Problem Sheet 4

Due date: 7th October 2008 **12:00 noon**

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News and .pdf-files of Problems also at *http://home.gwu.edu/˜hgrie/lectures/math-methods08/math-methods08.html.*

1. EULER'S PROBLEM (**7P**): Determine the stability criterion for the buckling of a slender column under a compressive load, as posed in [StoI, problem 1.4.a)].

   Hint: If you think a bit, you do not have to extend the Euler-Lagrange equation to the case that the functional depends on higher derivatives, $y''(x)$ etc. If you use this extension, you must prove it.

2. MORE FIELDS (**2P**): The Physics of fields Ψ_i is in Classical Mechanics described by minimising the action $S = \displaystyle\int \mathrm{d}t\ \mathrm{d}^3r\ \mathcal{L}[\Psi(t,\vec{r})]$, where the LAGRANGE DENSITY or LAGRANGEAN $\mathcal{L}$ is a function of the field variables and their derivatives $\dot{\Psi}_i$, $\vec{\nabla}\Psi_i$.

   a) COMPRESSION WAVES (**1P**) in an isotropic, homogeneous and elastic medium are described by

   $$\mathcal{L} = \frac{1}{2}\,\rho_0\,[\dot{\vec{\Phi}}^2 - c_s^2(\vec{\nabla}\cdot\vec{\Phi})^2]\ .$$

   The field $\vec{\Phi}(t,\vec{r})$ is a three-component vector (one component for each direction in which the medium can be compressed). The two constants are: the density coefficient ρ_0; and the speed of sound c_s. Derive the Euler-Lagrange equations of this problem.

   b) STANDARD TOY-MODEL OF HIGH-ENERGY PHYSICS (**1P**): We now explore the complex scalar field $\Phi(x^\mu)$ of a spin-zero particle with mass m, i.e. $\Phi = \Phi_R + \mathrm{i}\Phi_I$ has both a real and imaginary part, and $\Phi^\dagger = \Phi_R - \mathrm{i}\Phi_I$ is its complex conjugate. The Lagrangean is

   $$\mathcal{L}_\Phi = \dot{\Phi}^\dagger\dot{\Phi} - \left(\vec{\nabla}\Phi^\dagger\right)\cdot\left(\vec{\nabla}\Phi\right) - m^2\Phi^\dagger\Phi\ .$$

   Derive the Euler-Lagrange equations for Φ and $\Phi^\dagger$. The field and its complex conjugate are best treated as independent variables.

3. ϵ-TENSOR AND KRONECKER-δ (**5P**): The totally anti-symmetric unit pseudo-tensor of rank 3 (Levi-Civitá symbol, "ϵ-tensor") and the Kronecker-δ are defined as

   $$\epsilon^{ijk} = \begin{cases} 1 & \text{for } (ijk) \text{ cyclical/even permutations of } (123) \\ -1 & \text{for } (ijk) \text{ anti-cyclical/odd permutations of } (123) \\ 0 & \text{otherwise} \end{cases} \quad , \quad \delta^{ij} = \begin{cases} 1 & \text{for } i=j \\ 0 & \text{for } i\neq j \end{cases}\ .$$

   With $\vec{a}$, $\vec{b}$, $\vec{c}$, $\vec{d}$ constant, arbitrary vectors, calculate or prove using the ϵ-tensor and Einstein's summation convention, and *not* using some other technique:

   a) (**1P**) $(\vec{a}\times\vec{b})_i = \epsilon_{ijk}a^j b^k$
   b) (**1P**) A "master formula" for the following problems: $\epsilon^{ijk}\epsilon_i{}^{lm} = \delta^{jl}\,\delta^{km} - \delta^{jm}\,\delta^{kl}$;
   c) (**1P**) The "bac-cab" rule from the previous rule $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c} = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$;
   d) (**1P**) $\epsilon^{ijk}\,a_i\,b_j\,c_k = \det(\vec{a}\,\vec{b}\,\vec{c})$, where $(\vec{a}\,\vec{b}\,\vec{c})$ is the matrix built of the row vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$;
   e) (**1P**) $\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right) = \vec{a}\cdot\left(\vec{b}\times(\vec{c}\times\vec{d})\right) = \left(\vec{a}\cdot\vec{c}\right)\left(\vec{b}\cdot\vec{d}\right) - \left(\vec{a}\cdot\vec{d}\right)\left(\vec{b}\cdot\vec{c}\right)$;

Please turn over.

4. GROUP OR NO GROUP (**3P**): Consider the following sub-sets of the group of invertible, real 3×3 matrices $M \in GL(3,\mathbb{R})$. Do these sets form groups under matrix multiplications? If yes, what is their identity element?

   a) (**1P**) The set of matrices which obeys $M^T = M$.

   b) (**2P**) The set of matrices which obeys $(M)_{ij} = 0$ for $i < j$ ("upper-triangular matrices").

5. AN ABSTRACT LIE ALGEBRA (**3P**): Show that the infinite-dimensional set of all first-order differential operators $\mathcal{L}[f] := f(x)\frac{\mathrm{d}}{\mathrm{d}x}$ forms an abstract Lie algebra, when the Lie bracket is the commutator $(\mathcal{L}[f_1]\mathcal{L}[f_2] - \mathcal{L}[f_2]\mathcal{L}[f_1])g(x)$ and $f(x), g(x)$ are arbitrary, infinitely often differentiable functions.

