1. **Frobenius’ Method (6P):** We will find a solution to the differential equation

\[ 4x f''(x) + 2(1-x)f'(x) - f(x) = 0 \]

by Frobenius’ ansatz

\[ f(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n \quad a_0 \neq 0 \]

a) (2P) Show that the indicial equation dictates that \( \alpha = 0 \) or \( \alpha = \frac{1}{2} \).

b) (4P) For the case \( \alpha = 0 \), derive the recurrence relation for the coefficients \( a_j \) and construct the closed answer.

**Note:** The case \( \alpha = \frac{1}{2} \) is straight-forward and therefore boring. Do not do it.

2. **A Two-dimensional Green’s Function (4P per correct, independent way)** Show that the Green’s function to the two-dimensional Poisson equation “without boundaries at infinity” is

\[ G(\vec{r}, \vec{r}') = \alpha \ln \frac{|\vec{r} - \vec{r}'|}{C}, \]

where \( C \) is an arbitrary constant. Determine the constant \( \alpha \). There are several ways to do this problem, given the toolchest developed in the last semester. **Each correct way gives 4 points.**

You might want to recall also some Green’s functions of one- and three-dimensional “empty” space.

3. **Image Charges and Green’s Function for Plates at an Angle (4P)** A point charge \( q \) is located at \( \vec{r}_0 = (a, a, 0) \) in front of two infinitely long, perfectly conducting, grounded plates which meet at the origin in an angle of 90°; see figure for details. The problem is three-dimensional. Use the method of image charges (you need 3 images).

a) (2P) Determine the potential everywhere.

b) (2P) Derive the induced surface charge density on the plates at \( z = 0 \). Sketch!

4. **Spherical Harmonics (7P):**

a) (2P) Determine the parity of \( Y_{lm}(\theta, \phi) \).

b) (3P) Inspired by a good book, prove the addition theorem for spherical harmonics:

\[ P_l(\cos \alpha) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}^*(\Omega) Y_{lm}(\Omega') = \sqrt{\frac{4\pi}{2l+1}} Y_{l0}(\alpha, 0), \]

where \( \alpha \) is the angle between \((\theta, \phi)\) and \((\theta', \phi')\), i.e. \( \cos \alpha = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi') \).

c) (2P) Construct the first three Legendre polynomials from the generating function.

Please turn over.
5. **Square with Boundary Conditions in Electrostatics (8P):** The sides of a square (length of sides $L$) are made of some material such that the boundary conditions on the potential are

$$\Phi(x, y) = \begin{cases} \Phi_0 \sin \frac{3\pi x}{L} & \text{on the upper side, } y = L \\ 0 & \text{on all other sides.} \end{cases}$$

The problem is two-dimensional. There are no charges inside the square; see figure.

a) **(1P)** Are these boundary condition of the Dirichlet or von-Neumann type?

b) **(5P)** Determine the potential $\Phi(x, y)$ everywhere inside the square.

**Hint:** Show that you need to solve differential equations (you need to determine the constant of separation $\alpha$):

$$\frac{d^2}{dx^2} X(x) = -\alpha^2 X(x), \quad \frac{d^2}{dy^2} Y(y) = \alpha^2 Y(y).$$

with the boundary conditions $X(x = 0) = X(x = L) = 0 = Y(y = 0) = 0$.

**Fail-safe point:** $\Phi(x, y) \propto \sin \frac{3\pi x}{L} \left[ e^{\frac{3\pi y}{L}} - e^{-\frac{3\pi y}{L}} \right]$.

c) **(2P)** Using a pillbox construction, determine the charge density on the lower plate, $y = 0$.

6. **Legendre Polynomials (2P)** Prove that

$$\int_{-1}^{1} dx \, P_l(x) = 0$$

for $l$ a positive integer.

7. **Electrostatic Multipole Moments (7P):** An infinitesimally thin, conducting, circular ring of radius $R$ carries the homogeneous line charge density $\mu$. It is centred at the origin in the $xy$ plane, see figure.

**Hints:** Determine all spherical multipole moments with respect to the origin.

If multipole moments vanish, you can substitute a calculation by a short argument.

Recall that $Y_{l0}(\theta, \phi) \propto P_l(\cos \theta)$, so that some results can also be written using Legendre polynomials at a given value, which you need to look up.

a) **(1P)** Convince yourself that the charge density can take the form $\rho(r, \theta, \phi) = \frac{\mu}{r} \delta(r - R) \delta(\cos \theta)$.

Is this a Dirichlet or von-Neumann boundary problem?

b) **(5P)** Determine all spherical multipole moments for $r \gg R$. Give explicit answers for the monopole, dipole and quadrupole moments.

c) **(1P)** Determine the scalar potential far from the origin as expansion in a suitable, small parameter.

d) **(3P)** Determine the scalar potential close to the origin as expansion in a suitable, small parameter.