Problem Sheet 3

Due date: 03 February 2016 16:00

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.

Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format. I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework. News and .pdf-files of Problems also at home.gwu.edu/~hgrie/lectures/nupa-16I/nupa-16I.html.

1. **Standard Toy Model of Low-Energy Nuclear Physics (5P):** At kinetic energies much smaller than the particle mass, we can resort to a non-relativistic theory. We thus explore the complex scalar field $\Phi(x)$ of a particle with mass $M$ and charge $Ze$, using the “cooking recipes” of the lecture. The Lagrangean is

$$\mathcal{L} = \Phi^\dagger [i\partial_0 + \vec{D}^2] \Phi - C (\Phi^\dagger \Phi)^2$$

where $\vec{D} = \vec{\partial} - iZe\vec{A}$ is the gauge-covariant derivative.

   a) (2P) Derive the propagator. It’s not relativistic!
   b) (3P) Derive the Feynman rules for all three interactions, with their corresponding diagrams.

2. **Positronium (5P):** is the hydrogen-like bound state between electron and positron, where the spins can couple to total spin 0 (para) or 1 (ortho). This state has a finite lifetime since the probability amplitude for both particles to overlap is nonzero when they are in a relative $S$-wave. Using discrete symmetries, consider:

   a) (2P) For the decay into photons only, at least how many photons are generated from para-positronium? How many from ortho-positronium?
   b) (1P) Estimate the ratio of the two life times.
   c) (2P) Show that the positronium decay into $\pi^0\pi^0$ is forbidden.

3. **Electron Scattering on Deuterons (4P):** In Spring 2014, the A1 collaboration at MAMI conducted an experiment on $d(e,e')d$ scattering to determine the deuteron form factor. The cross section for the process is on the 10% level predicted as (see next lectures for details):

$$\frac{d\sigma}{d\Omega}_{\text{lab}} = \left(\frac{\alpha}{2E\sin^2\frac{\theta}{2}}\right)^2 \cos^2\frac{\theta}{2}$$

According to the MAMI beam schedule (online), a beam of 20 $\mu$A and 1 cm$^2$ cross section is focused on a liquid-deuterium target with $1.0 \times 10^{23}$ deuterons per cm$^2$ of beam cross section. (Yes, they constructed the target to give a nice and round number!) The experiment is run at many angles and energies, but we pick a 200 MeV beam and 100° scattering angle. The scattered electrons are detected with one of the A1 spectrometers. Its opening is 28 msr (opening must be very small because the momentum resolution must be very good). Let’s say the detector system has an efficiency of 50%. Confirm that you expect a cross section of about 0.15 $\mu$b/sr; determine the luminosity in MHz/$\mu$b; find the number of events counted per second; and finally calculate the runtime to achieve a measurement with a statistical accuracy of 1%.

Please turn over.
4. **Form Factor for a Uniformly Charged Sphere (3P):** Find the form factor of a homogeneously charged sphere with total charge $Ze$, i.e. $\rho = \text{const.}$ for $r < R$, zero otherwise. This is useful for heavy nuclei. Show that this form factor has a zero at $qR \approx 4.5$, and that the relation between root-mean-square radius and sphere size is $\langle r^2 \rangle = \frac{5}{3} R^2$. Calculate also the relation between $R$, $q$ and the second zero of the form factor.

5. **Current Conservation in Feynman Rules (2P):** We saw in the lecture that the coupling of one photon to the Klein-Gordon or Dirac field, as described by minimal substitution, can be written as $A_\mu j^\mu_{\text{em}}$, where $j^\mu_{\text{em}}$ is the electromagnetic current of the field. It is classically conserved, $\partial_\mu j^\mu_{\text{em}} = 0$, if the fields on the “legs” describe “on-shell” particles ($p^2 = m^2$). We translated the interaction of current and gauge field into Feynman rules, so these in turn should obey current conservation. Instead of doing the Fourier transform of $\partial_\mu j^\mu_{\text{em}} = 0$, let’s go about this pragmatically.

a) (1P) Consider the Feynman rule for the $\Phi^2 A$ vertex of the complex Klein-Gordon field with a photon which carries away a four-momentum $q$: $(-ie)(p_\mu + p'_\mu)$. Show that the contraction of $q$ with the vertex Feynman rule is zero for on-shell particles, i.e. show $q^\mu (-ie)(p_\mu + p'_\mu) = 0$.

b) (1P) The same for the Feynman rule describing the interaction of a Dirac particle with a photon.

6. **Fill in the Details: Lepton Tensor (2P):** In the lecture, the lepton tensor was defined as

$$e^2 L_{\mu\nu} = \frac{1}{2s+1} \sum_{s,s'} j^\mu_{s,s'}(k,k') j^{\nu*}_{s',s}(k,k')$$

and calculated for massless leptons. Derive now the result for the case $m \neq 0$.

7. **Fill in the Details: Mott Cross Section (7P):** For electron scattering on a massive point-particle without spin, the lecture jumped from the averaged and squared matrix element straight to the lab cross section. So, fill in the details of the following (take $L_{\mu\nu}$ as given):

$$|\tilde{M}|^2 = \frac{(Ze^2)^2}{q^4} L_{\mu\nu} (p+p')^\mu (p+p')^\nu \implies \left. \frac{d\sigma}{d\Omega} \right|_{\text{lab}} = \left( \frac{Z\alpha}{2E\sin^2 \frac{q}{2}} \right)^2 \cos^2 \frac{\theta}{2} \frac{E'}{E}.$$

8. **$e^+ e^- \rightarrow \mu^+ \mu^-$, Part I (2P):** Write down the matrix element describing the annihilation of an electron (spin $s$, 4-momentum $k$) and positron (spin $r$; 4-momentum $p$) into a muon and anti-muon of spins $s'$ and $r'$, respectively, at lowest non-vanishing order. We need the result for the next HW.

In the successor to the LHC, the particles will be accelerated by Chuck Norris’ roundhouse kicks.