DEPENDENCE RELATIONS IN COMPUTABLY RIGID COMPUTABLE VECTOR SPACES

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ABSTRACT. We construct a computable vector space with the trivial computable automorphism group, but with the dependence relations as complicated as possible, measured by their Turing degrees. As a corollary, we answer a question asked by A. Morozov in [7].

The authors are very thankful to the referee for important suggestions. The last two authors also acknowledge support of the NSF binational grant DMS-0075899.

1. INTRODUCTION

Metakides and Nerode [5] showed that the study of effective vector spaces can be reduced to the study of $\mathcal{V}_\infty$, an $\aleph_0$-dimensional vector space over a computable field F, consisting of all finitely nonzero ω-sequences of elements of F, under pointwise operations. Clearly, these operations can be performed algorithmically. Every element α in F can be identified with its Gödel code $\#\alpha$, which is a natural number. A standard basis E for $\mathcal{V}_\infty$ consists of vectors $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$, where ε_i, $i \geq 0$, is the ω-sequence with the ith term 1 and all other terms 0. The support of v with respect to E, denoted by $supp(v)$, is the set $\{\varepsilon_{i_j} : j \in \{0, \ldots, t\}\}$, where

$$v = \sum_{j=0}^{t} \lambda_j \varepsilon_{i_j}$$

and $(\forall j \leq t)[\lambda_j \in F - \{0\}]$. Every vector v in V_∞ can be identified with its Gödel code $\#v$. We will assume that the coding of vectors is such that if $\varepsilon_i \in supp(v)$, then $\#v > i$.

A subspace $\mathcal{V}$ of $\mathcal{V}_\infty$ is *computable* if its domain V is a computable subset of V_∞. A subspace $\mathcal{V}$ of $\mathcal{V}_\infty$ is *computably enumerable* (c.e.) if its domain V is a c.e. subset of V_∞. We also say that a quotient space $\frac{\mathcal{V}_\infty}{\mathcal{V}}$ is computable (c.e., respectively) if $\mathcal{V}$ is a computable (c.e., respectively) space. Metakides and Nerode [5] showed that every c.e. vector space is computably isomorphic to $\frac{\mathcal{V}_\infty}{\mathcal{V}}$, where $\mathcal{V}$ is a c.e. subspace of $\mathcal{V}_\infty$.

For a vector space $\mathcal{V}$, its automorphism group, $Aut(\mathcal{V})$, consists of all automorphisms of $\mathcal{V}$. For a computable vector space $\mathcal{V}$, its *computable automorphism group, $Aut_c(\mathcal{V})$*, consists of all computable automorphisms of $\mathcal{V}$. An automorphism f of a vector space $\mathcal{V}$ is *trivial* if it maps every 1-dimensional subspace of $\mathcal{V}$ into itself. Such f also maps every subspace of $\mathcal{V}$ into itself. We will show that if f is a trivial automorphism of $\mathcal{V}$, then $f = f_\alpha$ for some $\alpha \in F - \{0\}$, where

$$(\forall v \in V)[f_\alpha(v) = \alpha v].$$

A computable vector space is called *computably rigid* if its computable automorphism group is trivial, that is, consists of only trivial automorphisms. Morozov [7] constructed a computable vector space $\mathcal{V}$ such that $\frac{\mathcal{V}_\infty}{\mathcal{V}}$ is computably rigid.

Let $S \subseteq V_\infty$. By S^* we denote the linear span of S, the set of all linear combinations of vectors in S. The structure $\mathcal{S}^*$ is the smallest (with respect to $\subseteq$) vector subspace of $\mathcal{V}_\infty$ whose universe contains the set S. For vector spaces $\mathcal{V}$ and $\mathcal{W}$, we define the domain of $\mathcal{V} \oplus \mathcal{W}$ to be $V \oplus W =_{def} (V \cup W)^*$. The structure of all c.e. subspaces of $\mathcal{V}_\infty$ is denoted by $\mathcal{L}(\mathcal{V}_\infty)$. It is a modular lattice under $\cap$ and $\oplus$. A space $\mathcal{V} \in \mathcal{L}(\mathcal{V}_\infty)$ is complemented if there exists $\mathcal{W} \in \mathcal{L}(\mathcal{V}_\infty)$ such that $V \cap W = \{0\}$ and $V \oplus W = V_\infty$.

A *dependence algorithm* for a vector space decides whether any finite set of its vectors is linearly dependent. For example, the space $\mathcal{V}_\infty$ has a dependence algorithm. Moreover, a c.e. vector space has a dependence algorithm iff it has a c.e. basis. Mal'tsev [4] established that there is a computable isomorphic copy of $\mathcal{V}_\infty$, which does not have a dependence algorithm. For $k \geq 1$, a *k-dependence algorithm* for a space decides whether any k-element set of its vectors is linearly dependent. Lytkina [3] constructed infinitely many nonisomorphic computable copies of $\mathcal{V}_\infty$, which have k-dependence algorithms, but do not have $(k+1)$-dependence algorithms.

