Problem Sheet 6

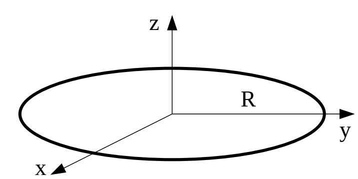
Due date: 24th February 2009 **13:00 noon**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at *http://home.gwu.edu/˜hgrie/lectures/edyn09/edyn09.html*.

1. ELECTRIC FIELD BETWEEN TWO DIPOLES (**3P**) Prove that the energy stored in the field between two dipoles $\vec{d}_1$, $\vec{d}_2$ which are separated by a distance $\vec{r}$ is

$$W_{dd} = -\frac{3(\vec{e}_r \cdot \vec{d}_1)\,(\vec{e}_r \cdot \vec{d}_2) - \vec{d}_1 \cdot \vec{d}_2}{r^3} + \frac{4\pi}{3}\,\vec{d}_1 \cdot \vec{d}_2\,\delta^{(3)}(\vec{r}) \ .$$

2. MAGNETIC FIELD OF A SUPERCONDUCTING RING (**11P**): A constant, electric current I circles in the mathematically positive sense inside a closed, infinitesimally thin, circular conductor of radius R. The ring is centered in the xy-plane at the origin, see figure. Determine in the following some characteristics of the magnetic field.



Hint: Item b) forms the basis for each of the sub-sequent questions, each of which can be treated independently of the others. You may use spherical or Cartesian coordinates as you please.

a) (**2P**) Discuss the (direction of the) magnetic field close to the ring. A calculation is not necessary, but a sketch helps.

Hint: Go back to Maxwell's equations.

b) (**3P**) Reduce the problem to a two-dimensional one by symmetry considerations. Determine the vector potential $\vec{A}$ as integral over the angle ϕ' parameterising the circumference of the ring.

c) (**1P**) Calculate from this the magnetic field on the z-axis. You can speed up the calculation enormously by first identifying its direction.

d) (**3P**) Calculate for each of the three components the leading non-vanishing term of the field $\vec{B}(x, 0, z)$ close to the origin.

e) (**2P**) Substituting the spherical expansion for $1/|\vec{r} - \vec{r}'|$ in the equation determining the vector potential, determine the leading non-vanishing contribution in the spherical multipole expansion for $\lim\limits_{r\to\infty} \vec{A}(\vec{r})$.

Please turn over.

3. CHARGED, ROTATING SPHERE (**8P**): A homogeneous sphere of radius R and total charge Q rotates about a fixed axis with constant angular velocity $\vec{\omega}$.

   a) (**1P**) Show that the current density can in Cartesian coordinates be written as

   $$\vec{j}(\vec{r}) = \sqrt{\frac{2\pi}{3}}\, r \begin{pmatrix} Y_{1,-1}(\Omega) - Y_{1,1}(\Omega) \\ \mathrm{i}(Y_{1,-1}(\Omega) + Y_{1,1}(\Omega)) \\ \sqrt{2}\, Y_{10}(\Omega) \end{pmatrix}$$

   b) (**3P**) Determine the vector potential $\vec{A}(\vec{x})$ outside the sphere. Show that it is a pure magnetic dipole field.

   c) (**2P**) Determine the magnetic dipole moment. Compute and interpret Landé's *gyromagnetic ratio* g_{L} between magnetic moment and angular momentum $\vec{l}$ of a sphere with mass M,

   $$\vec{m} = g_{\mathrm{L}}\, \frac{Q}{2Mc}\, \vec{l}\ .$$

   d) (**2P**) Determine the gyromagnetic ratio when the mass M is concentrated at the surface of the sphere, while the current is (as before) distributed homogeneously over the whole sphere. Could this serve as model for the proton or neutron?

4. PARALLEL ELECTRIC AND MAGNETIC FIELDS FROM LORENTZ BOOSTS (**7P**): Consider arbitrary electric and magnetic fields which are not perpendicular to each other at some point in a given inertial frame **I**. We will show: There exists a reference frame **II** in which the electric and magnetic fields are parallel at that point.

   Hint: The two sub-problems are independent of each other.

   a) (**3P**) Express the field magnitudes in frame **II** in terms of Lorentz invariants.

      Hint: Consider Lorentz invariants built from the field-strength tensor.

   b) (**4P**) Consider the Poynting vector of the electromagnetic field to find the velocity of frame **II** with respect to **I**. The relative velocity depends on the magnitudes of $\vec{E}$ and $\vec{B}$ and the angle between them, all given in frame **I**. Can one always find an inertial frame **II**?

      Hint: You may assume without proof that observer **II** moves perpendicular to both the electric and magnetic fields.