Problem Sheet 8

Due date: 10th March 2009 **13:00 noon**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at *http://home.gwu.edu/˜hgrie/lectures/edyn09/edyn09.html*.

1. Another Radiation Formula (**3P**): We derived in the lecture the general solution of a radiative problem as

$$\vec{A}(\vec{r},t) = \frac{1}{c}\int \mathrm{d}^3r' \,\frac{1}{|\vec{r}-\vec{r}'|}\,\vec{j}(t-\frac{|\vec{r}-\vec{r}'|}{c},\vec{r}')$$

Show now that at large distances from the source:

$$\vec{A}(\vec{r},t) \approx \frac{1}{cr}\int \mathrm{d}^3r' \left[\vec{j}(\vec{r}',t-\frac{r}{c}) + \frac{\vec{r}\cdot\vec{r}'}{cr}\,\frac{\partial}{\partial t}\,\vec{j}(\vec{r}',t-\frac{r}{c})\right]$$

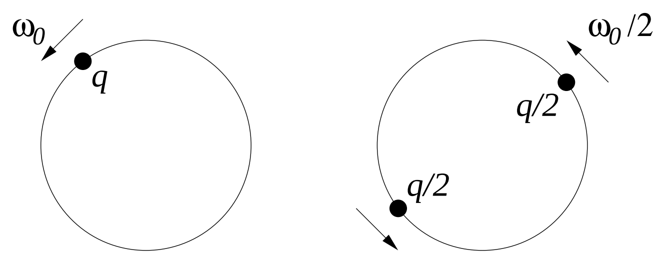
Convince yourself that the first term describes electric dipole radiation, while the second one encodes electric quadrupole and magnetic dipole radiation – no intricate details necessary. Discuss however in detail how the long-wavelength approximation must be invoked to make sense of the expansion.

2. No Radiation (**2P**): Consider a spherically symmetric charge distribution which oscillates only radially with time. Show: This system does not radiate.

Hint: One of many ways uses a symmetry argument on the electric and magnetic fields to show that the Poynting vector vanishes.

3. Atomic Dipole- and Quadrupole-Radiation, Part I (**21P**): Electric quadrupole radiation in atoms is considerably weaker than electric dipole-radiation. It can therefore be neglected – except of course if electric dipole radiation is forbidden e.g. because of selection rules. We now discuss a classical analogon to this phenomenon.

Point-charges move in the xy-plane on a circle with radius l in the mathematically positive sense with constant, non-relativistic angular velocity. The atomic nucleus rests at the centre, making the atom as a whole electrically neutral. The electric dipole is then described by one point-charge q which rotates with angular velocity ω_0. The quadrupole can be thought of as two point-charges $q/2$ which are always opposite to each other and rotate with angular velocity $\omega_0/2$; see figures. We consider only the far-zone, and the lowest non-vanishing multipole approximation.



We first solve the *dipole*, and next week the quadrupole: Concentrate now on the *left* figure.

Hint: You can of course just copy equations from textbooks. Or you use the opportunity to start from the wave-equation and derive step-by-step the solution by yourself, understanding radiation theory in this simple example. In sub-problems b), d) and f), arguments can substitute calculations.

Please turn over.

a) (**2P**) Show that the position of the charge q at time t can be written as the complex vector $\vec{R}(t) = l\,(\vec{e}_x + \mathrm{i}\vec{e}_y)\,\mathrm{e}^{-\mathrm{i}\omega_0 t}$. Its real part is the physically relevant component.

b) (**3P**) Determine the time-dependent, Cartesian dipole and quadrupole moments in the left figure. Show that your result implies that you can neglect the quadrupole radiation of the single charge.

c) (**1P**) Determine the frequency of the emitted dipole and quadrupole radiations, respectively.

d) (**2P**) Determine the retarded vector-potential $\vec{A}_{\mathrm{ret}}$ in the far-zone. A possible answer in ω-space:

$$\vec{A}^{(E1)}(\omega,\vec{r}) = \frac{\mathrm{e}^{\mathrm{i}kr}}{r}\,\frac{\omega_0}{c}\,lq\,(\vec{e}_y - \mathrm{i}\vec{e}_x)\,\delta(\omega-\omega_0)$$

e) (**1P**) Is the retarded scalar potential Φ_{ret} an independent quantity? Is knowing it imperative to determine the electric and magnetic fields of the radiation?

f) (**5P**) Determine the *time-dependent* electric and magnetic field in the far-zone. A possible answer:

$$\vec{E}^{(E1)}(t;r,\theta,\phi) = \frac{lq\omega_0^2}{c^2}\,\frac{1}{r}\,\mathrm{Re}[\exp[\mathrm{i}(kr+\phi-\omega_0 t)]\,(\vec{e}_\theta\cos\theta + \mathrm{i}\vec{e}_\phi)]$$

Hint: You might have to derive

$$\vec{e}_r \times (\vec{e}_x + \mathrm{i}\,\vec{e}_y) = \mathrm{e}^{\mathrm{i}\phi}\,(\vec{e}_\phi\,\cos\theta - \mathrm{i}\vec{e}_\theta)$$

g) (**2P**) Discuss the polarisation of the radiation, depending on the observer's position. The polar angles $\theta = 0,\ \frac{\pi}{2},\ \pi$ are particularly interesting.

h) (**5P**) Show that the time-averaged power radiated into an arbitrary solid angle is

$$\frac{\mathrm{d}P^{(E1)}}{\mathrm{d}\Omega} = \frac{q^2 l^2 \omega_0^4}{8\pi c^3}\,\left(\cos^2\vartheta + 1\right) \quad .$$

Determine also the total radiated power. Discuss and sketch the radiation characteristics (angular distribution) of this kind of dipole radiation. Compare in particular to Hertz' dipole discussed in the lecture. Could you use one or more Hertz' dipoles to get the same characteristics?

4. Radiating Hydrogen Atom (**4P**): Using the results you derived above, estimate the life-time τ of the $2p$-state in hydrogen, assuming that the system is classical and the frequency is unchanged despite of radiative losses, cf. the Bohr-Sommerfeld model of the atom. Compare to the experimentally measured value $\tau \approx 1.6 \times 10^{-9}$ s. About how many rotations does the electron make in that time-span?

   How long would it classically take for the hydrogen atom to radiate all of its ground-state energy? That's a clear signal that classical electrodynamics has to break down on atomic scales.