{A^e}(A, I^p \otimes J^q)$. Thus we have

$$\begin{aligned} \mathrm{Cotor}^n_C(M,N) &= H^n(I^* \Box J^*) \\ &= H^n \mathrm{Hom}_{A^e}(A, I^* \otimes J^*) \\ &= H^n_{gr}(A, M \otimes N) \quad \blacksquare \end{aligned}$$

Remark 5.7 Eilenberg and Moore actually use homological indexing, defining $\mathrm{Cotor}^C_n(M,N)$ to be the product (over $p+q=n$) of the $\mathrm{Cotor}_{p,q}(M,N)$, which in our notation is the homological degree p part of the graded vector space $\mathrm{Cotor}^{-q}_C(M,N)$; see [EM, p. 207].

Example 5.8 Let A^* denote the Steenrod algebra over $\mathbb{F}_p$. If X is a topological space, its cohomology $H^*(X)$ is a bounded-below module over A^* and its homology $H_*(X)$ is a graded comodule over the dual A_*. We can view $H_*(X)$ as cohomologically bounded above, and hence by 5.1 as a bounded-above module over A^*, with $H^*(X) = DH_*(X)$. Because $H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$ is a bicomodule over A^*, we have

$$H_*(X) \Box_{A_*} H_*(Y) = \mathrm{Hom}_{A^e}\left(A^*, H_*(X \times Y)\right).$$

and

$$\mathrm{Cotor}^n_{A_*}\left(H_*(X), H_*(Y)\right) = H^n\left(A^*, H_*(X \times Y)\right).$$

6 The Differential Graded Case

We now pass to the differential graded setting, using essentially the same notation as in the graded case. By a differential graded algebra (or DG-algebra) we will mean a graded algebra A with a differential of degree one, satisfying the usual Leibniz relation for $d(aa')$. We will regard A as a cochain complex:

$$0 \to A^0 \to A^1 \to \cdots.$$

We use cochain notation M^* for DG A-modules, so their differential has degree one.

By a differential graded coalgebra (or DG-coalgebra) C we will mean a positively graded coalgebra with a differential of degree -1, satisfying the co-Leibniz relation. We will regard C as a chain complex:

$$\cdots \to C_1 \to C_0 \to 0.$$

For example, $C = DA$ is a DG-coalgebra with $C_n = \mathrm{Hom}(A^n, k)$. We use chain complex notation for DG C-comodules, so their differential has degree -1.

Proposition 6.1 *If A is of finite type, then the category of right DG DA-comodule is equivalent to the category of locally-finite left DG A-modules.*

Proof. The only point which must be added to the proof for the graded case (Proposition 5.1) is that the same differential is used for both A-modules and DA-comodules. ∎

If M and N are right and left DG-comodules, then their cotensor product $M\Box_C N$ is naturally a DG-vector subspace of $M \otimes N$. Yet again, the proof of Proposition 3.1 goes through to prove the following DG analogue.

Proposition 6.2 *Let A be a DG algebra of finite type, with dual coalgebra C. If M and N are locally finite left and right DG A-modules, respectively, then there is a natural isomorphism of DG vector spaces:*

$$M\Box_C N \;\cong\; \mathrm{Hom}_{A^e}(A, M \otimes N).$$

The cotorsion groups $\mathrm{Cotor}_n^C(M, N)$ were defined by Eilenberg and Moore in [EM, p. 206] as follows. If $M \to I^*$ and $N \to J^*$ are injective DG comodule resolutions, then $I^*\Box_C J^*$ is a triple chain complex whose (i, j, k) entry is $(I^{-i}\Box_C J^{-j})_k$. Here, the index k refers to the total internal grading of the DG comodules I^{-i}, J^{-j}. The group $\mathrm{Cotor}_n^C(M, N)$ is defined to be the nth homology of the (product) total complex of $(I^{-i}\Box_C J^{-j})_k$. The homological indexing reflects not only the applications to homology of spaces (see 6.4 below), but the fact that we want $\mathrm{Cotor}_n^C(C, C) \cong H_n(C)$ to be positively indexed.

If M and N are left and right DG A-modules, their tensor product $M\otimes N$ may be viewed as a DG bimodule. As such, we may form the differential graded Hochschild cohomology $H^p_{DG}(A, M \otimes N)$.

Theorem 6.3 *Suppose A is a DG algebra of finite type with dual coalgebra C and M, N are bounded-above DG A-modules. Then for each n there is a natural isomorphism:*

$$\mathrm{Cotor}_n^C(M, N) \cong H_{DG}^{-n}(A, M \otimes N).$$

Proof. Both Lemma 5.3 and 5.5 carry over to the DG setting with almost no change. Thus we can choose injective objects I^* and J^* in the category of DG-modules over $A = DC$ such that $M \otimes N \to I^* \otimes J^*$ is an injective resolution in the category of DG-bimodules.

By definition, $H_{DG}^n(A, M \otimes N)$ is the nth cohomology of the (product) total complex of the triple cochain complex $\mathrm{Hom}_{A^e}(A, I^* \otimes J^*)$. The (i, j, k) entry of this complex is

$$\mathrm{Hom}_{A^e}(A, I^i \otimes J^j)^k = (I^i \Box_C J^j)_{-k}.$$

The standard conversion between chain complexes and cochain complexes ($V^n = V_{-n}$) identifies this triple complex with the triple chain complex used above to define $\mathrm{Cotor}^C(M, N)$. ∎

Example 6.4 (Eilenberg-Moore) Suppose given a cartesian square of CW complexes

$$\begin{array}{ccc} E' & \to & E \\ \downarrow & & \downarrow \pi \\ B' & \to & B \end{array}$$

in which B is simply connected, with finitely many cells in each dimension, and π is a fibration. Let C_*B be the cellular complex of B with coefficients in a field k, and $C^*B = \mathrm{Hom}(C_*B, k)$ its dual algebra. This is a DG algebra of finite type. The main result (theorem 12.1) of [EM], combined with our Theorem 6.3 (and the Eilenberg-Zilber theorem) states that the homology of E' is isomorphic to

$$H_*(E'; k) \cong \mathrm{Cotor}^{C_*B}(C_*E, C_*B') \cong H_{DG}^*(C^*B, C_*(E \times B')).$$

In particular, if F is the fiber of the fibration $E \to B$, then (taking B' to be a point) we have

$$H_*(F; k) \cong \mathrm{Cotor}^{C_*B}(C_*E, k) \cong H_{DG}^*(C^*B, C_*E \otimes k).$$

Finally, if E is contractible then $C_*E \cong k$ and

$$H_*(\Omega B; k) \cong \mathrm{Cotor}^{C_*B}(k, k) \cong H_{DG}^*(C^*B, k).$$

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