Problem Sheet 6

Due date: 28th October 2008 **12:00 noon**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at *http://home.gwu.edu/~hgrie/lectures/math-methods08/math-methods08.html*.

1. DIV, ROT, GRAD (**7P**): The following problems are independent of each other.

   a) (**1P**) Calculate *in spherical coordinates* with $\vec{r} = x\,\vec{e}_x + y\,\vec{e}_y + z\,\vec{e}_z$, $r = \sqrt{x^2+y^2+z^2}$, $\vec{e}_r = \frac{\vec{r}}{r}$:

   $$\vec{\nabla}\cdot\vec{r},\quad \vec{\nabla}\frac{1}{r}$$

   b) (**2P**) Does the following field have sources/sinks or curls? Sketch or plot the vector-function in a suitable way! (That involves thinking what a suitable way could be...)

   $$\vec{A}(\vec{r}) = x\,\vec{e}_y + y\,\vec{e}_x$$

   Brain-Cracker (for 3 extra points): Reason whether you can determine this field from its sources and curls using Helmholtz' theorem as we discussed it in the lecture.

   c) (**4P**) Determine all exponents α for which the spherically symmetric vector field $\vec{A}(\vec{r}) = |\vec{r}|^\alpha\,\vec{e}_r$ is source-free everywhere.

2. A STRANGE VECTOR POTENTIAL (**3P**): Consider the vector potential $\vec{A}(\vec{x}) = \vec{e}_\varphi\,\dfrac{\Phi}{2\pi r}$ in cylindrical coordinates (r,ϕ,z), where Φ is some constant. You can without fear of reprimand copy the form of differential operators in these coordinates from textbooks; they are orthogonal curvilinear. Calculate and discuss direction and strength of the magnetic field $\vec{B} = \vec{\nabla}\times\vec{A}$ and the flux it describes through a disk of arbitrary radius in the $(z=0)$-plane. As in the lecture for $\Delta\frac{1}{r}$, pay particular attention to those points where $\vec{A}$ is not well-defined, using Stokes' theorem.

3. COVARIANT ϵ-TENSOR (**3P**): We saw in the lecture that $\tilde{\epsilon}^{ijk}\sqrt{\det\tilde{g}} = \epsilon^{ijk}\sqrt{\det g}$ under a change of basis for the *contravariant* components. $\det g$ is shorthand for the determinant of the covariant metric tensor with components g_{ij}.

   a) (**2P**) Calculate first in Cartesian coordinates $\epsilon^{ijk}\epsilon_{ijl}$ and from that $\epsilon^{ijk}\epsilon_{ijk} = 6$. These formulae can be quite handy in a number of contexts.

   b) (**1P**) Determine now the transformation of the *covariant components* ϵ_{ijk}.

   Hint: Sub-section a) was not in vain.

4. TENSORS AND PSEUDO-TENSORS (**4P**): We saw that the magnetic field $\vec{B}$ is a rank-1 pseudo tensor.

   a) (**2P**) The electric charge density $\rho(\vec{r})$ is obviously invariant under coordinate transformations, and also looks the same in a mirror. Using Gauß' law $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$, show that the electric field $\vec{E}$ must be a rank-one (polar) tensor, not a pseudo/axial tensor.

   b) (**2P**) A point charge q which moves with speed $\vec{v}$ is subject to the electro-magnetic LORENTZ FORCE $\vec{F}_\mathrm{L} = q[\vec{E} + \vec{v}\times\vec{B}]$. Determine wether $\vec{F}_\mathrm{L}$ is a "real" or pseudo-tensor, or a mixture, and find its rank.

Please turn over.

5. HYPERBOLIC COORDINATES (**13P**): The relation between hyperbolic coordinates (u, v, ϕ) and Cartesian ones in three-dimensional Euclidean space is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cosh u \ \cosh v \ \cos\phi \\ \cosh u \ \cosh v \ \sin\phi \\ \sinh u \ \sinh v \end{pmatrix} .$$

I know you know you can find the solution in wikipedia, but if I would be interested in the solution, I would do the problem myself. I am interested in how you *get* the solution.

a) (**3P**) Derive the transformation matrix from the change $\mathrm{d}s^i_{\text{Cartes}} = \bar{d^i}_j \, \mathrm{d}s^j_{\text{hyp}}$ of the line element.

b) (**2P**) When you calculate the metric tensor, you find that these are orthogonal curvilinear coordinates. Show that the scale factors are $h_{(u)} = h_{(v)} = \sqrt{\cosh^2 u - \cosh^2 v}$, $h_{(\phi)} = \cosh u \, \cos v$.

c) (**2P**) Sketch the "coordinate grid", i.e. the lines mapped out when all but one of the coordinates u, v and ϕ are kept fixed. Pay particular attention to their intersections.

d) (**2P**) Determine the invariant volume element. Is there a coordinate singularity, i.e. a point at which the coordinate transformation is not well-defined?

e) (**4P**) Show that the most general function $\psi(u)$ which *only* depends on u and which satisfies LAPLACE'S EQUATION $\Delta\psi(u) = 0$ is

$$\psi(u) = A + B \arctan \mathrm{e}^u \quad , \text{ where } A, B \text{ are constants.}$$