Problem Sheet 9

Due date: 25th November 2008 **12:00 noon**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at *http://home.gwu.edu/˜hgrie/lectures/math-methods08/math-methods08.html.*

1. Spherical Harmonics (**7P**):

   a) (**2P**) Determine the parity of $Y_{lm}(\theta,\phi)$. This will also tell you which rank the tensor Y_{lm} has.

   b) (**3P**) Inspired by a good book, prove the addition theorem for spherical harmonics:

   $$P_l(\cos\alpha) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l} Y^*_{lm}(\Omega)\, Y_{lm}(\Omega') = \sqrt{\frac{4\pi}{2l+1}}\, Y_{l0}(\alpha,0) \ ,$$

   where α is the angle between (θ,ϕ) and (θ',ϕ'), i.e. $\cos\alpha = \cos\theta\ \cos\theta' + \sin\theta\ \sin\theta'\ \cos(\phi-\phi')$.

   c) (**2P**) Construct the first three Legendre polynomials from the generating function.

2. Symmetries of Multipole Moments (**3P**): Given a localised charge distribution. Show that a skilful choice of the coordinate system reveals the following properties of the spherical multipole moments q_{lm}, which will come in handy later.

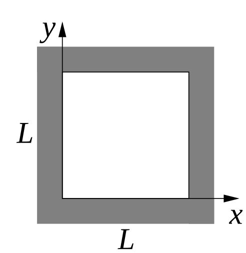
   Hint: Investigate, for example, the effect of each symmetry operation on the coordinates (θ,ϕ).

   a) (**1P**) $\rho(\vec{x})$ rotationally symmetric with respect to an axis $\Longrightarrow q_{lm} = \delta_{m0}\, q_{l0}$;

   b) (**1P**) $\rho(\vec{x})$ spherically symmetric $\Longrightarrow$ only monopole moment non-zero;

   c) (**1P**) $\rho(\vec{x})$ mirror-symmetric with respect to a plane $\Longrightarrow q_{lm} = q_{l,-m}$.

3. Square with Boundary Conditions (**8P**): The sides of a square (length of sides L) are made of some material such that the boundary conditions on the potential are

   $$\Phi(x,y) = \begin{cases} \Phi_0 \sin\frac{3\pi x}{L} & \text{on the upper side, } y=L \\ 0 & \text{on all other sides.} \end{cases} \quad .$$

   The problem is two-dimensional. There are no charges inside the square; see figure.

   a) (**1P**) Are these boundary condition of the Dirichlet or von-Neumann type?

   b) (**5P**) Determine the potential $\Phi(x,y)$ everywhere inside the square.

   Hint: Show that you need to solve differential equations (you need to determine α):

   $$\frac{\mathrm{d}^2}{\mathrm{d}x^2}X(x) = -\alpha^2\ X(x)\ \ ,\ \ \frac{\mathrm{d}^2}{\mathrm{d}y^2}Y(y) = \alpha^2\ Y(y)\ \ .$$

   c) (**2P**) Using a pillbox construction, determine the charge density on the lower plate, $y=0$.

Please turn over.

4. GREEN'S FUNCTION OF THE GROUNDED SQUARE (**7P**): Determine the Green's function for the geometry of the previous problem when $\Phi = 0$ on *all* sides. Discuss to which physical situation this mathematical problem is equivalent.

   Hint: Recal that you can construct a Green's function from the eigenfunctions and eigenvalues of the corresponding differential operator. The final result is an infinite series.

5. MULTIPOLE MOMENTS ON THE SURFACE OF A SPHERE (**5P**): The following potential is given on the surface of a sphere with radius R:

$$\Phi(\vec{R}) = \frac{\Phi_0}{2R^2} \left(3(\vec{e}_z \cdot \vec{R})^2 - \vec{R}^2\right) \quad ,$$

   where $\vec{e}_z$ is the unit vector in the z-direction, $\vec{R}$ the radial vector from the centre of the sphere to a point to the surface and Φ_0 some constant. The origin is at the centre of the sphere.

   Determine the potential *both inside and outside* the sphere, and the first three spherical multipole moments (monopole, dipoles and quadrupoles) with respect to the given coordinate system. Take into account that the potential should disappear at infinity and be finite as $r \to 0$.

   Hint: Upon close inspection, the potential has a form similar to one of the Spherical Harmonics.