Problem Sheet 1

Due date: 16th September 2008 **12:00 noon**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at *http://home.gwu.edu/~hgrie/lectures/math-methods08/math-methods08.html.*

1. MECHANICAL SIMILARITY (**2P**): We investigate a bit more the potential which is a homogeneous function of degree ν, i.e. $U(\alpha\vec{r}) = \alpha^\nu U(\vec{r})$. We showed in the lecture that if a curve $\Gamma = \{\vec{r}\}$ is a solution to the equation of motion, then the re-scaled curve $\alpha\Gamma = \{\alpha\vec{r}\}$ solves the rescaled equation when one re-scales the time variable appropriately.

   a) (**1P**) We now consider more carefully the initial condition: Γ is as solution to the initial condition $\vec{r}(t=0) = \vec{r}_0$, $\dot{\vec{r}}(t=0) = v_0$. What are the initial conditions on $\alpha\Gamma$ for a scaled solution?

   b) (**1P**) Consider now re-scaling not in $\vec{r}$ but in time t, keeping $\vec{r}$ un-scaled. Find the ratio of the times on a path when you re-scale the particle masses but keep their potential energy the same.

2. A FAMILIAR LAGRANGE FUNCTION (**2P**): Derive the equation of motion from the one-dimensional Lagrange function of a mass m at point x with velocity $\dot{x}$, and ω a real constant: $L = \frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2 x^2$. Which physical system does it describe?

3. GALILEIAN RELATIVITY AND CHANGING THE FUNCTIONAL (**4P**): The Lagrange function of a mechanical system is not unique, while the equation of motion is.

   a) (**2P**) Show: The Euler-Lagrange equation of $J[f] = \int \mathrm{d}t\ f$ is invariant under a change by a total derivative, $f(t;\vec{r},\dot{\vec{r}}) \to f(t;\vec{r},\dot{\vec{r}}) + \frac{\mathrm{d}}{\mathrm{d}t}g(t;\vec{r})$. Here, g is an arbitrary function which depends only on t and $\vec{r}(t)$, but not on $\dot{\vec{r}}(t)$, and does not change the boundary conditions imposed on f.

   b) (**2P**) Consider now a Lagrange function $L = \frac{m}{2}\dot{\vec{r}}^2 - V(\vec{r})$. Using the previous result, show that a change of variables $(t \to t; \vec{r} \to \vec{r}' = \vec{r} + \vec{v}t)$, $\vec{v}$ a constant, does not change the equations of motion when $V(\vec{r}) = V(\vec{r}')$, i.e. as long as the potential only depends on the distance between mass points, and not on their distance to the origin.

4. SEVERAL INDEPENDENT VARIABLES (**5P**): Consider a functional

$$J[\Phi] = \int \mathrm{d}^n t\ f\left(t_1,\dots,t_n;\Phi(t_1,\dots,t_n),\frac{\partial\Phi(t_1,\dots,t_n)}{\partial t_1},\dots,\frac{\partial\Phi(t_1,\dots,t_n)}{\partial t_n}\right)$$

   which contains several independent variables $t_1,\dots,t_n$ but only one dependent variable $\Phi(t_1,\dots,t_n)$ and the derivatives of Φ with respect to the independent variables.

   a) (**2P**) Derive the Euler-Lagrange equation for this generalisation,

$$\sum_{i=1}^{n}\frac{\partial}{\partial t_i}\frac{\partial f}{\partial\frac{\partial\Phi}{\partial t_i}} - \frac{\partial f}{\partial\Phi} = 0\ ,$$

   by studying the response of J against infinitesimal variations $\Phi \to \Phi + \epsilon\alpha(t_1,\dots,t_n)$, $\epsilon \searrow 0$, which do not change the boundary conditions. Carefully state which variables are to be kept constant in each of the partial derivatives. There are *no* total derivatives in this expression.

   b) (**3P**) As an application, we look at the electro-static field $\vec{E} = -\vec{\nabla}\Phi$ inside a given volume with fixed boundary conditions on the scalar potential $\Phi(x,y,z)$. Derive the Euler-Lagrange equation for the condition that the total field energy should be minimal. In natural units, $\frac{1}{2}\vec{E}^2(\vec{r})$ is the energy density in the volume. Under which other name do you know this equation?