For every $n \geq 2$, we let $\langle x_0, \ldots, x_{n-1} \rangle$ be a computable bijection from ω^n onto ω, which is strictly increasing with respect to each coordinate, and such that for every $j < n$, $x_j \leq \langle x_0, \ldots, x_{n-1} \rangle$. Let $\mathcal{V} \in \mathcal{L}(\mathcal{V}_\infty)$. For $k \geq 1$, the k-dependence relation *modulo* V $(\operatorname{mod} V)$, in symbols $D_k(V)$, is defined to be

$$D_k(V) = \{\langle v_0, \ldots, v_{k-1} \rangle : v_0, \ldots, v_{k-1} \text{ are dependent over } V\}.$$

Clearly, if $\mathcal{V} \in \mathcal{L}(\mathcal{V}_\infty)$, then the set $D_k(V)$ is c.e. A k-dependence algorithm $\operatorname{mod} V$ decides whether any k-element set of vectors in V_∞

is linearly dependent over V. Obviously, there is a k-dependence algorithm mod V iff $\frac{\mathcal{V}_\infty}{\mathcal{V}}$ has a k-dependence algorithm. Let

$$D(V) =_{def} \bigcup_{k\geq 1} D_k(V).$$

A space $\mathcal{V} \in \mathcal{L}(\mathcal{V}_\infty)$ is complemented iff $D(V)$ is computable.

By $\leq_T$ we denote Turing reducibility of sets, and by $\equiv_T$ Turing equivalence of sets. The Turing degree of a set X is denoted by $deg(X)$. The dependence degree mod V is $deg(D(V))$. The k-dependence degree mod V is the Turing degree of $D_k(V)$. Clearly, $D_1(V) = V \leq_T D(V)$. If $\mathcal{V}$ is a vector space over a finite field, then we have $V \equiv_T D(V)$. Clearly,

$$(D_k(V) \leq_T D(V)) \wedge (D_k(V) \leq_T D_{k+1}(V)),$$

uniformly in k. Shore established that, except for this necessary condition on a sequence of c.e. degrees, the dependence degrees *modulo* a c.e. subspace can be made arbitrary.

Theorem 1.1. ([9]) *Let the space $\mathcal{V}_\infty$ be over an infinite computable field. Assume that $A_1, A_2, A_3, \ldots, A_0$ is a (simultaneously) c.e. sequence of c.e. sets such that for $k \geq 1$, $A_k \leq_T A_0$ and $A_k \leq_T A_{k+1}$, uniformly in k. Then there is a c.e. subspace $\mathcal{V}$ of $\mathcal{V}_\infty$ such that for $k \geq 1$,*

$$(D_k(V) \equiv_T A_k) \wedge (D(V) \equiv_T A_0).$$

The proof of Theorem 1.1 is based on the following key combinatorial lemma.

Lemma 1.2. ([9]) *Let $k \in \omega$ and $v_0, \ldots, v_k \in V_\infty$. Assume that $\mathcal{V}$ is a finite dimensional vector subspace of $\mathcal{V}_\infty$, and that $v_0, \ldots, v_k$ are linearly independent over V. Let X be a finite set (of codes) of sequences of vectors in V_∞ of length $\leq k$ such that*

$$X \cap D(V) = \emptyset.$$

Then there are $\lambda_0, \ldots, \lambda_k \in F$ such that

$$X \cap D((V \cup \{\lambda_0 v_0 + \ldots + \lambda_k v_k\})^*) = \emptyset.$$

In [6], Metakides and Nerode generalized Theorem 1.1 to any regular computable infinite dimensional Steinitz closure system. A Steinitz closure system is *regular* if it does not have a finite dimensional closed subset that is the union of a finite number of its proper closed subsets. Nerode and Remmel also established the following strengthening of Theorem 1.1. A c.e. subspace $\mathcal{V}$ of $\mathcal{V}_\infty$ is called *supermaximal* if $\frac{\mathcal{V}_\infty}{\mathcal{V}}$ is infinite dimensional, and for every c.e. space $\mathcal{W}$ with $\mathcal{V} \subseteq \mathcal{W} \subseteq \mathcal{V}_\infty$, we have that $\mathcal{V}_\infty = \mathcal{W}$ or $\frac{\mathcal{W}}{\mathcal{V}}$ is finite dimensional.

Theorem 1.3. ([8]) *Let the space $\mathcal{V}_\infty$ be over an infinite computable field. Assume that $A_1, A_2, A_3, \dots, A_0$ is a c.e. sequence of c.e. sets such that A_0 is not computable and for $k \geq 1$, $A_k \leq_T A_{k+1}$ and $A_k \leq_T A_0$, uniformly in k. Then there are supermaximal subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ of $\mathcal{V}_\infty$ such that for $k \geq 1$,*

$$D_k(V_1) \equiv_T D_k(V_2) \equiv_T A_k,$$
$$D(V_1) \equiv_T D(V_2) \equiv_T A_0,$$

and there is no automorphism Φ of the lattice $\mathcal{L}(\mathcal{V}_\infty)$ such that $\Phi(V_1) = V_2$.

