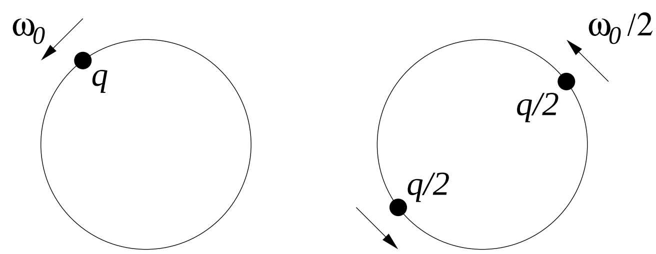
Problem Sheet 9

Due date: 24th March 2009 **13:00 noon**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at *http://home.gwu.edu/˜hgrie/lectures/edyn09/edyn09.html.*

1. ATOMIC DIPOLE- AND QUADRUPOLE-RADIATION, PART II (**13P**): Electric quadrupole radiation in atoms is considerably weaker than electric dipole radiation. It can thus be neglected – except if electric dipole radiation is forbidden e.g. due to selection rules. We continue to discuss a classical analogon.

   Point-charges move in the xy-plane on a circle with radius l in the mathematically positive sense with constant, non-relativistic angular velocity. The atomic nucleus rests at the centre, making the atom as a whole electrically neutral. We dealt with the electric dipole last week. The quadrupole can now be thought of as two point-charges $q/2$ which are opposite to each other and rotate with angular velocity $\omega_0/2$; see figures. Consider only the far-zone, in the lowest non-vanishing multipole approximation.



   After last week's dipole, we now concentrate on the the quadrupole, *right* figure.

   Hint: You can again copy equations from textbooks. Or you use the opportunity to start from the wave-equation and derive step-by-step the solution by yourself, understanding radiation theory in this simple example. You may also choose to solve the problem with the alternative radiation formula in coordinate space presented in problem 1 last week.

   In sub-problems b) and e), arguments can substitute calculations.

   a) (**3P**) Find the time-dependent, Cartesian dipole and quadrupole moments in the right figure.
   b) (**1P**) Determine the frequency (or wave-length) of the emitted radiation.
   c) (**2P**) Determine the retarded vector-potential $\vec{A}_{\text{ret}}$ in the far-zone.
   d) (**4P**) Determine the *time-dependent* electric and magnetic field in the far-zone.

      Hint: You might have to derive

$$\vec{e}_r \times (\vec{e}_x + \mathrm{i}\,\vec{e}_y) = \mathrm{e}^{\mathrm{i}\phi}\ (\vec{e}_\phi\ \cos\theta - \mathrm{i}\vec{e}_\theta)$$

   e) (**1P**) Discuss the polarisation of the radiation, depending on the observer's position. Polarisation close to the polar angles $\theta = 0,\ \frac{\pi}{2},\ \pi$ is particularly interesting.
   f) (**3P**) Determine the time-averaged power radiated into an arbitrary solid angle, and the total radiated power. Discuss and sketch the radiation characteristics (angular distribution) of this quadrupole. Compare in particular to last week's dipole.

      Some possible answers in Cartesian and spherical coordinates (r, ϑ, φ), respectively:

$$\vec{A}^{(E2)}(t,\vec{r}) = -\frac{ql^2\omega_0^2}{4rc^2}\ (\vec{e}_x + \mathrm{i}\vec{e}_y)\,[\vec{e}_r\cdot(\vec{e}_x + \mathrm{i}\vec{e}_y)]\exp[\mathrm{i}(kr - \omega_0 t)]$$

$$\frac{\mathrm{d}P^{(E2)}}{\mathrm{d}\Omega} = \frac{q^2 l^4\omega_0^6}{128\pi c^5}\ \sin^2\vartheta\left(\cos^2\vartheta + 1\right)$$

Please turn over.

2. NOW WE COMBINE WHAT WE LEARNED ABOUT THESE SYSTEMS (**3P**): Compare the total, time-averaged power of dipole and quadrupole radiation for typical atomic extensions and wave-lengths: $l \sim 1$ Å, $\lambda \sim 1000$ Å. Compare to the case that atomic radius and wavelength are comparable.

3. RADIATION FIELD OF A ROTATING SPHERE (**14P**): Consider a spherical shell of radius R with its centre located at the origin. The shell is infinitely thin, and the total charge Q is distributed uniformly across the shell. The sphere rotates non-relativistically around the z-axis with a given, time-dependent frequency $\omega(t)$. A point charge $-Q$ is located at the centre of the sphere.

   a) (**1P**) Show that the current density $\vec{j}(t,\vec{r}\,) = \rho(r\,)\,\vec{\omega}(t) \times \vec{r}$ has a vanishing divergence. Show from this that the radiation cannot be electric in origin.

   b) (**3P**) Calculate the time-dependent vector potential of this distribution at large distances from the sphere, taking into account retardation at leading non-vanishing order:

   $$\vec{A}(\vec{r},t) = \frac{QR^2\dot{\omega}(t-\frac{r}{c})}{3rc^2}\,\vec{e}_z \times \vec{e}_r$$

   Hint: You may use: $\vec{A}(\vec{r},t) \approx \frac{1}{cr}\int \mathrm{d}^3r' \left[\vec{j}(\vec{r}\,',t-\frac{r}{c}) + \frac{\vec{r}\cdot\vec{r}\,'}{cr}\frac{\partial}{\partial t}\,\vec{j}(\vec{r}\,',t-\frac{r}{c})\right].$

   The first term describes electric dipole radiation, the second one represents electric quadrupole and magnetic dipole radiation.

   c) (**1P**) Calculate the time-dependent electric or magnetic field.

   Fail-safe point: With Γ a constant involving Q, R, c, a result can be cast into the form

   $$\vec{E}(t,\vec{r}\,) = \frac{\Gamma}{r}\,\ddot{\omega}(t-\frac{r}{c})\,\vec{e}_r \times \vec{e}_z \ .$$

   d) (**2P**) Discuss the polarisation of the radiation field emitted in the following directions:
      (i) in (very close to) the xy-plane;
      (ii) in (very close to) the positive or negative z-direction.

   e) (**3P**) Determine and sketch the angular distribution of the emitted radiation. Compare to radiation from a Hertz dipole.

   f) (**1P**) Evaluate the total radiated power at time t.

   g) (**2P**) At which frequency does the total radiated power have its maximum? Can one answer the question without knowing more about $\omega(t)$?

   h) (**1P**) Discuss and interpret your result for constant angular velocity.