Simple and immune relations on countable structures

S. S. Goncharov
Academy of Sciences, Siberian Branch
Mathematical Institute
630090 Novosibirsk, Russia
gonchar@math.nsc.ru

V. S. Harizanov
Department of Mathematics
The George Washington University
Washington, D.C. 20052, U.S.A.
harizanv@gwu.edu

J. F. Knight
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556, U.S.A.
julia.f.knight.1@nd.edu

C. McCoy
Department of Mathematics
University of Wisconsin, Madison
Madison, WI 53706, U.S.A.
mccoy@math.wisc.edu

Abstract

Let $\mathcal{A}$ be a computable structure and let R be a new relation on its domain. We establish a necessary and sufficient condition for the existence of a copy $\mathcal{B}$ of $\mathcal{A}$ in which the image of R ($\neg R$, resp.) is simple (immune, resp.) relative to $\mathcal{B}$. We also establish, under certain effectiveness conditions on $\mathcal{A}$ nad R, a necessary and sufficient condition for the existence of a computable copy $\mathcal{B}$ of $\mathcal{A}$ in which the image of R ($\neg R$, resp.) is simple (immune, resp.).

1 Introduction and Notation

We investigate Post-type computability-theoretic properties of an additional relation on the domain of a countable structure. The domain of any infinite countable structure can be identified with an infinite subset of ω, the set of all natural numbers. Thus, such a domain is equipped with an ordering. We denote structures by script letters, and their domains by corresponding capital Latin letters. Unless otherwise stated, we assume that L is a computable relational language. If L is the language of a structure $\mathcal{B}$, then $L(B)$ is the language expanded by adding a constant symbol for every $b \in B$. Let $\mathcal{B}_B = (\mathcal{B}, b)_{b \in B}$ be the natural expansion of $\mathcal{B}$ to the language $L(B)$.

The *atomic diagram* of $\mathcal{B}$, denoted $D(\mathcal{B})$, is the set of all atomic and negated atomic sentences of $L(B)$ which are true in $\mathcal{B}_B$. We can identify $D(\mathcal{B})$ with a subset of ω by using a suitable Gödel coding of sentences. The degree of a structure $\mathcal{B}$ is the Turing degree of its atomic diagram $D(\mathcal{B})$. We say that a set X is computably enumerable (c.e.) relative to $\mathcal{B}$ if X is c.e. relative to $D(\mathcal{B})$. A structure is *computable* if its domain is a computable set and its atomic diagram is computable. Equivalently, a structure is computable iff its Turing degree is $\mathbf{0}$. By $F : \mathcal{A} \cong \mathcal{B}$ we denote that F is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. We call any structure isomorphic to $\mathcal{A}$ a *copy* of $\mathcal{A}$.

Throughout the paper, we will denote by $\mathcal{A}$ an infinite computable structure, and by R a new infinite co-infinite relation on A. A relation on the domain of $\mathcal{A}$ is *new* if it is not named in the language of $\mathcal{A}$. Without loss of generality, we assume that R is unary. We are interested in syntactic conditions under which there is a computable copy of $\mathcal{A}$ in which the image of R is simple. We may also ask when the image of $\neg R$ is only immune. Recall (see [12] and [10]) that a set is *immune* if it is infinite and contains no infinite c.e. subset. A set is *simple* if it is c.e. and its complement is immune.

Problem 1. *Under what syntactic conditions is there an isomorphism F from $\mathcal{A}$ onto a computable copy such that $\neg F(R)$ is immune?*

Problem 2. *Under what syntactic conditions is there an isomorphism F from $\mathcal{A}$ onto a computable copy such that $F(R)$ is simple?*

