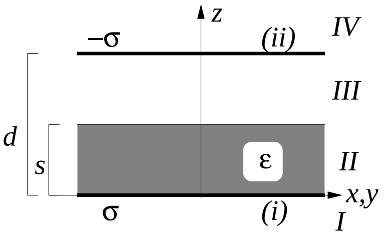
Problem Sheet 12

Due date: 14th April 2009 **13:00 noon**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at *http://home.gwu.edu/~hgrie/lectures/edyn09/edyn09.html.*

1. PARTIALLY FILLED PLANAR CAPACITOR (**10P**): Two infinitely long, infinitesimally thin, perfectly conducting plates carry opposite charge densities and are mounted parallel to each other at distance d. The lower plate is identical with the xy-plane and carries a surface charge density $\sigma_{(i)} = \sigma > 0$. As shown in the figure, it is coated such that the space II between the plates is filled by a dielectric $\varepsilon > 1$ with thickness s. The other part of the capacitor, III, is evacuated, as are the exterior regions I and IV below and above, respectively.



Hint: You may choose to switch the order of e.g. b) and c).

   a) (**1P**) List the boundary conditions of the electric field and dielectric displacement on each plate. Determine the boundary conditions for the tangential and normal components of the electric field and dielectric displacement at the surface between II and III.

   b) (**3P**) Determine the electric field and dielectric displacement *between and outside* the plates. Sketch $\vec{D}$ and $\vec{E}$, paying particular attention to the free charges and polarisation charges.

   c) (**1P**) Determine and sketch the electro-static potential Φ, with Φ continuous everywhere.

   d) (**1P**) Determine the surface charge density of the polarisation charges at the surface II/III.

   e) (**1P**) Show that the field energy per unit surface area of the plates is (determine $X > 0$!):

$$u = X\,\sigma^2\,[d + s\,(\frac{1}{\varepsilon} - 1)]\ .$$

   f) (**3P**) Determine the numerical value of the electro-static pressure (force per surface unit) on the boundary II/III for a thin film of distilled water, $\varepsilon = 80$, $\sigma = 10^{-3}$ C/m^2. Assuming that the medium ε is a perfect fluid, would it flow in or out of the capacitor? Under which condition can you fill the capacitor completely with the medium when the capacitor lies on your lab table on Earth?

2. CAPACITOR WITH AND WITHOUT MEDIUM (**2P**): The capacitance is defined as $C := Q/\Delta\Phi$, with $\Delta\Phi$ the potential difference between the plates and Q the free surface charge on the plates. Consider now a planar capacitor with capacitance C_{vac} when the space between the plates is empty. Compare to the capacitance C_ε when the capacitor is completely filled with a dielectric $\varepsilon > 1$

   a) (**1P**) when the charges on the plates are kept constant as the medium is inserted;

   b) (**1P**) when the potential difference between the plates is kept constant as the medium is inserted.

3. CAVITY IN POLARISED MEDIUM (**6P**): A spherical cavity of radius R is cut out of an infinite medium with uniform permanent polarisation density $\vec{P}$. Determine the electric field inside and outside the cavity, and the resulting surface charge density. Compare to the complementary case discussed in the lecture: A uniformly polarised sphere in vacuum.

   Hint: No, there is *no* external electric field applied here.

Please turn over.

4. WAVEGUIDES (**12P**): The electric and magnetic fields of a "free wave" which propagates only in a finite volume are *not always* perpendicular to its propagation directions. As an example, consider an infinitely long waveguide, i.e. the evacuated interior of an infinitely long tube whose walls are made of a perfectly conducting material. It has a quadratic profile with dimensions $z \in]-\infty;\infty[$, $x, y \in [0;L]$. We will show that waves which travel along the positive z-direction also have a *longitudinal* component.

   a) (**2P**) Show that the components of the electric field $\vec{E}$ tangential to the surface must vanish at the surface. Show the same for the components of the magnetic field $\vec{B}$ normal to the surface.

   b) (**4P**) *Confirm* that the following electric and magnetic fields solve all of Maxwell's equations in vacuum and obey *all* boundary conditions:

$$\vec{E}(t,\vec{r}) = \mathrm{e}^{-\mathrm{i}(\omega t - kz)} \begin{pmatrix} C_x \cos\frac{n_x \pi x}{L} \sin\frac{n_y \pi y}{L} \\ C_y \sin\frac{n_x \pi x}{L} \cos\frac{n_y \pi y}{L} \\ C_z \sin\frac{n_x \pi x}{L} \sin\frac{n_y \pi y}{L} \end{pmatrix}$$

$$\vec{B}(t,\vec{r}) = \frac{c}{\omega}\,\mathrm{e}^{-\mathrm{i}(\omega t - kz)} \begin{pmatrix} \left(-\mathrm{i}C_z \frac{\pi n_y}{L} - C_y k\right) \sin\frac{n_x \pi x}{L} \cos\frac{n_y \pi y}{L} \\ \left(\mathrm{i}C_z \frac{\pi n_x}{L} + C_x k\right) \cos\frac{n_x \pi x}{L} \sin\frac{n_y \pi y}{L} \\ \frac{\mathrm{i}\pi}{L}\left(C_x n_y - C_y n_x\right) \cos\frac{n_x \pi x}{L} \cos\frac{n_y \pi y}{L} \end{pmatrix}$$

   Here, C_x, C_y, C_z are constants, n_x, n_y non-negative integers, k real and positive if the wave travels in the positive z-direction, and

$$\omega = c\sqrt{k^2 + \left(\frac{n_x \pi}{L}\right)^2 + \left(\frac{n_y \pi}{L}\right)^2}\ ,\quad C_x \frac{\pi n_x}{L} + C_y \frac{\pi n_y}{L} - \mathrm{i}C_z k = 0$$

   These are a lot of conditions, so be sure you do not miss to check any! This is indeed the *most general* solution, but you do not have to show that (you may, of course, for extra credit).

   Hint: This may be a bit tedious, so I suggest you at first take my word for it and solve the following sub-problems. Come back to this one when you have the time.

   c) (**4P**) Now, deduce from all this information: If $E_z = B_z = 0$, then $\vec{E} = \vec{B} = 0$, i.e. if the wave would be purely transversal, it would disappear altogether.

   Hint: There is a very clever way which shows that the electric and magnetic fields have in that case neither curl nor divergence if $E_z = B_z = 0$. If you start from the expressions above and use another way, you need to differentiate between three cases: (i) $n_x = 0$; (ii) $n_y = 0$; or (iii) $C_z = 0$.

   This means we can classify all waves in the guide as either of two:

   (i) transverse electric (TE) modes: $E_z = 0$ and $\left.\frac{\partial B_z}{\partial n}\right|_S = 0$;

   (ii) transverse magnetic (TM) modes: $B_z = 0$ and $E_z|_S = 0$.

   $\frac{\partial}{\partial n}$ denotes the derivative normal to the surface S of the waveguide.

   d) (**2P**) Finally, show that minimum frequencies exist for both waves: $\omega_{\text{min}}^{\text{TE}} = \frac{c\pi}{L}$, $\omega_{\text{min}}^{\text{TM}} = \sqrt{2}\,\frac{c\pi}{L}$.

   Note: So, a rectangular waveguide can serve as frequency filter for electro-magnetic waves. This result depends however on the geometry: For example, no critical frequency exists for a round profile, i.e. for a coaxial cable.