### Supplement: Comparison between Dirac’s $\langle \text{Bra} | \text{Ket} \rangle$ Notation and Einstein’s $\sum$ Summation Convention

| Quantity | Dirac’s $\langle \text{Bra} | \text{Ket} \rangle$ Notation | Einstein $\sum$ Summation Convention |
|----------|----------------------------------------------------------|--------------------------------------|
| abstract vector in vector space $\mathcal{M}$ | $|x\rangle$ ket $\ddagger$ clear difference | $x^* = \bar{x}$ column vector $\ddagger$ no clear distinction |
| abstract vector in dual space $\mathcal{M}^*$ | $\langle x\rangle$ bra | $x^* = \bar{x}^T$ row vector (when $x$ real; otherwise $^* = \bar{x}^T$) |
| basis of $\mathcal{M}$ | $|i\rangle$, $i = 1, \ldots, \dim \mathcal{M}$ no distinction between basis & vectors kets can be basis vectors | $e_i$, $i = 1, \ldots, \dim \mathcal{M}$ clear separation of vector components and basis |
| basis of $\mathcal{M}^*$ | $\langle i\rangle$, $i = 1, \ldots, \dim \mathcal{M}^*$ | $e^i$, $i = 1, \ldots, \dim \mathcal{M}^*$ |
| ortho-normality | $\langle i|j \rangle = \delta_{ij} \forall i, j = 1, \ldots, \dim \mathcal{M}$ if $\mathcal{M} \simeq \mathcal{M}^*$ | $\langle e^1|e^j \rangle = \delta^j_j$ |
| closure/completeness | $\sum |i\rangle \langle i| = \text{id}$ ”inserting unity”, used often | $e^i \otimes e_i = 1$ tensor product, used infrequently |
| scalar product $\langle y, x \rangle$ of $x \in \mathcal{M}$ and $y \in \mathcal{M}^*$ | $\langle y|x \rangle = (\langle x|y \rangle)^*$ usually complex | $y_i x^i = y^j x_j$ usually real |
| metric $g_{ij} =$ | ??? | $\langle e_i, e_j \rangle$ |
| contravariant components | $\langle i|x \rangle$ | $x^i$ |
| covariant/dual components | $\langle x|i \rangle$ | $x_i$ |
| relation between the two | $\langle x|i \rangle = (\langle i|x \rangle)^*$ | $x_i = g_{ij} x^j$ |
| basis change | $\langle n|x \rangle = \sum_j \langle n|j \rangle \langle j|x \rangle$ | $x^i = d^i_j x^j$ |
| components of matrix $M$ | $\langle i|M|j \rangle$ | $M^i_j$ (index ordering important) |
| abstract matrix $M =$ from components | $\sum_i |i\rangle \langle i|M|j \rangle \langle j|$ | $e_i M^i_j e^j$ |
| Further typical features | prefers abstract vectors e.g. “states” in QM can deal with infinite-dim. space and continuous basis can deal with complex components non-orthogonal bases hard to describe order matters: $|i\rangle \langle j| \neq \langle j|i\rangle$ no summation, e.g. projector on eigenstate: $|i\rangle \langle i| \neq \sum_i |i\rangle \langle i| = \text{id}$ ambiguous operator action $\langle y|Mx \rangle \neq \langle My|x \rangle \neq \langle y|M|x \rangle$ e.g. selfadjoint vs. Hermitian | prefers concrete representations e.g. locally orthog. coord., GRT prefers finite-dim. space prefers real components made for non-orthogonal bases & metrics components commute: $x_i y^j = y^j x_i$ summation “mandatory” operator action cannot be addressed |