Problem Sheet 11

Special Due date: Monday 15th December 2008 12:00 noon

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.

I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.

News and .pdf-files of Problems also at *http://home.gwu.edu/˜hgrie/lectures/math-methods08/math-methods08.html*.

1. Complex Integration (**4P**):

   a) (**2P**) Evaluate the following integrals, with $\mathcal{C}$ the circle $|z| = 3$:

   $$\oint_{\mathcal{C}} \frac{\mathrm{d}z}{z^3(z+5)} \quad , \quad \oint_{\mathcal{C}} \mathrm{d}z\ \frac{\mathrm{e}^z}{(z-1)(z-2)}$$

   b) (**2P**) Turn the following integral into a contour integral around the unit circle and evaluate:

   $$\int_0^{2\pi} \mathrm{d}\vartheta\ \frac{\sin^2\vartheta}{a+\cos\vartheta} \quad , \quad a > 1$$

2. More Complex Integrations (**12P**): You can check your final results with an algebraic manipulation programme. If you use contour integration with neglecting an arc at infinity, ***discuss in detail that your function vanishes indeed on that arc.***

   a) (**1P**) Evaluate the integral $\displaystyle\int_0^{\infty} \mathrm{d}x\ \frac{2x^2+1}{x^4+5x^2+6}$

   b) (**2P**) Evaluate the integral $\displaystyle\int_0^{\infty} \frac{\mathrm{d}x}{x^4+4x^2+4}$

   c) (**2P**) Determine $\displaystyle\int_{-\infty}^{\infty} \mathrm{d}x\ \frac{\mathrm{e}^{\mathrm{i}kx}}{x+\mathrm{i}}$, $k > 0$.

   d) (**3P**) Determine $\displaystyle\int_{-\infty}^{\infty} \mathrm{d}x\ \frac{\mathrm{e}^{-\mathrm{i}ax}}{x^4+6x^2+8}$ for $a > 0$ and for $a < 0$.

   e) (**4P**) Show using a contour which is not a complete semi-circle at infinity: $\displaystyle\int_0^{\infty} \frac{\mathrm{d}x}{1+x^n} = \frac{\pi}{n\sin\frac{\pi}{n}}$, $n = 2, 3, 4, \ldots$.

3. Branch Points (**6P**) Given the function $(z-a)^{1/n}$, $n \in \mathbb{N} \setminus \{0\}$.

   a) (**2P**) Discuss under which conditions it is well-defined/analytic in the complex plane: If the phase of $z - a$ is defined in the interval $\theta \in]-\pi;\pi[$, locate the position of the branch cut and determine by how much the answers differ if one evaluates the function on either side of the branch cut.

   b) (**1P**) How many Riemann sheets do you need to represent the function?

   c) (**3P**) Calculate the integral along a contour which encircles m times the branch point a and interpret your result.

Please turn over.

4. PRINCIPAL VALUE INTEGRAL (**2P**) Find the principal value of $\int\limits_{-\infty}^{\infty} \mathrm{d}x \; \frac{\sin x}{(x^2+4)(x-1)}$.

5. THE YUKAWA POTENTIAL, AGAIN (**8P**) We come back to problem 8.2: If the photon had a mass m, the Poisson equation of Electrostatics would read:

$$\left[\Delta - m^2\right] \Phi(\vec{r}) = 4\pi \; \rho(\vec{r})$$

Now, we caluclate the Green's function for this problem via the Fourier transform.

a) (**1P**) Construct the Green's function in momentum space: $\tilde{G}(\vec{k}) = \alpha \; \frac{1}{\vec{k}^2 + m^2}$ (determine $\alpha \in \mathbb{R}$).

b) (**2P**) To alleviate the inverse Fourier transform to coordinate space, show: When $\tilde{f}(k)$ only depends on the magnitude of $\vec{k}$ but not on its angle, then in three dimensions:

$$f(r) = \gamma \int\limits_0^{\infty} \mathrm{d}k \; \frac{k}{r} \; \sin kr \; f(k) \quad , \quad \text{and determine the real number } \gamma.$$

c) (**4P**) Find $G(\vec{r}, \vec{r}')$ in coordinate space using contour integration. You know the result.

d) (**1P**) Compare to the well-known result in the limit $m \to 0$.

6. DISPERSON RELATION (**3P**): Apply the Kramers-Kronig relation to a medium which shows no absorption except at a frequency ω_0. That means the imaginary part of the frequency-dependent dielectric susceptibility is found to be strongly peaked around ω_0: $\mathrm{Im}[\chi(\omega)] = \alpha\delta(\omega - \omega_0)$. Determine the real part of $\chi(\omega)$ from your observation. You may assume that $\chi(\omega)$ has no poles on the real axis and is causal.