Problem Sheet 13

Due date: 21st April 2009 **13:00 noon**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at *http://home.gwu.edu/˜hgrie/lectures/edyn09/edyn09.html*.

1. Microscopic Response Models (**3P**): Response functions $\chi_{\rm el}(t)$ are better understood in time than in frequency space. For the following responses, determine the dielectric function $\varepsilon(\omega)$ in frequency space and plot its real and imaginary part. Plot its singularities in the complex ω-plane.

   Hint: $\theta(t)$ is Heaviside's step-function and χ_0, $\Gamma > 0$, ω_0, ω_p are real. Recall $\lim\limits_{\eta\searrow 0} \dfrac{\eta}{\eta^2+x^2} = \pi\delta(x)$.

   a) (**1P**) Non-dispersive medium: $\chi_{\rm el}(t) = \chi_0\ \delta(t)$, i.e. immediate response to applied impulse.
   b) (**1P**) Damped medium: $\chi_{\rm el}(t) = \chi_0\ \theta(t)\ \mathrm{e}^{-\Gamma t}$, i.e. response relaxes slowly back to original state.
   c) (**1P**) Perfectly conducting medium: $\chi_{\rm el}(t) = \chi_0\ \theta(t)$, i.e. no restoring force drives response back.

2. Dielectric Constant of the Hydrogen Atom (**9P**): We return to the hydrogen atom already discussed in Problem Sheet 5, number 1. The charge density in its ground state is (Bohr radius a):

   $$\rho(\vec{r}) = q\ \delta^{(3)}(\vec{r}) - \frac{q}{\pi a^3}\ \mathrm{e}^{-\frac{2r}{a}}$$

   This leads to field strengths of the order of 10^9 V/m inside the atom. At first order in perturbation theory, an external electric field $\left|\vec{E}_{\rm ext}\right| \lesssim 10^6$ V/m shifts therefore only the electron's charge density without deformation by a vector $\vec{r}_0$ against the proton.

   a) (**1P**) Determine the induced dipole moment $\vec{d}_{\rm ind}$ in the field $\vec{E}_{\rm ext}$ as function of $\vec{r}_0$.
   b) (**3P**) Show that this shift of the electron cloud leads to a restoring force on the proton:

   $$\vec{F}_{\rm restore} = -\frac{4q}{3a^3}\ \vec{d}_{\rm ind} \quad \text{for } r_0 \ll a$$

   Hint: The electric field of the unperturbed electron cloud was determined in Problem 5.1.
   c) (**2P**) The equilibrium condition between the force exerted by the external field and $\vec{F}_{\rm restore}$ gives $\vec{d}_{\rm ind}$ as function of $\vec{E}_{\rm ext}$, and hence the (classical) polarisability of the hydrogen atom.
   d) (**3P**) Determine finally the dielectric constant for a gas of N hydrogen *molecules*, homogeneously distributed in a volume V. Compare to the estimate in the lecture (from $\alpha_{\rm est}(0) \approx a^3$), the QM result (from $\alpha_{\rm QM}(0) = \frac{9}{2}\ a^3$), and explain the discrepancy to the experimental value, $\varepsilon_{H_2}(\omega = 0) = 1.00026$ at room temperature and normal pressure.

Please turn over.

3. SPHERES IN EXTERNAL FIELDS (**14P**): A neutral, homogeneous sphere with radius R and dielectric constant ε is set into an external, originally homogeneous electric field $\vec{E}_0$.

   a) (**1P**) Show that the boundary conditions on the surface of the sphere can be written as von-Neumann boundary conditions between the potentials Φ_i inside and Φ_o outside:
   $$-\frac{1}{R}\frac{\partial\Phi_i}{\partial\vartheta}\bigg|_{r=R} = -\frac{1}{R}\frac{\partial\Phi_o}{\partial\vartheta}\bigg|_{r=R} \quad , \quad -\varepsilon\frac{\partial\Phi_i}{\partial r}\bigg|_{r=R} = -\frac{\partial\Phi_o}{\partial r}\bigg|_{r=R}$$

   b) (**2P**) Determine the potential, electric field and dielectric displacement inside and outside.

   Hint: Symmetries! An ansatz $\Phi(\vec{r}) = \sum\limits_l [A_l r^l + B_l r^{-(l+1)}]\ P_l(\cos\vartheta)$ might be helpful. Motivation?

   c) (**1P**) Discuss the fields inside and outside. Are there induced fields/screened fields? Sketch!

   d) (**2P**) Find the induced polarisation and total dipole moment of the sphere in vacuum; answer in 4.b). In which direction does it point? Derive and sketch the induced surface charge density.

   e) (**2P**) By how much is the total energy changed when the dielectric is brought in?

   f) (**2P**) Determine now the fields, polarisation and surface charge density in the complementary situation of a spherical hole of vacuum inside a dielectric medium ε', with external field. In which direction does the polarisation density point now?

   g) (**4P**) What changes when you consider instead a sphere of magnetic permeability μ in an external, originally homogeneous magnetic field $\vec{B}_0$? Discuss in particular the magnetic field and induction, the magnetisation density and the direction of the magnetisation with respect to $\vec{B}_0$ for dia- and para-magnetic media. Calculate and sketch the resulting surface current density.

   Hints for f) and g): Compare field equations and boundary conditions with the ones in a).

4. OPTICAL TWEEZERS (**5P**) are a standard tool in Bio-Physics to manipulate microscopic dielectric media of electric susceptibility χ_{el}, like individual cells and their components.

   a) (**1P**) As a warm-up, show that the energy density of an induced dipole in a static electric field $\vec{E}$ is (α the polarisability):
   $$\mathcal{H}_{\text{el}} = -\frac{1}{2}\alpha\,\vec{E}^2 \quad .$$

   Hint: The formula $\mathcal{H}_{\text{el}} = \vec{d}\cdot\vec{E}$ is valid only for the energy of a *permanent* dipole $\vec{d}$ in an external field. If the dipole is induced, $\vec{d}[\vec{E}]$, consider the infinitesimal change in energy which comes from changing the electric field.

   b) (**2P**) Derive now the total energy for a dielectric sphere $\varepsilon > 1$ with radius R:
   $$H_{\text{el}} = -\frac{1}{2}\frac{\varepsilon-1}{\varepsilon+2}\,\vec{E}^2\,R^3 \quad .$$

   Hint: You derived in 3.d) above that $\vec{P} = \dfrac{3}{4\pi}\dfrac{\varepsilon-1}{\varepsilon+2}\,\vec{E}$ for the dielectric sphere.

   c) (**2P**) Show: The medium is attracted to regions in which $\left|\vec{E}\right|$ is large, irrespective of its direction.

   Notes: This is the principle used to drag dielectrics by laser beams. Lasers usually have a parabolic intensity profile $I = I_0\left(1 - r^2/R^2\right)\theta[R-r]$, where I_0 is the intensity at the centre, R the beam radius, r the (transverse) distance from the beam centre and $\theta[x]$ Heaviside's step-function. A sphere of mass M describes therefore close to the centre of the beam oscillations around the beam centre with a frequency $\omega^2_{\text{tweezer}} = 4\pi\,\dfrac{\varepsilon-1}{\varepsilon+2}\,\dfrac{I_0 R}{Mc}$. For water (i.e. cells), $\varepsilon(\omega = \text{light}) \approx 1.8$ gives a frequency $f_{\text{res}} \approx 4$ kHz for $R = 1\,\mu$m, $I_0 = 1$ mW, which a tabletop laser can provide.