Start-Quiz

Special due date: Friday 5th September 2008 12:00 noon

This quiz allows you and me to judge your math skills. I hope you easily solve it in less than 3 hours. The quiz covers topics which are essential for this course but which I assume you have well-understood and mastered in your undergraduate studies – although this is by far not a complete list. It will allow me to gauge how solid your background is, in order to fine-tune the course accordingly. If you have problems or identify areas unfamiliar to you, contact me as soon as possible and prepare for remedial self-study.

This quiz does not count towards your grade, and I will keep only an anonymised record. Nonetheless, you should immediately get accustomed to hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers.

Solve the probles with pen/pencil and paper. Do *not* use any electronic devices – that would defy the purpose. You may use books to refresh your memory, but make sure you understand each term and know *how* to do the problems.

1. MATRICES: Given the matrix $\mathcal{M} = \begin{pmatrix} \frac{\lambda+3}{4} & \frac{\lambda-1}{4} & \frac{\lambda-1}{2\sqrt{2}} \\ \frac{\lambda-1}{4} & \frac{\lambda+3}{4} & \frac{\lambda-1}{2\sqrt{2}} \\ \frac{\lambda-1}{2\sqrt{2}} & \frac{\lambda-1}{2\sqrt{2}} & \frac{\lambda+1}{2} \end{pmatrix}$, with $\lambda \in \mathbb{C}$ arbitrary.

   a) Determine all eigenvalues and normalised eigenvectors. **Hint:** At least one eienvalue is 1.
   b) Determine a (the?) matrix which diagonalises $\mathcal{M}$ and write down the diagonalised version of $\mathcal{M}$.
   c) Determine the inverse of $\mathcal{M}$. Are there λ for which the inverse does not exist?
   d) Under which condition on λ is $\mathcal{M}$ (i) unitary; (ii) orthogonal; (iii) special-unitary; (iv) special-orthogonal? **Hint:** This problem is independent of a) to c).
   e) More generally: Is the trace of a matrix invariant under unitary transformations? **Hint:** Independent of a) to d).

2. ROTATE the vector $\vec{r} = (1, 2, 3)$ counter-clockwise about the y-axis by $60°$.

3. BASIC ANALYSIS (**1P**):

   a) Calculate the first two non-trivial terms (i.e. the first two terms which are not just pure numbers) in the Taylor series around $x = 0$ of: (i) $\dfrac{1}{\sqrt{1+x^2}}$, (ii) $\dfrac{\tan x}{x}$, (iii) $\dfrac{1-\cos x}{x^2}$
   b) Bring the following expressions into the form $a + \mathrm{i}b$, where $\mathrm{i}^2 = -1$ the imaginary unit and a, b real:
   $$\text{(i) } \sqrt{\mathrm{i}} \ , \quad \text{(ii) } \frac{1+\mathrm{i}}{1-\mathrm{i}} \ , \quad \text{(iii) } \sin\frac{\pi \mathrm{i}}{2}$$
   c) What is the Fourier Series of $f(x) = x$ on the interval $-\pi \leq x \leq \pi$?
   d) What is the Fourier transform of $f(x) = \exp[-\alpha x^2]$, $\alpha > 0$?
   e) Determine $\sum\limits_{n=0}^{\infty} x^n$ for $|x| < 1$.

4. BASIC VECTOR ANALYSIS (**1P**):

   a) Show $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - \vec{c}\,(\vec{a} \cdot \vec{b})$.
   b) Calculate with $\vec{r} = x\,\vec{e}_x + y\,\vec{e}_y + z\,\vec{e}_z$, $r = \sqrt{x^2+y^2+z^2}$, $\vec{e}_r = \frac{\vec{r}}{r}$: (i) $\vec{\nabla} \cdot \vec{r}$, (ii) $\vec{\nabla} \times \vec{r}$, (iii) $\vec{\nabla}\dfrac{1}{r}$

c) Calculate $\Delta\frac{1}{r}$ everywhere.

d) Does the field $\vec{A}(\vec{r}) = \frac{1}{1+x^2}\,\vec{r}$ have sources/sinks or curls? If so, where?

e) Write the vector (x, y, z) in spherical coordinates.

f) Write the integral $\int\limits_{-\infty}^{\infty} \mathrm{d}^3r\; f(\vec{r})$ in cylindrical coordinates. Carefully state the integration limits!

g) A function has the form $f(x, y, z)$, where $y(x), z(x)$ implicitly depend on x. Determine the (i) total; (ii) partial derivative of f with respect to x.

h) Use Gauß' Theorem to show: The integral of the field $\vec{A}(\vec{r}) = \vec{r}$ over the surface of an arbitrarily-shaped volume V is given by $3V$.

i) Verify Stokes' theorem by explicit calculation in the example field $\vec{A} = (x^2y, 2yz, 3xz)$. Pick as surface of integration a disk of radius R, centred at the origin in the xy-plane.

5. Basic Quantum Mechanics: Let's look at the infinite-square well.

a) Write down the normalised ground-state wave-function.

b) Is the operator $p_x x$ Hermitean? Is $[p_x, x]$ Hermitean?

c) What are the ground-state expectation values of the operators in b)?

6. Differential Equations:

a) Solve the ordinary differential equation $\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) + \lambda^2 x^2 = a^2$ with a and $f(x)$ real, and the boundary conditions $f(x=0) = 1$, $\frac{\mathrm{d}}{\mathrm{d}x} f(x=0) = 0$.

b) Find the most general solution to the ordinary differential equation $\frac{\mathrm{d}}{\mathrm{d}x} f(x) = x\,f(x)$.

7. How Much *Beyond* Undergraduate Math Do You Know? The following problems cover topics which we will expand upon in the course. Therefore, I do not expect you to be able to solve them. But they give me some hint how far beyond the "simplest" math you have looked. Please indicate if you are not familiar with any of the concepts.

a) Determine using the residue theorem: $\int\limits_{-\infty}^{\infty} \mathrm{d}x\; \frac{1}{x^2 + a^2}$, $a > 0$.

b) Solve $\int\limits_{-\infty}^{\infty} \mathrm{d}x\; (2x+3)\;\delta(2x-3)$

c) Sketch how you would solve the following partial differential equation: $\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}^2}{\mathrm{d}y^2}\right) f(x, y) = 0$.

d) Does the set of even numbers form a group under additions?

8. A very Important Question: Which undergraduate textbooks did you use in
(i) Mathematical Methods; (ii) Electrodynamics; (iii) Quantum Mechanics?

9. Another very Important Question: Which line of research are you interested in?