For a computable linear order $\mathcal{A}$, Hird [6] determined which co-c.e. intervals have immune image on some computable copy: those of order type ω with no supremum in $\mathcal{A}$; those of order type ω^* with no infimum in $\mathcal{A}$; those of order type $\omega^* + \omega$ and with neither supremum nor infimum in $\mathcal{A}$. Remmel [11] established that if $\mathcal{A}$ is a computable Boolean algebra with infinitely many atoms, then there is a computable copy $\mathcal{B}$ of $\mathcal{A}$ such that the set of all atoms of $\mathcal{B}$ is immune. Hird [7] and Ash, Knight and Remmel [1] investigated a related notion, the so-called *quasi-simplicity* of relations on computable structures. Hird proved that, under certain decidability condition on $\mathcal{A}$ and R, there is an isomorphism F from $\mathcal{A}$ onto a computable copy $\mathcal{B}$ such that $F(R)$ is quasi-simple. Ash, Knight and Remmel gave effectiveness conditions on $\mathcal{A}$ and R, which are sufficient for obtaining such a quasi-simple relation $F(R)$ in an arbitrary nonzero c.e. Turing degree. Certain quasi-simple relations coincide with simple relations. However, there are computable structures which contain no simple substructures but have quasi-simple substructures in every non-zero c.e. Turing degree. A well studied example of such a structure is $\mathcal{V}_\infty$, a computable $\aleph_0$-dimensional vector space over a computable field, such that for every $n \in \omega$, $\mathcal{V}_\infty$ has a computable n-ary dependence relation. If $\mathcal{V}$ is an infinite c.e. subspace of $\mathcal{V}_\infty$, then the set V is not a simple subset of V_∞. Assume that $V \neq V_\infty$. Let $a \in V_\infty - V$. Then $a + V =_{def} \{a + v : v \in V\}$ is a c.e. set such that $(a + V) \cap V = \emptyset$.

Results establishing various equivalences of syntactic and corresponding semantic conditions in computable copies of $\mathcal{A}$ usually involve additional effectiveness conditions, expressed in terms of $\mathcal{A}$ and R. To discover syntactic conditions governing the algorithmic properties of images of R in computable copies of $\mathcal{A}$, it is sometimes helpful to consider arbitrary copies of $\mathcal{A}$ and relative versions of the algorithmic properties. One advantage is that we may use the forcing method instead of the priority method—the latter is more complicated. In addition, the relative results should require no additional effectiveness conditions, which often mask the syntactic conditions. Examples of such relative results are presented in [2] and [3].

Definition 1. (i) *A new relation on a countable structure $\mathcal{B}$ is immune relative to $\mathcal{B}$ if it is infinite and contains no infinite subset that is c.e. relative to $\mathcal{B}$.*
(ii) *A new relation on a countable structure $\mathcal{B}$ is simple relative to $\mathcal{B}$ if it is c.e. relative to $\mathcal{B}$ and its complement is immune relative to $\mathcal{B}$.*

Thus, we are led to also consider the following problems.

Problem 3. *Under what syntactic conditions is there an isomorphism F from $\mathcal{A}$ onto a copy $\mathcal{B}$ such that $\neg F(R)$ is immune relative to $\mathcal{B}$?*

Problem 4. *Under what syntactic conditions is there an isomorphism F from $\mathcal{A}$ onto a copy $\mathcal{B}$ such that $F(R)$ is simple relative to $\mathcal{B}$?*

Let $W_0, W_1, W_2, \ldots$ be a fixed effective enumeration of all c.e. sets. Let $X \subseteq \omega$. Then $W_0^X, W_1^X, W_2^X, \ldots$ is a fixed effective enumeration of all sets that are c.e. relative to X. For a structure $\mathcal{B}$, $W_e^{\mathcal{B}}$ stands for $W_e^{D(\mathcal{B})}$. By $\leq_T$ we denote Turing reducibility, and by $\equiv_T$ Turing equivalence. We write $\mathcal{B} \leq_T X$ if $D(\mathcal{B}) \leq_T X$.

By $\vec{c}$ we denote a finite sequence (tuple) of elements; we write $a \in \vec{c}$ to indicate that $a \in ran(\vec{c})$, and $\vec{c} \cap \vec{d} = \emptyset$ to denote that $ran(\vec{c}) \cap ran(\vec{d}) = \emptyset$. A sequence of variables displayed after a formula includes all of its free variables. If a formula is in prenex normal form, then the matrix of the formula is its part after the quantifiers. *Almost all* means all but finitely many.

