Problem Sheet 7

Due date: 3rd March 2009 **13:00 noon**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at *http://home.gwu.edu/˜hgrie/lectures/edyn09/edyn09.html*.

1. Pressure of Solar Radiation (**5P**): The total power radiated by the sun is 3.8×10^{26} W. A portion of that hits Earth, at a distance of $r = 1.5\times10^{11}$ m. Thus, Earth experiences a pressure. Assume that all radiation is absorbed. (radius of the Earth $R = 6000$ km, average density $\rho = 5.5$ g cm^{-3})

   The planar electro-magnetic wave

   $$\vec{E} = A\,\vec{e}_x\;\cos[kz-\omega t]\;\;,\;\;\vec{B} = A\,\vec{e}_y\;\sin[kz-\omega t]\;\;,$$

   is a solution of Maxwell's equations in vacuum if $\omega = kc$.

   a) (**1P**) Confirm for this wave explicitly the relation between Poynting vector and energy-density.
   b) (**2P**) Show that the time-averaged magnitude of the magnetic field of solar radiation close to Earth is 0.024 Gauß.
   c) (**2P**) Determine the time-averaged force on Earth. Set it in relation to the gravitational force on Earth, exerted by the sun.

2. Wave-Equation of the Field-Strength Tensor (**2P**): Show that the electric and magnetic fields obey – in the absence of charges or currents – the same wave-equation as the gauge field in the Lorentz gauge.

3. Einstein and the Light-Ray (**7P**): A mono-chromatic light-wave of frequency $\omega_{\mathbf{I}}$ and energy density $\mathcal{H}_{\mathbf{I}}$ travels along the $\vec{e}_x$-axis in an inertial frame **I**.

   a) (**4P**) Determine the frequency and energy density in another inertial frame **II** which moves at velocity $\vec{v}$ relative to **I** (usually $\vec{v} \not\propto \vec{e}_x$!).
   b) (**3P**) Young Einstein wanted to know what a light-ray would look like as one rides along it with the same speed. Answer him!

Please turn over.

4. Energy-Momentum Tensor of the Free Field (**9P**):

   a) (**4P**) Determine the energy-momentum tensor of a free, mono-chromatic electro-magnetic wave travelling in the x-direction, in terms of its energy density $\mathcal{H}$. Calculate and interpret its "trace" in Minkowski-space, $T^{\mu\nu}\, g_{\mu\nu}$.

   Hint: You can do it brute-force, but a more elegant method is: Recall the relation between energy density and energy current density for electromagnetic fields; then, show that $\vec{k}, \vec{E}, \vec{B}$ are eigenvectors of Maxwell's stress tensor; calculate their eigenvalues; and finally recall that eigenvectors to different eigenvalues are orthogonal to each other.

   b) (**3P**) Repeat for a free, mono-chromatic, real scalar field of the same energy density but mass m.

   Hint: To simplify the problem, you may consider only time-averaged quantities in part b). Do *not necessarily* apply this trick in part a) for the electro-magnetic wave, as this would make it *more* complex than necessary! To write your result in compact form, use that the group-velocity of the free massive scalar field is $v_g = \partial\omega(k)/\partial k = c^2\, k/\omega(k)$, where $k = |\vec{k}|$.

   c) (**2P**) Compare and discuss: When a massive and a massless wave hit a mirror and both carry the same energy density, which exerts the greater pressure? Why?

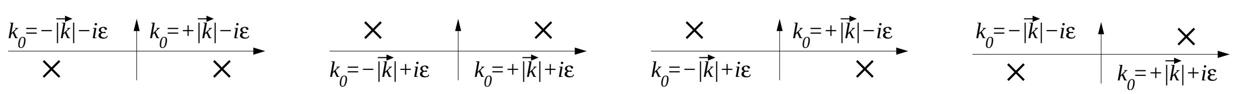
5. Advanced, Retarded and Feynman's Green's Function (**7P**): We now explore another way to construct the Green's function of the wave equation with sources, $\partial_\mu \partial^\mu G(x) = \delta^{(4)}(x)$, namely by Fourier transformation.

   a) (**1P**) Show that the Green's function is in 4-momentum space $G(k) = -\frac{1}{k^\mu k_\mu}$. Is it unique?

   To first perform the Fourier back-transformation of the time-component only, re-write

$$G(k) = \frac{1}{2|\vec{k}|}\left[\frac{1}{k_0 + |\vec{k}|} - \frac{1}{k_0 - |\vec{k}|}\right] \quad .$$

   We now have to make the k_0-integration in each term meaningful. We can do this by shifting each pole $k_0 = |\vec{k}|$, $k_0 = -|\vec{k}|$ separately away from the real axis either into the upper or lower half-plane by an infinitesimal amount, $k_0 \to k_0 \mp i\epsilon$, $\epsilon \searrow 0$. This leads to four combinations to distribute the two poles in the complex k_0-plane:



   b) (**3P**) First, consider both poles to lie in the lower half-plane. Construct the Green's function in coordinate space by Fourier-transformation. Which (advanced, retarded, other) did you obtain?

   c) (**3P**) Let's turn now to the three other cases. While you should perform the Fourier back-transformation to time to interpret the physical scenarios for these cases, you do not need to perform the momentum-space integrals. In which case are the components $\omega > 0$ transformed forward in time, while those with $\omega < 0$ are non-zero only for $t < 0$? This is Feynman's propagator, to be explored in QM II.