For $u, v \in V_\infty$, we write $u =_{\mathrm{mod}\, V} v$ if $\frac{u}{V} = \frac{v}{V}$. To simplify the notation, for $s \in \omega$, $v \in V_\infty$, and $\alpha \in F$, we will also write $v \leq s$ and $\alpha \leq s$ instead of $\#v \leq s$ and $\#\alpha \leq s$. In the text that follows, computability theoretic concepts and notation are as in [10]. Let $\varphi_0, \varphi_1, \dots$ be a standard computable enumeration of all partial computable functions. For $u, v \in V_\infty$, we write $\varphi_{e,s}(u) = v$ if $u, v < s$, and $\varphi_e(u) = v$ in fewer than s steps. For a partial function φ we write $\varphi(x)\downarrow$ to denote that $x \in dom(\varphi)$. For a set $Y \subseteq \omega$ and $c \in \omega$, let $Y \upharpoonright c =_{def} Y \cap \{0, \dots, c-1\}$. The cardinality of Y is denoted by $|Y|$.

2. Computable Automorphisms and Dependence Degrees

We will assume that the computable field F is infinite. In [7], Morozov asked whether it is possible to obtain for every $k \geq 2$ a computable vector space $\mathcal{V}$ such that $\frac{\mathcal{V}_\infty}{\mathcal{V}}$ is computably rigid, has the k-dependence algorithm $\mathrm{mod}\, V$, does not have the $(k+1)$-dependence algorithm $\mathrm{mod}\, V$, and its dependence algorithm $\mathrm{mod}\, V$ has an arbitrary nonzero c.e. Turing degree. We answer this question positively by establishing a general result. Clearly, if $deg(D(V)) = \mathbf{0}$, then $\frac{\mathcal{V}_\infty}{\mathcal{V}}$ has a computable basis and, hence, the computable automorphism group of $\frac{\mathcal{V}_\infty}{\mathcal{V}}$ is nontrivial. First, we establish the following lemma for the nontrivial automorphisms of vector spaces.

Lemma 2.1. *Let φ be a total function such that $\varphi : V_\infty \to V_\infty$. If φ does not induce a trivial automorphism of $\frac{V_\infty}{V}$, then one of the following conditions hold:*

(1) *There exist $u, v \in V_\infty$ and $\alpha, \beta \in F$ such that*

$$\varphi(\alpha u + \beta v) \neq_{\mathrm{mod}\, V} \alpha\varphi(u) + \beta\varphi(v),$$

(2) *There exists $w \in V_\infty - V$ such that $\varphi(w) \in V$,*

(3) *There exists $w \in V_\infty - V$ such that the set $\{w, \varphi(w)\}$ is independent* $\mathrm{mod}\, V$.

Proof. If (1) holds, then φ does not induce a linear transformation of $\frac{V_\infty}{V}$. If (2) holds, then φ does not induce a $1-1$ linear transformation of $\frac{V_\infty}{V}$. We will prove that if φ induces an automorphism of $\frac{V_\infty}{V}$ that does not satisfy (3), then

$$(\exists \lambda \in F - \{0\})(\forall w \in V_\infty)[w =_{\operatorname{mod} V} \lambda\varphi(w)].$$

Obviously, the negation of (3) implies that for every $w \in V_\infty - V$, we have $w =_{\operatorname{mod} V} \lambda_w \varphi(w)$ for some $\lambda_w \in F - \{0\}$. We will prove that for every $w_1, w_2 \notin V$, we have $\lambda_{w_2} = \lambda_{w_1}$. Consider the following two cases.

Case 1. Assume that w_1 and w_2 are independent $\operatorname{mod} V$. Then

$$\begin{aligned} w_1 + w_2 &=_{\operatorname{mod} V} \lambda_{w_1+w_2}\varphi(w_1+w_2) = \\ &=_{\operatorname{mod} V} \lambda_{w_1+w_2}[\varphi(w_1)+\varphi(w_2)] \\ &=_{\operatorname{mod} V} \frac{\lambda_{w_1+w_2}}{\lambda_{w_1}} w_1 + \frac{\lambda_{w_1+w_2}}{\lambda_{w_2}} w_2. \end{aligned}$$

Therefore, $\lambda_{w_1+w_2} = \lambda_{w_1} = \lambda_{w_2}$.

Case 2. Assume that $w_1 =_{\operatorname{mod} V} \mu w_2$ for some $\mu \in F - \{0\}$. Then

$$\begin{aligned} w_1 &=_{\operatorname{mod} V} \lambda_{w_1}\varphi(w_1) \\ &=_{\operatorname{mod} V} \lambda_{w_1}[\varphi(\mu w_2)] =_{\operatorname{mod} V} \lambda_{w_1}\mu\varphi(w_2) \\ &=_{\operatorname{mod} V} \frac{\lambda_{w_1}\mu}{\lambda_{w_2}} w_2 =_{\operatorname{mod} V} \frac{\lambda_{w_1}}{\lambda_{w_2}} w_1. \end{aligned}$$

Therefore, $\lambda_{w_1} = \lambda_{w_2}$. □

We now state and prove our main result. Recall that $\{\varepsilon_0, \varepsilon_1, \varepsilon_2, ...\}$ is a standard basis for $\mathcal{V}_\infty$.

Theorem 2.2. *Let A_0 be a noncomputable c.e. set, and let $A_1, A_2, A_3, \ldots$ be a c.e. sequence of c.e. sets such that A_1 is computable, and*

$$A_1 \leq_T A_2 \leq_T \ldots \leq_T A_k \leq_T A_{k+1} \leq_T \ldots \leq_T A_0,$$

uniformly in k. Then there is a computable subspace $\mathcal{V}$ of $\mathcal{V}_\infty$ such that $\frac{\mathcal{V}_\infty}{\mathcal{V}}$ is computably rigid, $D_k(V) \equiv_T A_k$ for $k \geq 1$, and $D(V) \equiv_T A_0$.

