Problem Sheet 5

Due date: 21st October 2008 **12:00 noon**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at *http://home.gwu.edu/~hgrie/lectures/math-methods08/math-methods08.html*.

1. Structure Constants as Lie Algebra (**4P**): There are *four* criteria which define a Lie algebra: closure, bi-linearity, anti-symmetry, Jacobi-identity. We now interpret the structure constants f^{abc} as matrices $(f^c)^{ab}$ which give the (ab) entry of the cth generator of the Lie algebra. Show that these matrices obey the definition of a Lie algebra. This representation is called the adjoint representation of the Lie algebra.

2. Mapping $SU(2)$ onto $SO(3)$ (**5P**): Using the sine-cosine representation of an arbitrary element of $SU(2)$ from HW 4.7, show: When $U[\alpha]$ rotates a 2-dimensional complex vector by an angle $\alpha = \sqrt{\alpha^a\alpha_a} \in \mathbb{R}$ about the α^a/α-axis, then the $SO(3)$ matrix

$$2\mathrm{tr}[U^\dagger[\alpha]\sigma^a U[\alpha]\sigma^b]$$

rotates a three-dimensional real vector r^a about the same axis by an angle 2α. If you are lazy and want to forfeit half of the points, you can consider only rotations of the form $\alpha\sigma^1$.

3. Multiple Products with Derivatives (**5P**): Evaluate using the ϵ-tensor and Einstein's summation convention:

   a) (**2P**) $\vec\nabla \times (\vec\nabla\Phi)$, $\vec\nabla \cdot (\vec\nabla \times \vec A)$, $\vec\nabla \times (\Phi\vec\nabla\Phi)$;
   b) (**1P**) $\vec\nabla \times \Big(\Phi(\vec x)\vec A(\vec x)\Big) = \Phi\,\vec\nabla \times \vec A - \vec A \times \vec\nabla\Phi$;
   c) (**1P**) $\vec\nabla \times (\vec\nabla \times \vec A) = \vec\nabla(\vec\nabla \cdot \vec A) - \Delta\vec A$;
   d) (**1P**) $\vec\nabla \cdot (\vec A \times \vec B)$

   Hint: While the ϵ-tensor is coordinate independent and hence commutes with derivatives, the arbitrary scalar function $\Phi(\vec r)$ and arbitrary vector functions $\vec A(\vec r), \vec B(\vec r)$ are not.

4. Quantum Mechanics and Scalar Products (**2P**): Show that

$$\langle f|g\rangle := \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}x}{(2\pi)}\; f^*(x)\, g(x)\, w(x)$$

   fulfills all criteria of a scalar product when $f(x)$, $g(x)$ are functions and $\langle f|g\rangle$ is finite. Under which conditions on $w(x)$ is the scalar product positive definite?

Please turn over.

5. A SKEWED BASIS (**4P**): Given the following basis of the two-dimensional Euclidean space:

$$\vec{e}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \;,\; \vec{e}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

   a) (**2P**) Construct the metric tensor g_{ij} and dual basis $\vec{\epsilon}^{\,i}$.

   b) (**1P**) Determine the *covariant* components of the vector r_i in this basis for $\vec{r} = 2\vec{e}_x - \vec{e}_y$, with $\vec{e}_x$, $\vec{e}_y$ the ordinary, orthonormal basis vectors of two-dimensional Euclidean space.

   c) (**1P**) Determine the length of a vector $\vec{r} = 3\vec{e}_1 + 2\vec{e}_2$ *without* using the Cartesian basis.

6. LORENTZ GROUP (**10P**): We consider the group of two-dimensional matrices Λ which leaves the MINKOWSKI PSEUDO-METRIC $g_{\mu\nu} = \text{diag}(1, -1)$ invariant. The matrices acts only on real vectors.

   a) (**1P**) Show: Interpreting the contravariant components of a vector in this metric as $x^\mu = \begin{pmatrix} ct \\ x \end{pmatrix}$, with t the temporal and x the spatial distance between two events, and $c = 1$ the speed of light, this is the metric in which the speed of light is the same in all inertial systems. Calculate to this end $x_\mu x^\mu$ in this metric, and interpret it!

   b) (**3P**) Derive that the most general matrix with *only positive entries on the diagonal* which leaves the Minkowski-metric invariant can be written as

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \quad , \quad \text{where } \gamma = \frac{1}{\sqrt{1-\beta^2}} \;,\; 0 \leq \beta < 1 \;.$$

   Aside: We interpret $\beta = v/c$ for an inertial system with velocity v relative to an observer...

   c) (**1P**) Is the resulting Lie group of all two-dimensional Lorentz-transformations compact?

   d) (**1P**) Determine the inverse of $\Lambda^\mu{}_\nu$ and the transformation matrix $\Lambda_\mu{}^\nu$ for covariant components.

   e) (**2P**) Construct a complete ortho-normal basis of the Lie algebra.

   f) (**2P**) Show that the group of *all* Lorentz transformations is obtained when one supplements the transformations above with the elements of the WEYL GROUP $\{\text{diag}(1,1); \text{diag}(1,-1); \text{diag}(-1,1); \text{diag}(-1,-1)\}$. Study the behaviour of the time- and space-component of the contravariant components of a vector under this four-element group.

   Aside: Lorentz transformations with $\Lambda^0_0 > 0$ are called ORTHO-CHRONOUS. What would you call a Lorentz transformation with $\Lambda^1_1 < 0$?