RESEARCH INTERESTS

My research centers on the dynamical aspects of representations of semigroups by operators on a Banach space and the connections these representations have with algebraic, topological and analytical properties of compact topological semigroups. The operators typically are derived from actions of semigroups on topological spaces or probability spaces.

This research has its origins in Bohr's work on almost periodic functions on the real line, later generalized by von Neumann, Bochner, Jacobs, Glickberg, de Leeuw, and Eberlein, among others. In 1925-26 Harald Bohr considered bounded continuous functions $f : \mathbf{R} \to \mathbf{C}$ with the following property: for each $\epsilon > 0$ there exists a number ℓ_ϵ such that every interval of length ℓ_ϵ contains a number p such that

$$|f(t+p) - f(t)| < \epsilon \ \text{ for all } t \in \mathbf{R}.$$

He called such a function *almost periodic*. Obviously, every continuous periodic function is almost periodic. In particular, the continuous characters $t \to e^{itr}$ are almost periodic, as are all linear combinations of these. For example, the function $t \to e^{it} + e^{i\sqrt{2}t}$ is almost periodic but not periodic. Bohr showed that the set $AP(\mathbf{R})$ of almost periodic functions on the real line is closed under addition, scalar multiplication, multiplication, and uniform limits, i.e., $AP(\mathbf{R})$ is a C^* algebra with identity under the sup norm.

In 1927 S. Bochner gave a functional analytic characterization of almost periodicity. For a real number t define the operator L_t on the space $C(\mathbf{R})$ of all continuous complex-valued functions on $\mathbf{R}$ by

$$L_t f(s) = f(t+s) \ \ s \in \mathbf{R}.$$

Bochner showed that a function f on $\mathbf{R}$ is almost periodic iff the *orbit* $L_{\mathbf{R}} f = \{L_t f \mid t \in \mathbf{R}\}$ of f is relatively compact in the norm topology. This led Bochner and von Neumann in 1934-5 to develop the theory of almost periodic functions on an arbitrary group. Subsequently, A. Weil (1935, 1940) and E.R van Kampen (1936) used compactifications to show that the theory of almost periodic functions on a discrete group may be reduced to the theory of continuous functions on a compact topological group.

Generalizations of the classical theory have taken two main directions. The first is the study of functions of "almost periodic type" on semigroups. The second is the study of of semigroups of operators of "almost periodic type." These are related by the notion of semigroup compactification.

To proceed we need some definitions. A semigroup S with a topology is a

- *right topological semigroup* if the mappings $t \to ts$ $(s \in S)$ are continuous,
- *left topological semigroup* if the mappings $t \to st$ $(s \in S)$ are continuous,
- *semitopological semigroup* if S is both right and left topological, and a
- *topological semigroup* if $(s, t) \to st$ is jointly continuous.

For a simple example, take X be a topological space and X^X the space of all functions on $X \to X$ with the product topology (pointwise convergence). Then X^X is a right topological semigroup with respect to function composition. The subsemigroup of all continuous functions is a semitopological semigroup, and if X is a uniform space with the topology of uniform convergence then any subsemigroup that is an equicontinuous family is a topological semigroup.

Let $C(S)$ denote the space of all bounded, continuous, complex-valued functions on a semitopological semigroup S. Functions on S of a particular almost periodic type typically form a C^*-subalgebra F of $C(S)$ with the property that the spectrum (maximal ideal space) S^F of F may be given a topology and a multiplication relative to which S^F is a compact right topological semigroup such that the evaluation mapping $S :\mapsto S^F$ is a continuous homomorphism with dense range. The semigroup S^F is called the *F-compactification of* S. Each F-compactification of S is universal with respect to some property $\mathcal{P}$ in the sense that if $\pi : S \mapsto X$ is a continuous homomorphism into compact semigroup with property $\mathcal{P}$, then there exists a continuous homomorphism $\overline{\pi} : S^F \mapsto X$ such that the following diagram commutes:

$$\begin{array}{ccc} S^F & \xrightarrow{\overline{\pi}} & X \\ \big\uparrow & & \big\uparrow \pi \\ S & = & S \end{array}$$

Here are some examples (R_s and L_s denote the right and left translation operators on $C(S)$):

- **Weakly almost periodic functions.** $WAP(S) = \{f : R_S f$ is rel. wkly compact $\}$. S^{WAP} is the universal semitopological semigroup compactification of S.
- **Almost periodic functions.** $AP(S) = \{f : R_S f$ is rel. norm compact $\}$. S^{AP} is the universal semitopological semigroup compactification of S.
- **Strongly almost periodic functions.** $F = SAP(S) =$ algebra generated by the finite dimensional unitary subspaces of $C(S)$. S^{SAP} is the universal topological group compactification of S.
- **Distal functions.** $F = D(S) =$ algebra generated by the distal functions in $C(S)$. S^D is the universal right topological group compactification of S.