2 Relatively Immune Relations

A Σ_1 *formula* $\varphi(\vec{x})$ is an infinitary formula of the form

$$\bigvee\!\!\!\bigvee_{i\in I} \exists \vec{u}_i \psi_i(\vec{x}, \vec{u}_i),$$

where for every $i \in I$, $\psi_i(\vec{x}, \vec{u}_i)$ is a finitary quantifier-free formula. We assume that the finitary quantifier-free formulas are coded by some effective Gödel numbering, and ψ_i is the i^{th} formula under this listing. If the index set I is c.e., then we have a *computable* Σ_1 formula. (We can define, by induction, computable Σ^0_α and Π^0_α formulas for all $\alpha < \omega_1^{CK}$. Such formulas are called *computable infinitary formulas.*) If we are to construct an isomorphic copy of $\mathcal{A}$ in which the image of $\neg R$ is relatively immune, there must be no infinite subset D of $\neg R$ definable in $\mathcal{A}$ by a computable Σ_1 formula $\varphi(\vec{c}, x)$ (with a finite tuple of parameters $\vec{c}$). This obvious necessary condition turns out to be sufficient.

Theorem 2.1. *Let $\mathcal{A}$ be a computable L-structure, and let R be a unary infinite and co-infinite relation on A. Then the following are equivalent:*

- (i) *For all copies $\mathcal{B}$ of $\mathcal{A}$ and all isomorphisms F from $\mathcal{A}$ onto $\mathcal{B}$, $\neg F(R)$ is not immune relative to $\mathcal{B}$.*
- (ii) *There are an infinite set D and a finite tuple $\vec{c}$ such that $D \subseteq \neg R$ and D is definable in $\mathcal{A}$ by a computable Σ_1 formula $\varphi(\vec{c}, x)$.*

Proof: The rest of this section consists of a proof that $(i) \Rightarrow (ii)$. We build a "generic" copy $(\mathcal{B}, S)$ of $(\mathcal{A}, R)$. Under the assumption that $\neg S$, the image of $\neg R$, is not immune relative to $\mathcal{B}$, we produce the set D and a tuple $\vec{c}$ as in (ii). Let B be an infinite computable set, the universe of $\mathcal{B}$. The *forcing conditions* are the finite $1-1$ partial functions from B to A. The set $\mathcal{F}$ of these conditions is partially ordered by extension $\subseteq$. We use letters p, q, r, etc. to denote elements of $\mathcal{F}$.

Let $\mathbf{R}$ be an additional unary relation symbol not in L. As a forcing language, we take a propositional language P in which the propositional variables are just the atomic sentences in the language $(L \cup \{\mathbf{R}\})(B)$. Let P' be the sublanguage consisting of atomic sentences that are in the language $L(B)$ (without $\mathbf{R}$). Let $\mathcal{T}$ be the set of computable infinitary sentences in the language P, and let $\mathcal{T}'$ be the set of computable infinitary sentences in the language P'.

Among the sentences are those expressing the following facts in $(\mathcal{B}, S)$, the copy of $(\mathcal{A}, R)$:

- $W_e^{\mathcal{B}}$ is infinite (expressed in $\mathcal{T}'$);
- $W_e^{\mathcal{B}} \subseteq \neg S$ (expressed in $\mathcal{T}$).

We consider only computable infinitary formulas in *normal* form—with negations occurring only in finitary open subformulas. We write (φ) for the computable infinitary sentence that is dual to φ —equivalent to the negation, but in normal form. The constants of a sentence φ are the constants appearing in the propositional variables in φ.

We define forcing—the relation $p \Vdash \varphi$, for φ in $\mathcal{T}$.

1. If φ is a finitary sentence of $\mathcal{T}$, then $p \Vdash \varphi$ iff the constants of φ are all in $dom(p)$, and p (under natural interpretation of constants) makes φ true in $(\mathcal{A}, R)$.
2. If φ is a disjunction $\bigvee\!\!\!\bigvee_{i\in I} \psi_i$, then $p \Vdash \varphi$ iff there is $i \in I$ such that $p \Vdash \psi_i$.
3. If φ is a conjunction $\bigwedge\!\!\!\bigwedge_{i\in I} \psi_i$, then $p \Vdash \varphi$ iff for every $q \supseteq p$ and every $i \in I$, there exists $r \supseteq q$ such that $r \Vdash \psi_i$.

