Problem Sheet 8

Due date: 18 November 2008 **12:00 noon**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at *http://home.gwu.edu/˜hgrie/lectures/math-methods08/math-methods08.html.*

1. STURM-LIOUVILLE THEORY (**12P**): This is a very important problem, but also one for which you can profit greatly by getting inspired by textbooks. Consider the one-dimensional second-order differential operator

$$L(x) := p_2(x)\frac{\mathrm{d}^2}{\mathrm{d}x^2} + p_1(x)\frac{\mathrm{d}}{\mathrm{d}x} + p_0(x) \ ,$$

where $p_i(x)$ are three real functions of x which (we assume) have no singularities in a domain $x \in [a;b]$, with a, b two real numbers which also can be $\pm\infty$. We equip the Hilbert space with the "standard" scalar product with weight factor 1.

   a) (**5P**) Show: The operator is Hermitean only if $\frac{\mathrm{d}}{\mathrm{d}x}p_2(x) = p_1(x)$. Recall that you need to check $\langle g|Lf\rangle = \langle Lg|f\rangle +$ surface terms. We will deal with the surface terms below.

   b) (**1P**) Show: If L is Hermitean, then $L = \frac{\mathrm{d}}{\mathrm{d}x}p(x)\frac{\mathrm{d}}{\mathrm{d}x} + q(x)$. Determine p, q via the p_i's.

   c) (**3P**) Show: In order for L to be self-adjoint, we need to impose

$$g^*(x)\frac{\mathrm{d}}{\mathrm{d}x}f(x) = f^*(x)\frac{\mathrm{d}}{\mathrm{d}x}g(x) \ \text{ at } x = a, b \quad \text{or } p(x) = 0 \ \text{ at } x = a, b \ .$$

   d) (**3P**) Consider now the radial part of the Laplace operator in three dimensional, spherical coordinates: $\frac{1}{r}\frac{\partial^2}{\partial r^2}r$. It appears, for example, in the Schrödinger equation for the hydrogen atom. For bound states, we obviously want that the wave-function f is finite everywhere and disappears for large distances (it's called "bound state", after all): $f(r=0)$ finite, $f(r=\infty) = 0$. Is the operator Hermitean on these functions? Is it self-adjoint?

   Caveat: Recall $\int \mathrm{d}^3r = \int \mathrm{d}r \ \mathrm{d}\cos\theta \ \mathrm{d}\phi \ r^2$!

2. YUKAWA POTENTIAL AS GREEN'S FUNCTION (**6P**): If the photon had a mass m, the Poisson equation of Electrostatics would read

$$\left[\Delta - m^2\right]\Phi(\vec{r}) = 4\pi \ \rho(\vec{r})$$

   a) (**4P**) Construct a Green's function for this case, assuming that there are no charges or surfaces around. The scalar potential shall vanish at infinity. Name the boundary condition this imposes.

   b) (**2P**) What is the solution Φ, given ρ? Is it unique?

Please turn over.

3. ONE-DIMENSIONAL GREEN'S FUNCTION AND THE SUPERPOSITION PRINCIPLE (**5P**): The superposition principle is very useful even for very simple charge distributions. You know the solution to this problem inside-out, so enjoy a slightly different angle: Consider a homogeneously charged, infinitely long but infinitesimally thin, perfectly conducting plate with constant surface charge density σ, aligned in the (xy)-plane.

   a) (**1P**) Constrain the form of the field $\vec{E}$ by symmetry arguments and reduce the problem to a one-dimensional one, with x the distance from the plate.

      As we discussed in the lecture, the Green's function to the one-dimensional operator $\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ "without boundaries at infinity" is

$$G(x;x') = \frac{1}{2}\,|x-x'| \ .$$

   b) (**2P**) Calculate now from your result the electrostatic potential Φ and $\vec{E}$ for the given surface charge density σ.

      Fail-safe point: For a plate located at x_0, the scalar potential is from $\Delta\Phi = -4\pi\rho$:

$$\Phi(x) = \begin{cases} -2\pi\sigma x & \text{for } x > x_0 \\ 2\pi\sigma x & \text{for } x < x_0 \end{cases}$$

   c) (**2P**) Determine now the electric field between and on either side of two parallel, homogeneously charged, infinitely long, perfectly conducting plates with constant surface charge densities σ_1, σ_2.

4. IMAGE CHARGES AND GREEN'S FUNCTIONS (**7P**): A point charge q is placed at a distance $a > R$ from a sphere of radius R which is centered at the origin. As the sphere is perfectly conducting and grounded, the scalar potential has a single value $\Phi(R) = 0$ at its surface. Construct $\Phi(\vec{r})$ everywhere. You may get inspired by your undergraduate electrostatics textbook.

   a) (**1P**) Show that you can choose the point charge to sit on the x-axis.

   b) (**1P**) Show that no matter what $\Phi(R)$, there is no electric field inside.

   c) (**3P**) Solve for the electrostatic potential $\Phi(\vec{r})$ *outside the sphere* by considering an image charge q' which also lies on the x-axis, but inside the sphere, i.e. inaccessible to the outside observer.

   d) (**2P**) Now, a charge density $\rho(\vec{r})$ is given outside the same sphere with the same boundary conditions. Determine the total force between it and the sphere. Is the force repulsive or attractive?