Problem Sheet 3

Due date: 24th September 2008 **12:00 noon**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at *http://home.gwu.edu/˜hgrie/lectures/math-methods08/math-methods08.html*.

0. FINISH THE SOAP-BUBBLE PROBLEM.

1. VARIATIONS WITH VARIABLE END-POINTS (**2P**): Show that the shortest path between a straight line and a point (in the plane containing the point and line) is the perpendicular between point and line.

2. A DRUMMER'S PROBLEM (**8P**): The shape of a distorted drum-skin is described by the the height $h(x,y)$ at point (x,y) by which the drum-skin is displaced from the flat, un-distorted skin.

   a) (**4P**) Show: The area of the distorted drum is

   $$A[h] = \int \mathrm{d}x\ \mathrm{d}y\ \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}\ \ .$$

   b) (**2P**) Show: For small distortions and with $\vec{\nabla}$ the gradient w.r.t. to x and y, the area reduces to

   $$A[h \ll 1] = \text{const.} + \int \mathrm{d}x\ \mathrm{d}y\ \left(\vec{\nabla} h(x,y)\right)^2\ \ .$$

   c) (**2P**) Verify that A is extremal when h obeys the two-dimensional Laplace equation.

3. FIELD LAGRANGEAN (**2P**): The sub-atomic particle called Pion can be described by a field $\Phi(t,\vec{r})$ which obeys the variational principle (v, μ are real constants)

   $$\delta \int \mathrm{d}t\ \mathrm{d}^3r\ \frac{1}{2}\left[(\partial_t\Phi)^2 - v^2(\vec{\nabla}\Phi)^2 - \mu^2\Phi^2\right] \stackrel{!}{=} 0\ \ ,$$

   where ∂_t is the time-derivative. Derive the equation of motion. Under which name do you know it?

4. RAYLEIGH-RITZ VARIATIONAL METHOD (**5P**): The Hamilton operator of the harmonic oscillator is in appropriate units

   $$H = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{1}{2}x^2\ \ .$$

   a) (**3P**) By varying the parameter c in the trial function

   $$\phi_0(x) = \begin{cases} (c^2 - x^2)^2 & \text{for } |x| < c \\ 0 & \text{for } |x| \geq c \end{cases}\ \ ,$$

   obtain an upper bound for the ground-state energy and wave-function. Notice that ϕ_0 is not yet normalised and that it vanishes outside the interval $[-c,c]$, which makes the integrals easier.

   b) (**2P**) Compare to the exact result, also by calculating the "overlap" between the trial and exact wave-function as in the lecture. You may not be able to do the integral exactly.

Please turn over.

5. GRAM-SCHMIDT ORTHO-NORMALISATION (**7P**): We now use the Gram-Schmidt ortho-normalisation procedure familiar from Linear Algebra to construct ortho-normal sets of functions, see e.g. [AW, Sects. 3.1 and 10.3]. While "ortho-normal" has a familiar meaning for vectors, we have to define what we mean by ortho-normal functions. To that end, *define* two functions f, g in n-dimensional space as

   orthogonal to each other when $\displaystyle\int \mathrm{d}^n r\; w(\vec{r})\; f^\dagger(\vec{r})\, g(\vec{r}) = 0$, normalised when $\displaystyle\int \mathrm{d}^n r\; w(\vec{r})\; |f(\vec{r})|^2 = 1$,

   where $w(\vec{r})$ is a "weight function" which is given, the integration region may be finite or infinite, and "†" denotes complex conjugation. This definition constitutes a functional and is in itself a reasonable extension of the notion of ortho-normal vectors. The Gram-Schmidt ortho-normalisation procedure aims now to construct a COMPLETE ORTHO-NORMAL BASIS of the space of all function on the integration region, from a set of trial functions. To show that this basis is complete can be lengthy, see the later chapter on Function Spaces.

   The most famous set of trial functions is the set $\{x^n, n \in \mathbb{N}_0\}$ of all non-negative powers of x. On the interval $r \in [-1; 1]$ and with $w(r) = 1$, taking this as the "seed" of the Gram-Schmidt procedure leads for example to the LEGENDRE POLYNOMIALS, see [AW, Example 10.3.1]. Other choices for w and intervals lead to other complete ortho-normal sets of special functions, which in general are called ORTHOGONAL POLYNOMIALS.

   a) (**5P**) Construct now the first 3 (three) ortho-normal functions for the seed $\{x^n\}$ on the positive real half-line, $r \in [0; \infty[$, with weight factor $w(r) = \mathrm{e}^{-x}$. These are some of the LAGUERRE-POLYNOMIALS which appear in the radial solution of the Hydrogen wave-function.

   b) (**2P**) After what you have learned about Gram-Schmidt ortho-normalisation, how would you improve your trial wave-function for the Rayleigh-Ritz problem above? An outline of your reasoning is enough. Do *not* provide a full-blown calculation.

6. A GROUP OF FUNCTIONS (**6P**): Given the functions below and the operation $f_i(x) \circ f_j(x) = f_i(f_j(x))$.

$$\left\{ f_1(x) = x;\; f_2(x) = 1 - x;\; f_3(x) = \frac{1}{z} \right\} \quad .$$

   a) (**2P**) Construct the three additional elements which must be added to make the whole set of size elements a group under this operation.

   b) (**1P**) Which function serves as the identity element?

   c) (**1P**) Is this group Abelian?

   d) (**2P**) Identify the inverse of each group element.