Proof. Let $k \geq 2$. Fix a computable enumeration $(A_{k,t})_{t\in\omega}$ of A_k such that $A_{k,0} = \emptyset$ and for every $t \in \omega$, $|A_{k,t+1} - A_{k,t}| = 1$. At every stage s of the construction, we will have a finite independent set B^s and an (infinite) independent set $E^s = \{\varepsilon_0^s, \varepsilon_1^s, \ldots\}$ such that:

(a) $B^s \cup E^s$ is a basis for $\mathcal{V}_\infty$,

(b) for all i, $\varepsilon_i^s = \varepsilon_{l(i,s)}$, where $l(0,s) < l(1,s) < ...$.

Let $V^s =_{def} (B^s)^*$. The domain of the desired space $\mathcal{V}$ will be $V =_{def} \bigcup\limits_{s\in\omega} V^s$.

Let s be any stage of the construction. For every $x \in V_\infty - V^s$, we define $supp(x, s)$ to be the index set J in the unique representation of x as

$$x = v + \sum_{j\in J} \lambda_j \varepsilon_j^s,$$

where $v \in V^s$, and $\lambda_j \in F - \{0\}$ for $j \in J$. We then let

$$\max(x, s) =_{def} \max\{j : \varepsilon_j^s \in supp(x, s)\}.$$

(If $x \in V^s$, then we set $supp(x, s) = \emptyset$ and $\max(x, s) = -1$.)

In the construction, we will either have $B^{s+1} = B^s$, or $B^{s+1} = B^s \cup \{x\}$ for some $x \notin V^s$. In the latter case, we set $\varepsilon_i^{s+1} = \varepsilon_i^s$ for $i < \max(x, s)$, and $\varepsilon_i^{s+1} = \varepsilon_{i+1}^s$ for $i \geq \max(x, s)$. That is, when at stage $s+1$ we enumerate x into B, we drop $\varepsilon^s_{\max(x,s)}$ from the complementary basis. This will ensure that $\max(x, s) \geq \max(x, s + 1)$.

If $\mathcal{V}$ is computable, then it will follow that $D_1(\mathcal{V}) \equiv_T A_1$. We will also satisfy the following coding requirements:

Q_k: Encode A_k into $D_k(V)$ for $k \geq 2$, and

Q_0: Encode A_0 into $D(V)$.

The coding requirements will be allowed to act at stages $s + 1$ of the form $s + 1 = \langle 2k - 1, t\rangle$. We next specify finite sets of vectors that we will use to satisfy Q_k for $k \geq 2$. For $i \in \omega$ and $k \geq 2$, we define a finite set $C_{k,i}$ of (standard) basis elements by

$$C_{k,i} = \{\varepsilon_{\langle 2k-1,ki\rangle}, \dots, \varepsilon_{\langle 2k-1,ki+k-1\rangle}\}.$$

We will use the elements from $C_{k,i}$ to encode into $D_k(V)$ whether $i \in A_k$ for $k \geq 2$. Note that for every i, $|C_{k,i}| = k$. For a stage s and $k \geq 2$, we let $X(k, s)$ denote the set of all $z = \langle x_1, \dots, x_j\rangle$ such that $z \leq s$, $1 \leq j < k$ and $z \notin D(V^s)$. Note that $X(k, s) \supseteq \{z \leq s : z \notin V^s\}$. By Lemma 1.2, there is a vector x, which is a linear combination of the vectors from $C_{k,i}$, such that

$$D((V^s \cup \{x\})^*) \cap X(k, s) = \emptyset.$$

We will enumerate the least such x into B_{s+1}.

Only at stages of the form $s + 1 = \langle 2k - 1, \cdot\rangle$, there might be some $x \in B_{s+1} - B_s$ with $supp(x, s) \subset \{\varepsilon_{\langle 2k-1,z\rangle} : z \in \omega\}$. Thus, we will have for every $i \in \omega$,

$$i \in A_k \ \Leftrightarrow\ \left\langle \varepsilon_{\langle 2k-1,ki\rangle}, \dots, \varepsilon_{\langle 2k-1,ki+k-1\rangle} \right\rangle \in D_k(V).$$

Hence, $A_k \leq_T D_k(V)$. Moreover, there will be only finitely many stages $s+1$, other than stages of the form $s+1 = \langle 2j-1, \cdot\rangle$ for $j < k$, such that $D(V^{s+1}) \cap X(k,s) \neq \emptyset$. Since for $j < k$, $A_j \leq_T A_k$, we will be able to establish that $D_k(V) \leq_T A_k$.

We will encode A_0 into $D_0(V)$ at stages of the form $\langle 1, t\rangle$, $t \in \omega$, using the (standard) basis elements of the form $\varepsilon_{\langle 1,z\rangle}$. We assume that $0 \notin A_0$. For $i \geq 1$, we define

$$C_{0,i} = \{\varepsilon_{\left\langle 1, \binom{i+1}{2}\right\rangle}, \dots, \varepsilon_{\left\langle 1, \binom{i+2}{2}-1\right\rangle}\}.$$

Note that $|C_{0,i}| = i+1$. The construction will guarantee that

$$i \in A_0 \Leftrightarrow \left\langle \varepsilon_{\left\langle 1, \binom{i+1}{2}\right\rangle}, \dots, \varepsilon_{\left\langle 1, \binom{i+2}{2}-1\right\rangle} \right\rangle \in D(V).$$

Hence, $A_0 \leq D(V)$. To obtain that $D(V) \leq_T A_0$, we will use permitting by A_0.

