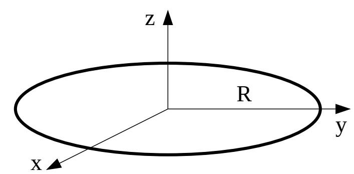
Problem Sheet 10

Due date: 2nd December 2008 **12:00 noon**

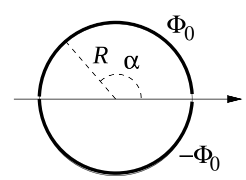
For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at *http://home.gwu.edu/˜hgrie/lectures/math-methods08/math-methods08.html*.



1. Electrostatic Multipole Moments (**11P**): An infinitesimally thin, conducting, circular ring of radius R carries the homogeneous line charge density μ. It is centred at the origin in the xy plane, see figure.

   Hints: Determine all multipole moments with respect to the origin.
   If multipole moments vanish, you can substitute a calculation by a short argument. Some results involve Legendre polynomials at a given value, which you need to look up.

   a) (**1P**) Convince yourself that the charge density can take the form $\rho(r,\theta,\phi) = \frac{\mu}{r}\,\delta(r-R)\,\delta(\cos\theta)$.
   b) (**1P**) Is this a Dirichlet or von-Neumann boundary problem?
   c) (**5P**) Determine all spherical multipole moments. Give explicit answers for the monopole, dipole and quadrupole moments.
   d) (**1P**) The electric field steming from the quadrupole component decreases at large distances with a power r^{-n}. Determine n.
   e) (**3P**) Determine the scalar potential close to the origin as expansion in a suitable, small parameter.



2. A Guillotine'd Cylinder (**8P**): An infinitely long, infinitesimally thin cylinder of radius R is separated into halves by cutting it along the cylinder axis. The potentials on the halves are constant and opposite, Φ_0 and $-\Phi_0$, see figure. Let's neglect any effects very close to the cuts.

   a) (**4P**) Determine the elementary solution of the Laplace equation $\Delta\Phi = 0$ in cylindrical coordinates. Show that the ansatz $\Phi(r,\phi,z) = U(r)\;\chi(\phi)$ leads to a separation of variables and the equation:
   $$\frac{r}{U(r)}\,\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r}U(r) = -\frac{1}{\chi(\phi)}\,\frac{\partial^2}{\partial\phi^2}\chi(\phi) = \mu^2$$
   Pay particular attention to the elementary solution for the value $\mu = 0$ of the separation constant, i.e. construct it and then show that is irrelevant for the problem at hand.
   b) (**4P**) Construct the potential *inside and outside* the cylinder. The solution is an infinite sum. Discuss where the solution converges well, and where it does not. Where would you trust your expansion?

Please turn over.

3. COMPLEX FUNCTIONS (**9P**): Let $z = x + \mathrm{i}y$ be complex, with x (y) its real (imaginary) part.

   a) (**1P**) Decompose the following complex function into real and imaginary part, $u + \mathrm{i}v$:

   $$\frac{1 + z^2}{\mathrm{i} - z^2}$$

   b) (**2P**) Let $u(x, y) = 2x(1-y)$ be the real part of a complex, differentiable function $f(z)$. Construct its imaginary part $v(x, y)$.

   c) (**6P**) Decompose the following complex functions into their real and imaginary part, $u + \mathrm{i}v$. Determine whether they obey the Cauchy-Riemann condition $\partial u/\partial x = \partial v/\partial y$, $\partial u/\partial y = -\partial v/\partial x$ everywhere, or at nearly all points (and if so, state at which they do not):

   $$\ln z \quad , \quad \sqrt{z}$$

   Hint: Sometimes, it helps to derive the form of u and v using the detour via the polar representation of a complex number, $z = |z|\,\mathrm{e}^{\mathrm{i}\phi}$. But that is a question of taste.

4. COMPLEX MAPPING (**2P**): A complex function maps complex numbers into complex numbers. Since complex numbers can be interpreted as coordinates in a 2-dimensional plane, it is natural to study how a geometric figure in 2 dimensions is mapped from the complex z-plane into the complex w-plane by a complex function w.

   Into which figure on the w-plane is the rectangle $\{z = x + \mathrm{i}y : 0 \leq x \leq 1,\ 0 \leq y \leq \pi\}$ mapped by the complex exponential $w = \mathrm{e}^z$?

   Point of information: This technique is again quite useful in two-dimensional Electrostatics. Say you have solved a problem without charges on that rectangle with given boundary conditions, getting $\Phi(x, y)$ as the real part of a complex function. Then you map this region via $z \to w(z)$ into a new region, and Φ to $\Phi(u(x, y), v(x, y))$. If the mapping w is analytic, the new Φ will again obey the Laplace equation, because the analytic function with real part $\Phi(x, y)$ is again analytic, i.e. its real and imaginary parts have to be harmonic. The "only" problem is to find that function w which maps your simple problem to the complicated problem you actually want to solve.