6. MATRIX RELATIONS (**10P**):

   a) (**2P**) Determine the real dimension of the Lie algebra of the group $U(N)$ by counting the number of independent entries in an anti-Hermitean matrix. You may *not* use that $\dim G = \dim L[G]$.

   b) (**2P**) Determine the real dimension of the Lie group $U(N)$ by counting the number of independent entries in a unitary matrix, similar to what we did in the lecture for $SO(N)$.

   c) (**2P**) Show $\det M = \exp \operatorname{tr} \ln M$ when M is diagonalisable.

   Hint: You may first want to look at a diagonal matrix.

   d) (**2P**) Assuming that X, Y are infinitesimal and elements of a Lie algebra, derive the first three terms of the Campbell-Baker-Hausdorff formula:

$$\exp X \, \exp Y = \exp[X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\left([X,[X,Y]] + [Y,[Y,X]]\right) + \dots]$$

   e) (**2P**) If $[A,B] = B$, calculate $\exp[\mathrm{i}\alpha A]\, B \, \exp[-\mathrm{i}\alpha A]$ with α commuting with A and B, e.g. a number. This relation is often used in QM.

7. PROPERTIES OF $SU(2)$ AND ITS LIE ALGEBRA (**10P**):

   a) (**1P**) Show that every element of $SU(2)$ can be written as $\begin{pmatrix} a & b \\ -b^* & a \end{pmatrix}$, where a, b are complex numbers and $|a|^2 + |b|^2 = 1$.

   b) (**1P**) Show that this implies that the group-manifold of $SU(2)$ is S^3, i.e. the three-dimensional surface of a sphere in $\mathbb{R}^4$.

   Hint: Interpret the 4 real parameters of $a, b \in \mathbb{C}$ as real components of a 4-dimensional vector.

   c) (**4P**) Prove the identity

$$\exp[\mathrm{i}\alpha_a \frac{\sigma^a}{2}] = 1\, \cos\frac{\alpha}{2} + \mathrm{i}\,\frac{\alpha_a \sigma^a}{\alpha} \sin\frac{\alpha}{2} \quad ,$$

   where $\alpha = \sqrt{\alpha^a \alpha_a}$ is the length of the vector α^a in the space spanned by the Pauli matrices σ^a.

   Hint: The properties of the Pauli matrices come in quite handy, as does the definition of the function of a matrix by its corresponding Taylor expansion.

   d) (**2P**) Show that the mapping of its Lie algebra on the Lie group $SU(2)$ is bijective (one-to-one and onto) for $\alpha \in [0; 2\pi]$. You have just proven by practical construction that: (i) the tangent space of $SU(2)$ in the identity suffices to construct the whole group $SU(2)$; (ii) $SU(2)$ is connected; (iii) as is S^3; and (iv) both are compact.

   e) (**2P**) The CENTRE Z OF A GROUP is defined as the set of all elements which commute with all group elements. Show that Z is always an Abelian sub-group of G. Determine the centre of $SU(2)$.

Please turn over.

8. STRUCTURE CONSTANTS (**8P**): We pick a set of elements of a Lie algebra $\{t^a\}$ such that we can expand every Lie algebra element in terms of these GENERATORS, i.e.

$$\forall X \in L[G]\ \exists \{\alpha_a\} \in (\mathbb{C}^{\dim G} \text{ or } \mathbb{R}^{\dim G}) \text{ such that } X = \mathrm{i}\alpha_a t^a \quad .$$

Notice that we pull out a factor i here, so that in a unitary representation of the Lie group (i.e. when all matrices are unitary), the generators are Hermitean, and not anti-Hermitean. That is just convenience. We also define the generators of a abstract Lie algebra with the same convention.

Because the Lie bracket operation renders Lie algebra elements, we can always expand the Lie bracket of two generators into the generators:

$$[t^a, t^b] = \mathrm{i}\, f^{abc} t_c$$

The expansion coefficients f^{abc} are called the STRUCTURE CONSTANTS of the Lie algebra. For $SU(2)$, the generators are for example the Pauli matrices (usually divided by 2), and $f^{abc} = \epsilon^{abc}$. The generators form a BASIS when they can be ortho-normalised as

$$\mathrm{tr}[t^a t^b] = \frac{1}{2}\delta^{ab} \quad .$$

Prove the following assertions:

a) (**1P**) The structure constants are anti-symmetric in the first two indices, $f^{abc} = -f^{bac}$, even when the generators do not form a basis. That implies they are zero for an Abelian group.

b) (**2P**) $f^{abc} f^{cde} + f^{adc} f^{bce} + f^{bdc} f^{cae} = 0$ even when the generators do not form a basis.

c) (**1P**) If the generators form a basis, then the coefficients of any Lie algebra element X with respect to this basis are $\alpha^a = -2\mathrm{i}\,\mathrm{tr}[X t^a]$.

d) (**2P**) If the generators form a basis, then $f^{abc} = -f^{acb}$, i.e. the structure constants are anti-symmetric in *all* indices.

   Hint: Marry the definition of the structure constant with the definition of a basis.

e) (**2P**) Consider the Lie algebra of strictly upper-triangular matrices. Find a possible set of generators and discuss whether they form a basis.

   Remark un-related to this exercise: One can show that only simple Lie groups (or their products), or the one-dimensional Lie group, admit a basis. This is key in the CARTAN-LIE CLASSIFICATION of all possible Lie groups and Lie algebras.