To ensure that V_∞/V is computably rigid, we will meet the following requirements for every $e \in \omega$,

R_e: If φ_e induces an automorphism of $\frac{V_\infty}{V}$, then φ_e is trivial.

The requirement R_e may act at stages of the form $\langle 2e, t\rangle$, and only if it is permitted by A_0. Moreover, if R_e acts at stage $s+1$, we will ensure that

$$D(V^{s+1}) \cap X(e+2, s) = \emptyset.$$

Hence, the action of R_e cannot affect $D_1(V), \dots, D_{e+1}(V)$. Every R_e can act only finitely many times. When considering the requirement R_e, we will check whether one of the following three conditions is satisfied:

(i) there exist $u, v \in V_\infty$ and $\alpha, \beta \in F$ such that $\varphi_e(\alpha u + \beta v) \neq_{\operatorname{mod} V} \alpha\varphi_e(u) + \beta\varphi_e(v)$,

(ii) there exists $w \notin V$ such that $\varphi_e(w) \in V$, or

(iii) there exists $w \notin V$ such that the set $\{w, \varphi_e(w)\}$ is independent $\operatorname{mod} V$.

If either (i) or (ii) happens, then we will try to preserve it by placing an appropriate marker. If (iii) occurs at stage $s+1$, then we will look for additional vectors $w_1, w_2, \dots, w_{e+1}$ with specific properties, which will guarantee that for any u that is a linear combination of w and $w_1, w_2, \dots, w_{e+1}$, the set $\{u, \varphi_e(u)\}$ will be independent $\operatorname{mod} V^s$. Since $w, w_1, w_2, \dots, w_{e+1}$ will be independent $\operatorname{mod} V^s$, by Lemma 1.2, we will find such u also satisfying

$$D((V^s \cup \{u\})^*) \cap X(e+2, s) = \emptyset,$$

and enumerate it into B^{s+1}. In addition, the vector u will be selected such that the higher priority R-requirements are preserved. If φ induces a linear transformation of $\frac{V_\infty}{V}$, we will have the marker Γ_e on $\varphi_e(u)$, and keep $\varphi_e(u)$ out of V with priority e. No coding requirement will be allowed to injure R_e. We will prove that if some R_e is not satisfied, then A_0 is computable, contrary to the assumption.

Construction

Stage 0. Let $B^0 = \emptyset$ and $\varepsilon_i^0 = \varepsilon_i$ for all $i \geq 0$. There are no vectors marked by Γ_n or $\tilde{\Gamma}_n$, for any $n \in \omega$.

Stage $s+1 = \langle 2k-1, t\rangle$ for some $k \geq 2$ and $t \in \omega$.

Suppose $i \in A_{k,t+1} - A_{k,t}$. By Lemma 1.2, there is a linear combination x of the vectors from $C_{k,i}$ such that

$$D((V^s \cup \{x\})^*) \cap$$
$$[X(k,s) \cup \{y \;:\; y \text{ is marked at } s \text{ by some } \Gamma_n \text{ or } \tilde{\Gamma}_n\}] = \emptyset.$$

We let $B^{s+1} = B^s \cup \{x\}$ for the least such x, and drop $\varepsilon^s_{\max(x,s)}$ from the complementary basis E^s.

Stage $s+1 = \langle 1, t\rangle$ for some $t \in \omega$.
Suppose $i \in A_{0,t+1} - A_{0,t}$. By Lemma 1.2, there is a linear combination x of the vectors from $C_{0,i}$ such that

$$D((V^s \cup \{x\})^*) \cap$$
$$[X(i+1,s) \cup \{y \;:\; y \text{ is marked at } s \text{ by some } \Gamma_n \text{ or } \tilde{\Gamma}_n\}] = \emptyset.$$

We let $B^{s+1} = B^s \cup \{x\}$ for the least such x, and drop $\varepsilon^s_{\max(x,s)}$ from the complementary basis E^s.

Stage $s+1 = \langle 2e, t\rangle$ for some $t \in \omega$. Assume that the marker $\tilde{\Gamma}_e$ is not placed on any vector. Otherwise, go to the next stage.

(i) Check if there exist $u, v \in V_\infty$ and $\alpha, \beta \in F$ such that:

$$u, v, \alpha, \beta \leq s,$$
$$\varphi_{e,s+1}(u)\downarrow,\ \varphi_{e,s+1}(v)\downarrow, \text{ and } \varphi_{e,s+1}(\alpha u + \beta v)\downarrow,$$
$$\varphi_{e,s+1}(\alpha u + \beta v) \neq_{\mathrm{mod}\, V^s} \alpha\varphi_{e,s+1}(u) + \beta\varphi_{e,s+1}(v).$$

If there are such u, v, α, β, then find the least such sequence, and put a $\tilde{\Gamma}_e$ marker on $\alpha\varphi_{e,s+1}(u) + \beta\varphi_{e,s+1}(v) - \varphi_{e,s+1}(\alpha u + \beta v)$. We say that R_e acts. Go to the next stage. If there are no such u, v, α, β, go to (ii).

