Problem Sheet 13

Due date: 30 November 2016 16:00

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.

Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.

I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.


1. Repeating the Lecture (6P): For this, you can very closely follow the example in class. A ball with radius $R$ holds a charge density $\rho(\vec{r}) = \theta(R - r) \ W (\vec{n} \cdot \vec{r})^2$. Determine $W$ such that the total charge is $Q$, and find all spherical charge multipoles and the scalar potential $\Phi(r, \theta, \phi)$ outside the ball.

2. A Guillotine’d Cylinder (8P): An infinitely long, infinitesimally thin cylinder of radius $R$ is separated into halves by cutting it along the cylinder axis. The potentials on the halves are constant and opposite, $\Phi_0$ and $-\Phi_0$, see figure. Let’s neglect any effects very close to the cuts.

   a) (4P) Determine the elementary solution of the Laplace equation $\Delta \Phi = 0$ in cylindrical coordinates. Show that the ansatz $\Phi(r, \phi, z) = U(r) \ \chi(\phi)$ leads to a separation of variables:

   $$\frac{r}{U(r)} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} U(r) = -\frac{1}{\chi(\phi)} \frac{\partial^2}{\partial \phi^2} \chi(\phi) = \mu^2$$

   Pay particular attention to the elementary solution for the value $\mu = 0$ of the separation constant, i.e. construct it and then show that is irrelevant for the problem at hand.

   b) (4P) Construct the potential inside and outside the cylinder. The solution is an infinite series. Discuss where the solution converges well, and where it does not.

   Note: The problem has actually an exact solution (and if you find it, you are very, very good at sums – but that’s not a part of the problem):

   $$\Phi(r, \alpha) = \frac{2\Phi_0}{\pi} \arctan \left[ \frac{2a}{1 - a^2} \sin \alpha \right] \text{ with } a = \begin{cases} \frac{r}{R} \text{ for } r < R \\ \frac{R}{r} \text{ for } r > R \end{cases}$$

   where $r, \alpha$ are the polar coordinates when the cylinder is centred at the origin, and the cut separating the halves is along the $x$-axis.

Please turn over.
3. **Why should I have to invent the problems? (9P):** This time, you invent the problem – and its solution. Find a charge distribution on a hollow sphere of radius \( R \) such that only the total charge and the spherical multipoles with \( l = 2, |m| = 1 \) contribute. Provide a complete and concrete solution for the surface charge density \( \rho \) and the electrostatic potential outside the sphere, in Cartesian coordinates. Is your choice unique?

**Example of a similar problem:** When \( \rho(\vec{r}) = \frac{3xq}{4\pi R^3} \delta(R - r) \) with \( q \) some quantity with the dimension of a charge, then only \( l = 1, |m| = 1 \) contributes, the net charge is zero, and \( \Phi(\vec{r}) = \frac{qxR}{(x^2 + y^2 + z^2)^{3/2}} \) for \( r > R \) (if I did not mess up factors).

4. **Complex Functions (7P):** Let \( z = x + iy \) be complex, with \( x \) (\( y \)) its real (imaginary) part.

a) \((1P)\) Decompose the following complex function into real and imaginary part, \( u + iv \): \( \frac{1 + z^2}{i - z^2} \)

b) \((2P)\) Let \( u(x,y) = 2x(1-y) \) be the real part of an analytic function \( f(z) \). Construct its imaginary part \( v(x,y) \).

c) \((4P)\) Decompose \( \ln z \) into its real and imaginary part, \( u + iv \). Determine whether it obeys the Cauchy-Riemann condition \( \partial u/\partial x = \partial v/\partial y, \partial u/\partial y = -\partial v/\partial x \) everywhere, or at nearly all points (and if so, state at which it does not). Determine the nature of each of its singularities.

**Hint:** Sometimes, it helps to derive the form of \( u \) and \( v \) using the detour via the polar representation of a complex number, \( z = |z|e^{i\phi} \). But that is a question of taste.