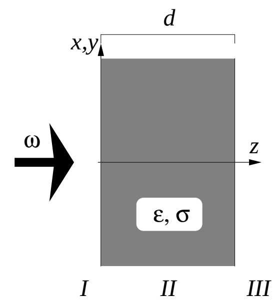
Problem Sheet 14

Special due date: Friday 1st May 2009 12:00h noon

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at *http://home.gwu.edu/˜hgrie/lectures/edyn09/edyn09.html*.

1. Transversality in An-Isotropic Media (**6P**) Consider a homogeneous, neutral, non-conducting medium with magnetic susceptibility $\mu(\omega) = 1$, but *an-isotropic and real* dielectric function, i.e. $\varepsilon(\omega) \neq 1$ is a real tensor. Show for a plane wave with frequency ω and wave-vector $\vec{k}$: (i) $\vec{H}$ is orthogonal to $\vec{D}$ and $\vec{E}$, (ii) $\vec{D}$ and $\vec{H}$ are transverse, but (iii) $\vec{E}$ is not. Discuss under which condition(s) $\vec{E}$ is transversal.



2. Transmission and Reflection of Electromagnetic Waves on Metal (**12P**): A plane, mono-chromatic wave with frequency ω is incident perpendicular on a homogeneous layer (II) with thickness d, dielectric constant $\varepsilon > 1$ and magnetic permeability $\mu = 1$, surrounded by vacuum (regions I and III). The layer is composed of a metal with conductivity σ, i.e. Ohm's law $\vec{j} = \sigma\,\vec{E}$ holds in (II).

   Hint: This is another problem in which algebraic manipulation programmes can save a lot of work.

   a) (**2P**) Show that the wave-equation inside the medium is
   $$\left(\nabla^2 - \frac{\varepsilon}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \frac{4\pi\sigma}{c^2}\frac{\partial\vec{E}}{\partial t}$$
   and that the index of refraction is determined by $n^2 = \varepsilon + \frac{4\pi \mathrm{i}\,\sigma}{\omega}$.

   b) (**2P**) Determine those solutions of the wave-equations for $\vec{E}$ (and $\vec{B}$) in each of the regions I, II, and III which are linearly polarised.

   c) (**2P**) Find the boundary conditions for $\vec{E}$ (and $\vec{B}$): They can be written as 4 linear equations for the non-vanishing amplitudes of $\vec{E}$.

   d) (**3P**) Compute the reflectivity R (ratio of intensities between incident and reflected waves in region I), the transmission coefficient T (ratio of intensities between incident wave in I and wave transmitted into III) and the constant X:
   $$R = X\left|\frac{1-n^2}{N}\sin nkd\right|^2 \quad , \quad T = X\left|\frac{n}{N}\right|^2 \quad \text{with } N := (1+n)^2\mathrm{e}^{-\mathrm{i}nkd} - (1-n)^2\mathrm{e}^{\mathrm{i}nkd}$$
   Discuss without an explicit calculation whether $R + T = 1$ for all σ.

   e) (**1P**) Discuss and sketch R as function of the thickness d of the layer for the case $\sigma = 0$.

   f) (**2P**) Calculate the ratio T/R for $\varepsilon \approx 1$, $\sigma/\omega \gg 1$. Discuss the limit in which the layer thickness is large (against what?).

Please turn over.

3. SIGNAL DELAY FROM PULSARS (**6P**): A pulsar is a star which regularly emits a whole spectrum of frequencies in one short pulse. On Earth, the arrival times of these bursts are delayed for low frequencies: The signal arrives as a "whistle" whose frequency changes with time. This is attributed to dispersion in the interstellar medium which is assumed to consist of fully ionised hydrogen. Given the density of the interstellar plasma, determine the signal delay for a wave which propagates with frequency ω through the plasma, compared to a freely propagating light-ray. Show that we can determine the distance D to the pulsar if we know the electron density in the interstellar medium and measure the rate of change (frequency versus time) of the signal:

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = -\frac{c}{D}\,\frac{\omega^3}{\omega_p^2}\left(1-\frac{\omega_p^2}{\omega^2}\right)^{\frac{3}{2}}$$

4. LARMOR'S FORMULA AND MODERN HIGH-ENERGY PHYSICS (**4P**): Will the frontiers of high-energy physics in future be pushed by linear accellerators or by storage rings?

   The relativistic version of Larmor's formula can be re-written as (Liénard 1898):

$$P = \frac{2q^2}{3c}\,\gamma^6\left[\left(\dot{\vec{\beta}}\right)^2 - \left(\vec{\beta}\times\dot{\vec{\beta}}\right)^2\right]$$

   a) (**2P**) Discuss radiation loss from ultra-relativistic particles in linear accelerators vs. storage rings when the magnitude of the force exerted on the particle is the same in both cases.

      Hint: Recall that the momentum is $\vec{p} = m\gamma(v)\,\vec{v}$. Is $|\vec{v}|$ constant for a storage ring/for a linear accelerator? Is therefore the acceleration the same in both cases, if the force exerted is the same?

   b) (**2P**) The Large Electron-Prositron-Storage Ring LEP-2 at CERN is with a circumference of 27 km likely to be the largest storage ring ever built: Its state-of-the-art superconducting magnets store electrons and positrons with an energy of up to 90 GeV, the injectors and accelerating cavities barely keeping up with radiation loss. Which fraction of the electron energy is per cycle lost to radiation?

      Note: At LEP-2, one discovered the W and Z bosons, i.e. the missing particles which mediate the unified force of electro-magnetic and weak interactions, and pinned down the "Standard Model of Fundamental Interactions" (Nobel 1979, 1984, 1992, 1999). With the LHC, we hope to find the Higgs-particle which gives mass to all other particles, and see beyond the Standard Model.

5. RELATIVISTICALLY OSCILLATING CHARGE (**12P**): A charge q with mass m oscillates harmonically about the origin on the z-axis, $z(t) = l\cos\omega t$, with possibly relativistic velocity.

   a) (**3P**) Before you start, discuss which radiation characteristics you would expect, based on your knowledge of the radiation characteristics of non-relativistically and relativistically oscillating dipoles. It may be useful to plot or discuss the time-dependent radiation characteristics for $\omega t = 0; \pi/2; \pi; 3\pi/2$.

   b) (**3P**) Show that the radiated power per unit solid angle is with $\beta := l\omega/c$:

$$\frac{\mathrm{d}P(t)}{\mathrm{d}\Omega} = \Gamma\,\frac{\sin^2\theta\,\cos^2\omega t}{(1+\beta\cos\theta\sin\omega t)^5}$$

      Determine $\Gamma(\beta, q, l)$. What is the physical meaning of the symbol β?

   c) (**3P**) Determine the time-averaged power radiated per unit solid angle. You may have to perform a contour integration about the unit circle, after a change of variables $\mathrm{e}^{\mathrm{i}\omega t} = z$.

   d) (**4P**) Compare the time-averaged radiation characteristics for the non-relativistic case to the relativistic one, and to Hertz' dipole. Include sketches of the characteristics at selected values of β. Also, plot the time-dependent characteristics at the times given in a).