(ii) Check whether there exists $w \leq s$ such that $w \notin V^s$, $\varphi_{e,s+1}(w)\downarrow$, and $\varphi_{e,s+1}(w) \in V^s$. If there is w, then put a $\tilde{\Gamma}_e$ marker on the least such w and go to the next stage. We say that R_e acts. If there is no such w, and if a Γ_e marker is not placed on any vector, go to (iii). Otherwise, go to the next stage.

(iii) Let

$$F_e^s =_{def} \{\varepsilon_{\langle 2e,t\rangle} : t \geq 0\} \cap E^s.$$

Check if there exists a vector $w_0 \leq s$ such that:

($\mathcal{C}_0$) $w_0 \notin V^s$, $\varphi_{e,s+1}(w_0)\downarrow$, and the set $\{w_0, \varphi_{e,s+1}(w_0)\}$ is independent mod V^s.

If there is no such w_0, go to the next stage. Otherwise, choose the least one. Check if there exist additional vectors $w_1, w_2, \ldots, w_{e+1} \leq s$ such that for every $i \in \{0, \ldots, e\}$, the following conditions are satisfied for $m_i^s =_{def} \max\{\max(w_i, s), \max(\varphi_{e,s+1}(w_i), s)\}$:

($\mathcal{C}_1$) $supp(w_{i+1}, s) \subset F_e^s$, and $\varphi_{e,s+1}(w_{i+1})\downarrow$,

($\mathcal{C}_2$) $supp(w_{i+1}, s) \cap \{\varepsilon_0^s, \ldots, \varepsilon_{m_i^s}^s\} = \emptyset$,

($\mathcal{C}_3$) $supp(\varphi_{e,s+1}(w_{i+1}), s) \cap \{\varepsilon_0^s, \ldots, \varepsilon_{m_i^s}^s\} = \emptyset$, and

($\mathcal{C}_4$) $\max(w_{e+1}, s) > j$, where $j \in A_{0,s+1} - A_{0,s}$.

If a sequence $w_1, \ldots, w_{e+1}$ does not exist, go to the next stage. Otherwise, find the least such sequence. By Lemma 1.2, there exist $\lambda_0, \lambda_1, \ldots, \lambda_{e+1} \in F - \{0\}$ such that for $u = \sum_{j=0}^{e+1} \lambda_j w_j$, we have

$$D((V^s \cup \{u\})^*) \cap$$
$$[X(e+2, s) \cup \{y \ : \ (\exists n < e)(y \text{ is marked at } s \text{ by } \Gamma_n \text{ or } \tilde{\Gamma}_n)\}] = \emptyset.$$

In this case, we will:

(a) enumerate the least such u into B^{s+1},

(b) drop $\varepsilon_{\max(u,s)}^s$ from the complementary basis E^s,

(c) put a Γ_e marker on $\sum_{j=0}^{e+1} \lambda_j \varphi_{e,s+1}(w_j)$, and

(d) remove all Γ_n and $\tilde{\Gamma}_n$ markers for $n > e$.

We say that R_e acts. Go to the next stage.

If the vectors $w_0, w_1, \ldots, w_{e+1}$ satisfy only ($\mathcal{C}_0$), ($\mathcal{C}_1$), ($\mathcal{C}_2$) and ($\mathcal{C}_3$), then we will say that R_e can possibly act at stage $s+1$ *via* the sequence $w_0, w_1, \ldots, w_{e+1}$ (if permitted by A_0).

End of the Construction.

Lemma 2.3. *The space $\mathcal{V}$ is computable.*

Proof. Let $v \in V_\infty$. It follows from the construction that if a stage s is such that $s \geq v$, then $(v \in V \Leftrightarrow v \in V^s)$. □

Lemma 2.4. *For $k \geq 2$, we have $A_k \leq_T D_k(V)$.*

Proof. Let $k \geq 2$. It is enough to prove that for every $i \in \omega$,

$$i \in A_k \Leftrightarrow \left\langle \varepsilon_{\langle 2k-1,ki\rangle}, \dots, \varepsilon_{\langle 2k-1,ki+k-1\rangle} \right\rangle \in D_k(V).$$

If $i \in A_k$, then $i \in A_{k,t+1} - A_{k,t}$ for some t. By the construction at the stage $\langle 2k-1, t\rangle$, we have that $\left\langle \varepsilon_{\langle 2k-1,ki\rangle}, \dots, \varepsilon_{\langle 2k-1,ki+k-1\rangle} \right\rangle \in D_k(V)$.

To prove the converse, notice that our construction is such that a vector u with $supp(u) \cap C_{k,i} \neq \emptyset$ can be enumerated in B either at a stage $s+1$ of the form $s+1 = \langle 2k-1, \cdot\rangle$ (when $i \in A_k$), or of the form $s+1 = \langle 2e, \cdot\rangle$. In the latter case, we have $\varepsilon^s_{\max(u,s)} = \varepsilon_{\langle 2e,t\rangle}$ for some t. Moreover, if a different such element u_1 is enumerated in B at a stage $s_1+1 = \langle 2e, \cdot\rangle$, and $\varepsilon^{s_1}_{\max(u_1,s_1)} = \varepsilon_{\langle 2e,t_1\rangle}$ for some t_1, then $t \neq t_1$. Enumeration of such vectors cannot cause $\left\langle \varepsilon_{\langle 2k-1,ki\rangle}, \dots, \varepsilon_{\langle 2k-1,ki+k-1\rangle} \right\rangle \in D_k(V)$. Therefore, if $i \notin A_k$, then $\left\langle \varepsilon_{\langle 2k-1,ki\rangle}, \dots, \varepsilon_{\langle 2k-1,ki+k-1\rangle} \right\rangle \notin D_k(V)$.