5. A Fata Morgana (**10P**): According to Fermat's Principle, in a medium with varying speed of light $v(\vec{r})$, light takes the path between two given points P and Q with the least travel time,

$$\text{i.e. minimises} \quad J[\vec{r}(s)] = \int\limits_P^Q \frac{\mathrm{d}s}{v(\vec{r})} \quad , \quad \text{where } s \text{ parameterises the path.}$$

   a) (**2P**) To simplify, we consider a two-dimensional problem, i.e. $\vec{r} = (x, y)$. In addition, we pick a medium in which v varies only with y, i.e. e.g. air whose light speed only changes with height (different layers/densities). Reduce the solution to the light path to a quadrature, i.e. find the parameterisation of the path $x(y)$ or $y(x)$ in a form which contains only one integral (which you cannot do in general). Start with a functional with independent variable x, not s.

   b) (**4P**) Consider now the special case $v(x, y) = v_0 + \gamma\, y$ with v_0, γ constants. This solution only makes sense for the range of y where $v > 0$. Construct the light path between two points on the surface of the (flat) Earth $P = (x = -L, y = 0)$ and $Q = (x = L, y = 0)$. Sketch, describe and discuss the form of the path. Is it unique? Differentiate between $\gamma > 0$ and $\gamma < 0$.

   c) (**4P**) Lawrence of Arabia is separated from an oasis 2km away by a 100m high dune which is right half-way to the oasis. At least by how much does the speed of light differ at the top of the dune from the value at the bottom, so that he sees the oasis as a Fata Morgana? Is v larger on the top or on the bottom of the dune?

   NB: The same solution can simulate a light ray which is trapped around a black hole.

6. Soap Bubbles: The Catenoid (**7P**): This is a special case of a "Classic", found in every textbook. But be careful to use *my* conventions for labelling variables, and not those of the book you get inspired from. Work through your own solution, especially when it comes to numerics and plots. No copying!

   Two rings with the same radius R are located concentric to the z-axis and parallel to the xy-plane, at distance $2L$ from each other and equi-distant from the origin. A thin film of soap stretches between them. Because of their surface tension, soap films want to minimise the area they cover. We now describe the form of this "minimal surface".

   a) (**1P**) Assume for brevity that you can parameterise the minimal surface by a monotonous function $z(x)$ plus rotational invariance. Show that for rings of different radii R_1 and R_2, the functional is

$$J[z(x)] = \int\limits_{R_1}^{R^2} \mathrm{d}x\; 2\pi\, x \sqrt{1 + (z'(x))^2} \; .$$

   Show by solving an appropriate differential equation that one can invert $z(x)$ to obtain $x(z) = c \cosh \frac{z}{c}$, where cosh is the hyperbolic cosine. Derive an (implicit) equation for the constant c from the boundary conditions with $R_1 = R_2 = R$.

   b) (**2P**) Determine the form of the surface of revolution as function of the parameter $\eta = L/R$. How many locally minimal surfaces do you find for which ranges of η? Draw the profile for some "typical" values of η which are interesting (make a good choice!).

   c) (**2P**) Find the area of the extremal surface, and plot this area for all solutions as function of η.

   d) (**2P**) There is another minimum of the action, called Goldschmidt's dis-continuous solution: two soap films, namely the two discs which are bounded by the two rings, with no connection between them. Compare the total area of this solution to the previous one to determine the *absolute minimum surface of rotation* as function of η. Show that Goldschmidt's solution is the absolute minimum for $\eta > 0.528\ldots$.

   Since the continuous solution found in problem a)-c) becomes for $\eta > 0.528\ldots$ a local minimum only, it is meta-stable. If you are really careful, you can still produce it in experiments.

7. Optional Questions to Brush Up Your Math

a) Matrices: Given the matrix $\mathcal{M} = \begin{pmatrix} 1-a & a \\ a & 1+a \end{pmatrix}$, with $a \in \mathbb{C}$ arbitrary.
Determine all eigenvalues and normalised eigenvectors, a matrix which diagonalises $\mathcal{M}$, the diagonalised version of $\mathcal{M}$ and $\mathcal{M}^{-1}$.Are there a for which the inverse does not exist?

b) Taylor Expansion: Expand up to the second (or first, if you want less work) non-trivial order:
$\cos[\pi\sqrt{1-x^2}]$ for $x \to 0$; $\exp[-x/(1+x^3)^{\frac{1}{4}}] - \exp[-x]$ for $x \to 0$.

c) Complex Numbers: Bring the following expressions into the form $a + \mathrm{i}b$, where $\mathrm{i}^2 = -1$ the imaginary unit and a, b real: (i) $\sqrt{\mathrm{i}+1}$, (ii) $\frac{1+2\mathrm{i}}{1-2\mathrm{i}}$, (iii) $\sin\frac{\pi\mathrm{i}}{4}$.

d) Differentiation in 3 Dimensions: Do the following fields have sources/sinks or curls? Sketch or plot the vector-function in a suitable way!

$$\vec{A}_2(\vec{x}) = x\,\vec{e}_y - y\,\vec{e}_x \qquad \vec{A}_3(\vec{x}) = \frac{1}{1+x^2}\,\vec{r}$$

e) Integral Theorems: Verify Stokes' theorem by explicit calculation in the field $\vec{A} = (xy, 4yz, 3x^4z)$. Pick as surface of integration a square with sides L, centred at the origin in the xy-plane.

f) Separable Ordinary Differential Equation: Show that $y' = a(x)\,b(x)$ is solved by $\int^{y(x)} \frac{\mathrm{d}t}{b(t)} = \int^x \mathrm{d}t\; b(t) + \text{const}$. Apply this to $y' = \exp[x + y(x)]$ with $y(x=0) = 1$.