Problem Sheet 2

Due date: 16 September 2015 16:00

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.

Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.

I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.


1. More Animal Physics (3P): Why does a horse run uphill slower than a dog, although both have the same running speed on level ground?

   **Hint:** Compare the energy an animal needs to lift itself by a height \( h \) to the power it can generate in its muscles. Why is muscular power proportional to the section of the muscle, not to its volume?

2. Dimensional Analysis (4P): By matching dimensions, we can discover some very fundamental relations of modern physics – without understanding the underlying theory.

   a) (2P) Let’s pretend that the mass of the electron is somehow generated by its charge via electromagnetic interactions (so electromagnetic fundamental constants can enter). Estimate the classical charge radius of the electron.

   b) (2P) It does not always work: We are looking for a “electrodynamic theory of gravitation”: Relate the gravitational mass and electromagnetic charge of the electron (and appropriate constants of Nature). Is something like this result found in any theory of fundamental forces?

3. Perihelion Shift of Mercury (7P): An important success of General Relativity is the correct prediction of the angle \( \Delta \phi = 43'' \) per century by which the perihelion (minimum distance to the Sun) of Mercury rotates. This effect, first discovered by Leverrier in 1859, could not be explained by Newtonian gravity. In 1915, Einstein found that exact value from the first post-Newtonian correction to the gravitational potential between two bodies. We now make a back-of-the-envelope estimate of the size of the effect – something which Einstein presumably did before he started on the hard part. The problem is also known as Feynman’s (Oral) Exam Question.

   In the lecture, we already talked about some aspects: We expect that the post-relativistic potential can be written as a Taylor expansion in squares of negative powers of the speed of light, \( c \); with \( r \): distance Sun-Mercury, \( M \): solar mass, \( m \): Mercury mass, \( G_N \): Newton’s gravitational constant:

   \[
   U(r) = -\frac{G_N M m}{r} \left[ 1 + \frac{\beta}{c^2} + O(c^{-3}) \right]
   \]

   a) (1P) Argue how the result depends on \( M, m, r \), period of orbit, other parameters and constants.

   b) (2P) Determine how the parameter \( \beta \) scales in terms of these parameters.

   c) (4P) If the potential were \( 1/r \), orbits would be closed, i.e. after one orbit \( \phi \to \phi + 2\pi \), the planet would return exactly to its previous position. The correction results in the orbit being shifted by an angle \( \delta \phi \) after each rotation. Estimate, using the values below, the precession \( \delta \phi \) after a full orbit from the post-Newtonian correction. From that, estimate the perihelion shift \( \Delta \phi \) per century. Compare with the experimental value.

   **Numbers:** Sun’s Schwarzschild radius: \( \frac{2G_N M}{c^2} = 3 \) km; Mercury’s revolution period: 0.249 years.

Please turn over.
4. **Understand the Difference between “Accuracy” and “Precision” (0P)**

Notice that mathematica stubbornly interchanges the two.

5. **Asymptotic Series (8P):** Consider the integral \( \int_{0}^{\infty} \frac{e^{-xt}}{1+t^2} \) for \( x \to \infty \).

   a) (4P) Using integration by parts or any other method, derive a series representation with remainder \( R_m(x) \).

      **Hint:** Depending on which approach you choose, it might be smart to not perform derivatives of \( 1/(1+t^2) \) explicitly but leave them in implicit form till very late. Then, look at a few derivatives and find a pattern.

   b) (1P) Prove that the series is an asymptotic series.

      **Hint:** If your remainder in a) is too complicated, start from \( R_m(x) = \int_{0}^{\infty} \frac{(-t^2)^{m+1}e^{-xt}}{1+t^2} \).

   c) (1P) Derive a criterion at which point the error in the series should start increasing again. What’s the “sweet spot”, as function of \( x \)?

   d) (2P) Numerically investigate for \( x = 10 \) the dependence on the number of terms \( m \) in the sum retained, up to suitable \( m \). Plot! Provide a prognosis for the “exact” integral at \( x = 10 \), including an error-bar.

6. **Saddle-Point Approximation (4P):** We approximately solve \( I(n) := \frac{\pi}{2} \int_{-\pi/2}^{\pi/2} \cos^n t \) for \( n \to \infty \).

   a) (2P) Apply the saddle-point approximation to the integrand. You see that its integral cannot be done in closed form. By plotting the integrand of the original and approximated function, shown that one can without significant error increase the integration of the approximated integral to have the limits \( \pm \infty \).

   b) (2P) Perform the approximation. Compare with the “exact” result, \( I(n) = \frac{\pi}{2^n} \frac{n!}{[(n/2)!]^2} \), \( n \) even.

7. **Root-Finding (4P):** Find the perturbative roots of the following polynomial to next-to-next-to-leading order and provide an error-assessment, dependent on \( \epsilon \) with |\( \epsilon \)| small:

   \[ \epsilon^2 x^7 - \epsilon x^5 + x - a = 0 \quad a \text{ real} \]

   For which \( a \) is this a good approximation, for which a bad one?