□

Lemma 2.5. *For $k \geq 2$, we have $D_k(V) \leq_T A_k$.*

Proof. Given $v = \langle v_1, \dots, v_k\rangle$, we will determine whether $v \in D_k(V)$, using oracle A_k. Fix a stage s_0 such that $s_0 \geq v$, $A_0 \upharpoonright k = A_{0,s_0} \upharpoonright k$, and no requirement R_e with $e < k-1$ acts after s_0. Using oracle A_k, find the least stage $s_1 > s_0$ such that

$$(\forall j)[2 \leq j \leq k \Rightarrow A_j \upharpoonright v = A_{j,s_1} \upharpoonright v].$$

We will prove that

$$\langle v_1, \dots, v_k\rangle \in D_k(V) \Leftrightarrow \langle v_1, \dots, v_k\rangle \in D_k(V^{s_1}).$$

At stages of the form $\langle 2j-1, \cdot\rangle$ for $2 \leq j \leq k$, greater than s_1, we can enumerate into V, for the sake of the coding requirement Q_j, only linear combinations of the vectors $\varepsilon_{\langle 2j-1,ji\rangle}, \dots, \varepsilon_{\langle 2j-1,ji+j-1\rangle}$ for some $i \geq v$. If at such a stage, we enumerate v into $D_k(V)$, then at least one of the vectors $\varepsilon_{\langle 2j-1,ji\rangle}, \dots, \varepsilon_{\langle 2j-1,ji+j-1\rangle}$ is in the support of some vector v_n among $v_1, \dots, v_k$. However, this is impossible because

$$v_n \leq \langle v_1, \dots, v_k\rangle = v \leq i < \langle 2j-1, ji\rangle\,.$$

If some element x is enumerated into V at any other stage $s+1 > s_1$, then the choice of s_1 and the construction guarantee that $D((V^s \cup \{x\})^*) \cap X(k+1,s) = \emptyset$. Hence, these stages cannot enumerate $\langle v_1, \dots, v_k\rangle$ into $D_k(V)$.

□

Lemma 2.6. $A_0 \leq_T D(V)$

Proof. It is similar to the proof of Lemma 2.4. $\square$

Lemma 2.7. $D(V) \leq_T A_0$

Proof. Given $v = \langle v_1, \ldots, v_k \rangle$ for some k, we would like to determine whether $v \in D(V)$, using oracle A_0. Although we have $A_k \leq_T A_0$, uniformly in k, because of the R-requirements, we cannot assume that $D_k(V) \leq_T A_k$, uniformly in k. This is why we also used permitting by A_0 in the construction. Using oracle A_0, find the least stage $s_0 \geq v$ such that $A_0 \upharpoonright k = A_{0,s_0} \upharpoonright k$, $A_0 \upharpoonright v = A_{0,s_0} \upharpoonright v$, and

$$(\forall j)[2 \leq j \leq k \Rightarrow A_j \upharpoonright v = A_{j,s_0} \upharpoonright v].$$

We will prove that

$$\langle v_1, \ldots, v_k \rangle \in D(V) \Leftrightarrow \ \langle v_1, \ldots, v_k \rangle \in D(V^{s_0}).$$

At stages of the form $\langle 2j-1, \cdot \rangle$ for $2 \leq j \leq k$, greater than s_0, we can enumerate into V only linear combinations of the vectors $\varepsilon_{\langle 2j-1,ji \rangle}, \ldots, \varepsilon_{\langle 2j-1,ji+j-1 \rangle}$ for some $i \geq v$ (by the choice of s_0). As in Lemma 2.5, we can show that these linear combinations cannot make $v_1, \ldots, v_k$ dependent over V. If at some stage $s+1 > s_0$, we enumerate a vector x into B for the sake of a coding requirement Q_j for $j > k$, then $D((V^s \cup \{x\})^*) \cap X(k+1, s) = \emptyset$. Thus, the action at such a stage cannot cause $\langle v_1, \ldots, v_k \rangle \in D(V)$. If at a stage $s+1 > s_0$ of the form $s+1 = \langle 2e, \cdot \rangle$, we enumerate some u into B then, by permitting by A_0, $\max(u, s) > v$. Hence, the action at this stage will not enumerate $\langle v_1, \ldots, v_k \rangle$ into $D(V)$. $\square$

