

Problem Sheet 7

Due date: 11th November 2008 **12:00 noon**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.

I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.

News and .pdf-files of Problems also at <http://home.gwu.edu/~hgrie/lectures/math-methods08/math-methods08.html>.

1. DIRAC'S δ -DISTRIBUTION (**10P**): Prove the following properties, i.e. show that they hold when multiplied with any square-integrable function $f(x)$ and integrated over all space.

- a) (**2P**) $\delta(ax) = \frac{1}{|a|}\delta(x)$. Use the representation of the δ -distribution as limit of your favourite sequence of "true" functions.
- b) (**2P**) Derive a representation of HEAVISIDE'S STEP-"FUNCTION" as limit of a sequence of functions using its definition $\theta(x) := \int_{-\infty}^x dy \delta(y) = \begin{cases} 1 & \text{for } x > 0 \\ 0 & \text{for } x < 0 \end{cases}$ and your favourite definition of the δ -distribution as limit of your favourite sequence of "true" functions.
- c) (**2P**) $\delta(f(x)) = \sum_{i=1}^N \frac{\delta(x-x_i)}{|\frac{df}{dx}(x_i)|}$ when $f(x)$ has only simple zeroes at $x = x_1, \dots, x_N$. You may use "heuristic" arguments by treating the distribution "like" a function, and assume that the Taylor expansion of $f(x)$ exists around all zeros x_i . Which conditions should $f(x)$ obey?
- d) (**2P**) Derive from the Cartesian version expressions for Dirac's δ -distribution in cylindrical and spherical coordinates. It may be useful to recall that $\int d^3r \delta^{(3)}(\vec{r}) = 1$, whatever coordinate system you use.
- e) (**2P**) As we expect to be able to recover Fourier series from Fourier integrals, proving the following result is very informative: When $f(x)$ is periodic with period L , then its Fourier transform is zero, except when $kL = 2\pi n$, where $n \in \mathbb{Z}$ is an integer. Can you write this result using Dirac's δ -distributions?

2. PRACTISING WITH DIRAC'S δ -DISTRIBUTION (**4P**):

- a) (**2P**): Show that $\frac{d^2}{dx^2}|x| = 2\delta(x)$ in the "distributional sense".
- b) (**2P**): Watching the integration limits, calculate:

$$\int_0^{\infty} dx \delta(x^2 - 1) e^{-x^2} \left(1 - \cos \frac{5\pi x}{2}\right)$$

Please turn over.

3. PROPERTIES OF FOURIER TRANSFORMS (**5P**): Prove for functions $f(x)$, $g(x)$ with Fourier transforms $F(\omega)$, $G(\omega)$:

a) FALTUNG/CONVOLUTION THEOREM (**2P**):
$$\int dy g(y) f(x-y) = \int \frac{d\omega}{2\pi} F(\omega) G(\omega) e^{-i\omega x}$$

b) (**1P**) The Fourier transform of a real function $f(x)$ obeys $F(-\omega) = F^*(\omega^*)$, ω complex.

c) (**2P**) The Fourier transform of an even function $f(x) = f(-x)$ is even. When $f(x)$ is also real, then so is its Fourier transform.

4. PRACTISING WITH FOURIER TRANSFORMS (**6P**):

a) (**3P**): Consider a Gaussian, $f(x) = \exp -x^2/(2\sigma_x^2)$, with width σ_x . When you calculate its Fourier transform, you will find that you get another Gaussian in momentum space, but with a different width σ_k . What is it? Make a connection to HEISENBERG'S UNCERTAINTY RELATION when you interpret $f(x)$ as a wave-packet in coordinate space.

b) (**2P**): Construct the solution to $\vec{\nabla}^2 G(\vec{r}) = \delta^{(d)}(\vec{r})$ in momentum space. Does $\tilde{G}(\vec{k})$ depend on the number of dimensions d you consider?

c) (**1P**): Combine this result with the result of 2.a) above to calculate the Fourier transform of $1/k^2$ in one dimension.

5. HELMHOLTZ'S FUNDAMENTAL THEOREM (**5P**):

a) (**2P**) Fill in the dots in the Fourier back-transformation in our "proof" of Helmholtz's Fundamental Theorem, i.e. calculate \mathcal{F}^{-1} of:

$$\Phi(\vec{k}) = i \frac{\vec{k} \cdot \vec{A}(\vec{k})}{|\vec{k}|^2} \quad , \quad \vec{a}(\vec{k}) = -i \frac{\vec{k} \times \vec{A}(\vec{k})}{|\vec{k}|^2}$$

b) (**3P**) Prove Helmholtz' theorem directly by applying "div" and "rot" to

$$\vec{A}(\vec{r}) = \frac{1}{4\pi} \int d^3r' \left[-\vec{\partial}_r \frac{\vec{\partial}_{r'} \cdot \vec{A}(\vec{r}')}{|\vec{r} - \vec{r}'|} + \vec{\partial}_r \times \frac{\vec{\partial}_{r'} \times \vec{A}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right] .$$