Problem Sheet 4

Due date: 10th February 2009 **12:00 noon**

For full credit, you should hand in a tidy and efficiently short presentation of your results and how they come about, in a manner that can be understood and reproduced by your peers. All problems and solutions are for your personal use only. Please do not pass solutions or problems on to incoming or other students who have not taken the course (yet). Noncompliance with these rules is a breach of academic integrity.
Handwritten solutions must be on 5x5 quadrille paper; electronic solutions must be in .pdf format.
I reserve the right to award zero points for any illegible, chaotic or irreproducible section of your homework.
News and .pdf-files of Problems also at *http://home.gwu.edu/˜hgrie/lectures/edyn09/edyn09.html*.

1. CARTESIAN AND SPHERICAL MULTIPOLE MOMENTS (**17P**): Three point-charges are located along a line: $q_1 = -q$ at $\vec{x}_1 = (0,0,a)$, $q_2 = -q$ at $\vec{x}_2 = (0,0,-a)$, and $q_3 = 2q$ at $\vec{x}_3 = (0,0,0)$.

   a) (**1P**) Calculate the exact solution for the potential $\Phi(\vec{x})$.

   We now consider the *Cartesian* multipole approximation of this problem.

   b) (**4P**) Determine the first three *Cartesian* multipole moments (monopole, dipoles, quadrupoles) with respect to the origin. Symmetries help to avoid some calculations!

   c) (**2P**) Construct the potential and electric field in that approximation.

   Consider now the *spherical* multipole approximation of the same problem.

   d) (**4P**) Determine *all* spherical multipole moments with respect to the origin. Symmetries!

   Fail-safe point: $Q_{lm} = \alpha\, q a^l \delta_{m0}\, [1 + (-1)^l - 2\delta_{l0}]$, with α to be determined.

   e) (**2P**) Construct the potential and electric field, also in quadrupole approximation.

   f) (**3P**) Compare your results to the solutions of c) and to the exact solution, a). From which distance on is the relative difference of the solutions less that e.g. 1%, e.g. along the positive z axis? Is you result in line with your expectation from the dimension-less expansion parameter?

   g) (**1P**) We return to problem 5 in HW 9 of the Mathematical Methods course, where the potential on the surface of a sphere with radius R

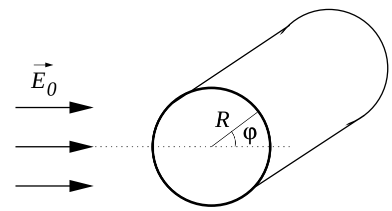
   $$\Phi(\vec{R}) = \frac{\Phi_0}{2R^2}\left(3(\vec{e}_z \cdot \vec{R})^2 - \vec{R}^2\right) \quad \text{led to} \quad \Phi(r,\theta,\phi) = \Phi_0 \left(\frac{R}{r}\right)^3 P_2(\cos\theta) \quad \text{for } r > R \ .$$

   Determine now the constant Φ_0 such that the field at large distances from the sphere is at leading non-trivial order identical to the one of the charge distribution above.

2. CHARGED SPHERE AND CLASSICAL CHARGE RADIUS OF THE ELECTRON (**3P**): Let's say the electron is a homogeneously charged sphere, $\rho(\vec{r})$ =const for $r \leq R$. Determine the size of the electron if you assume that all of its rest mass comes from the energy of its electric field. First, you need to determine at least the scalar potential inside the sphere which is generated by this charge distribution. Some possible solutions need even more information.

Please turn over.

3. CYLINDER IN AN EXTERNAL FIELD (**10P**): A uncharged, grounded, infinitely long cylinder of radius R is inserted into an orginally homogeneous electric field $\vec{E}_0$ which is directed perpendicular to the cylinder axis, see figure. Determine the modification of the electric field by constructing a useful Green's function along the following steps.



a) (**2P**) Determine the elementary solution of the Laplace equation $\Delta\Phi = 0$ in cylindrical coordinates. Show that the ansatz $\Phi(r, \phi, z) = U(r)\,\chi(\phi)$ leads to a separation of the variables and the equation:

$$\frac{r}{U(r)}\frac{\partial}{\partial r}\,r\,\frac{\partial}{\partial r}U(r) = -\frac{1}{\chi(\phi)}\frac{\partial^2}{\partial\phi^2}\chi(\phi) = \mu^2$$

Pay particular attention to the elementary solution for the value $\mu = 0$ of the separation constant. It contains a solution which is irregular at $r = 0$.

Hint: Re-visit HW 10.2 of Mathematical Methods last semester.

b) (**3P**) Determine the potential and electric field outside and inside the cylinder. Sketch both fields!

Hints: The field at infinity is $\vec{E}_0$ (why?). A partial solution is: $\Phi(r, \phi) = -|\vec{E}_0|\left(r - \frac{R^2}{r}\right)\cos\phi$

c) (**1P**) How big are Φ and $\vec{E}$ inside the cylinder?

d) (**2P**) Determine and sketch the induced surface charge density on the cylinder. What is the induced total charge?

e) (**2P**) Determine the dipole moment, per unit length, induced by the external electric field.