Lemma 2.8. *For every $e \in \omega$, the requirement R_e is satisfied.*

Proof. Assume otherwise. Let e be the least number such that R_e is not satisfied. Let s_0 be a stage such that no R_i for $i < e$ acts after s_0. Note that if R_e acts after s_0, then it will be satisfied. By Lemma 2.1, one of the conditions (1), (2), (3) must be satisfied. If (1) or (2) is satisfied, then the conditions (i) or (ii) will be satisfied at some stage $s+1 > s_0$ of the form $s+1 = \langle 2e, \cdot \rangle$, and R_e will act at the least such stage. Suppose that φ_e induces a $1-1$ linear transformation of $\frac{V_\infty}{V}$ that satisfies condition (3). Since R_e acts at most finitely often, some cofinite subset of $\{\varepsilon_{\langle 2e,t \rangle} : t \geq 0\}$ is independent $\operatorname{mod} V$. Let $\varepsilon_n^\infty =_{def} \lim\limits_{s \to \infty} \varepsilon_n^s$, $n \in \omega$. These vectors are independent over $\mathcal{V}$. For every $x \in V_\infty - V$, we define $supp(x, \infty)$ to be the index set J in the unique representation of x as

$$x = v + \sum_{j \in J} \lambda_j \varepsilon_j^{\infty},$$

where $v \in V$, and $\lambda_j \in F - \{0\}$ for $j \in J$. We then define

$$\max(x, \infty) = \max\{j : \varepsilon_j^{\infty} \in supp(x, \infty)\}.$$

Let w_0 be the least vector such that the set $\{w_0, \varphi_e(w_0)\}$ is independent mod V. Let

$$m_0 = \max\{\max(w_0, \infty), \max(\varphi_e(w_0), \infty)\}.$$

Let $s_1 > s_0$ be the least stage by which $\varepsilon_{m_0}^{\infty}$ has reached its limit, $\varphi_{e,s_1}(w_0) \downarrow$, and such that at every stage $s > s_1$, w_0 is the least vector with the set $\{w_0, \varphi_{e,s}(w_0)\}$ independent mod V^s. Recall that $F_e^{s_1} = \{\varepsilon_{\langle 2e,t \rangle} : t \geq 0\} \cap E^{s_1}$. Since R_e does not act after s_1, our construction guarantees that F_e^s remains unchanged at stages $s \geq s_1$. Let $F_e =_{def} F_e^{s_1}$. Fix distinct vectors $v_0, v_1, \dots, v_{m_0+1} \in F_e$ such that for every $j \in \{0, \dots, m_0 + 1\}$, we have $supp(v_j, \infty) \cap \{\varepsilon_0^{\infty}, \dots, \varepsilon_{m_0}^{\infty}\} = \emptyset$. Since the transformation induced by φ_e is linear, there is a vector w_1, obtained as a linear combination of the vectors $v_0, v_1, \dots, v_{m_0+1}$, such that $supp(\varphi_e(w_1), \infty) \cap \{\varepsilon_0^{\infty}, \dots, \varepsilon_{m_0}^{\infty}\} = \emptyset$. Similarly, we can find $w_2, \dots, w_{e+1}$ such that for every $i \in \{0, \dots, e\}$, the following conditions are satisfied for $m_i =_{def} \max\{\max(w_i, \infty), \max(\varphi_e(w_i), \infty)\}$:

($\mathcal{C}_1^{\infty}$) $\quad supp(w_{i+1}, \infty) \subset F_e$,

($\mathcal{C}_2^{\infty}$) $\quad supp(w_{i+1}, \infty) \cap \{\varepsilon_0^{\infty}, \dots, \varepsilon_{m_i}^{\infty}\} = \emptyset$, and

($\mathcal{C}_3^{\infty}$) $\quad supp(\varphi_e(w_{i+1}), \infty) \cap \{\varepsilon_0^{\infty}, \dots, \varepsilon_{m_i}^{\infty}\} = \emptyset$.

Since such vectors exist for the space V, there are a stage $s_2 > s_1$ and vectors $w_1, \dots, w_{e+1}$ such that ($\mathcal{C}_1$), ($\mathcal{C}_2$) and ($\mathcal{C}_3$) are satisfied at s_2. That is, R_e can possibly act *via* $w_0, w_1, \dots, w_{e+1}$. The reason R_e does not act is that the enumeration of A_0 does not permit it, i.e., ($\mathcal{C}_4$) from the construction is not satisfied. Also, notice that $w_1, \dots, w_{e+1}$ can be chosen so that $\left|\bigcup_{j=1}^{e+1} supp(w_j)\right|$ is arbitrarily large. Thus, assuming that the condition ($\mathcal{C}_4$) is not satisfied, we will have a decision procedure for computing A_0. Given $z \in \omega$, to determine whether $z \in A_0$, we effectively find $s_2 > s_1$ such that R_e can possibly act at s_2 *via* some sequence $w_0, w_1, \dots, w_{e+1}$ for which $\left|\bigcup_{j=1}^{e+1} supp(w_j)\right| > z$. It is clear that R_e can possibly act *via* the same sequence $w_0, w_1, \dots, w_{e+1}$ at any stage $s > s_2$. Thus, since $\bigcup_{j=1}^{e+1} supp(w_j) \subset F_e$, we have that $\max(w_{e+1}, s) \geq \left|\bigcup_{j=1}^{e+1} supp(w_j)\right| > z$ for every $s > s_2$. It must be that $A_0 \restriction (z+1) = A_0^{s_2} \restriction (z+1)$, or R_e would actually act *via* $w_0, w_1, \ldots, w_{e+1}$ if a number $\leq z$ is enumerated into A_0 after stage s_2.

$\square$

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