For some concrete examples, let G is be a locally compact non compact topological group. Then $C_0(G) \subset WAP(G)\backslash AP(G)$. If ds is left Haar measure and $1 < p, q < \infty$ are conjugate exponents then the function $t \to \int_S g(t^{-1}s)h(s)\,ds$ is weakly almost periodic for each $g \in L^p(G)$ and $h \in L^q(G)$. Coefficients of finite dimensional unitary representations are stongly almost periodic (e.g., characters); Coefficients of unitary representations are wap (positive definite functions).

A class of compactification problems that has interested me is the following: Given a collection of semitopological semigroups combined in some way to form a semitopological semigroup S, (e.g., direct products, semidirect products, Zappa products, inductive limits, projective limits) express a compactification of S in terms of compactifications of the component semigroups. For example Glicksberg and deLeeuw proved that if S and T are semitopological semigroups with identities then $(S \times T)^{AP} = S^{AP} \times T^{AP}$. Some of my papers concern generalizations of this result.

The Jacobs-Glicksberg-deLeeuw theory of weakly almost periodic operators is the second direction the classical theory of almost periodic functions has taken. Let S be a semitopological semigroup and $s \mapsto T_s$ a representation of S by uniformly bounded operators on $\mathcal{X}$. A vector $f \in \mathcal{X}$ is said to be (*weakly*) *almost periodic* if the set $T_S f := \{T_s f : s \in S\}$ is relatively compact in the norm (weak) topology of $\mathcal{X}$. The collection of (weakly) almost periodic vectors is denoted by $\mathcal{X}_{ap}$ ($\mathcal{X}_{wap}$.) A T_S-invariant subspace $\mathcal{U}$ of $\mathcal{X}$ is *unitary* if the semigroup $T_S|_{\mathcal{U}}$ is contained in a uniformly bounded group of operators on $\mathcal{U}$ with identity the identity operator. The space $\mathcal{X}_{sap}$ of *strongly almost periodic vectors* is defined as the closed linear subspace of $\mathcal{X}$ generated by the finite dimensional unitary subspaces. The spaces $\mathcal{X}_{ap}$, $\mathcal{X}_{wap}$, and $\mathcal{X}_{sap}$ are T_S-invariant closed linear subspaces of $\mathcal{X}$ with $\mathcal{X}_{sap} \subseteq \mathcal{X}_{ap} \subseteq \mathcal{X}_{wap}$. The representation T is said to be (*weakly*) *almost periodic* if $\mathcal{X} = \mathcal{X}_{ap}$ ($\mathcal{X} = \mathcal{X}_{wap}$). The set of *flight vectors* is defined as $\mathcal{X}_0 := \{f \in \mathcal{X} \mid 0 \in \overline{T_S f}^w\}$. For example, the right translation representation $s \mapsto R_s$ is weakly almost periodic on $WAP(S)$ and almost periodic on $AP(S)$. If T is weakly almost periodic then the closure $\overline{T_S}^w$ of T_S in the weak operator topology of the space of bounded linear operators on $\mathcal{X}$ is a compact semitopological semigroup under composition. If $s \mapsto T_s$ is continuous in the weak operator topology, then the representation lifts to a weakly continuous representation $x \mapsto T_x : S^{wap} \mapsto \overline{T_S}^w$ such that the following diagram commutes:

$$\begin{array}{ccc} S^{wap} & \xrightarrow{T} & \overline{T_S}^w \\ \uparrow & & \uparrow T \\ S & = & S \end{array}$$

The structure theory of compact semitopological semigroups may then be used to study the dynamical properties of the representation T. For example, if $WAP(S)$ has an invariant mean then the representation is asymptotically periodic in the sense that for each $x \in \mathcal{X}$ there exists a unique vector $x_m \in \mathcal{X}_{sap}$ and a net $T_{s_\alpha} x$ asymptotically converges to the vector x_m (which is periodic in some sense):

$$T_{s_\alpha} x - T_{s_\alpha} x_m \to 0.$$

We can now consider representations of subsemigroups, direct products, semidirect products, Zappa products, group extensions, projective limits, etc. Lifting these representations to compactifications of these structures gives information about the representations in terms of component representations. Some of my research considers these issues.