We say that q *decides* φ if q forces either φ or $\neg(\varphi)$. We have the usual forcing lemmas.

Lemma 2.2. *For any φ, and any p and q, if $p \Vdash \varphi$ and $q \supseteq p$, then $q \Vdash \varphi$.*

Lemma 2.3. *For any φ and p, it is not the case that ($p \Vdash \varphi$ and $p \Vdash \neg(\varphi)$).*

Lemma 2.4. *For any φ and p, there is some $q \supseteq p$ such that $q$decides φ.*

A *complete forcing sequence*, abbreviated as *c.f.s.*, is a chain $(p_n)_{n\in\omega}$ of forcing conditions, such that for each $\varphi \in \mathcal{T}$, there is some n such that p_n decides φ; for each $a \in A$, there is some n such that $a \in ran(p_n)$; and for each $b \in B$, there is some n so that $b \in dom(p_n)$. Lemma 2.4 implies the existence of a c.f.s. Given a c.f.s. $(p_n)_{n\in\omega}$, we obtain a $1-1$ function $\cup_n p_n$ from B onto A. Let $F =_{def} (\cup_n p_n)^{-1}$. Then F induces on B a copy $(\mathcal{B}, F(R))$ of $(\mathcal{A}, R)$. A sentence φ in forcing language P is propositional, but we may also think of it as a predicate sentence in the language $(L \cup \{\mathbf{R}\})(B)$.

We have the following “Truth-and-Forcing” lemma.

Lemma 2.5. *For any $\varphi \in \mathcal{T}$, $(\mathcal{B}_B, F(R)) \models \varphi$ iff there is $n \in \omega$ such that $p_n \Vdash \varphi$.*

To complete the proof of Theorem 2.1, we just need the following lemma.

Lemma 2.6. *The relation $\neg F(R)$ is immune relative to $\mathcal{B}$.*

Proof of Lemma 2.6. Suppose otherwise. Then there is $e \in \omega$ such that $W_e^{\mathcal{B}}$ is infinite and $W_e^{\mathcal{B}} \subseteq \neg F(R)$. By the Truth-and-Forcing Lemma, there is $p \in \mathcal{F}$ ($p = p_n$ for some n) such that p forces statements which express these two facts. Let p map $\vec{d}$ onto $\vec{c}$. We consider the set D consisting of all $a \in A$ for which there exist $b \in B - \{\vec{d}\}$ and $q \supseteq p$ such that $q(b) = a$ and $q \Vdash$ “$b \in W_e^{\mathcal{B}}$”.

(a) The set D is infinite, since it includes the set $F^{-1}(W_e^{\mathcal{B}} - \{\vec{d}\})$.

(b) The set D contains no element of R, since $p \Vdash$ “$W_e^{\mathcal{B}} \subseteq \neg F(R)$”.

(c) The set D is definable in $\mathcal{A}$ by a computable Σ_1 formula $\varphi(\vec{c}, x)$ of L.

To see (c), let us analyze what it means for $q \supseteq p$ to force "$b \in W_e^{\mathcal{B}}$". There must be a halting computation of oracle machine with Gödel index e on input b which uses only a finite oracle σ. This σ expresses information about $\mathcal{B}$ expressed by an open formula $\psi_\sigma(\vec{d}, b, \vec{b}_1)$ of $L(B)$ that q makes true in $\mathcal{A}$. We may assume, without loss of generality, that $b \not\in \vec{d}$, $b \not\in \vec{b}_1$, and $\vec{d} \cap \vec{b}_1 = \emptyset$ and that ψ_σ expresses these additional facts. Let $\theta_b(x)$ be the following infinitary formula of L: $\bigvee\!\!\!\bigvee_{\{\sigma : b \in W_e^\sigma\}} \exists \vec{a}_1 \psi_\sigma(\vec{c}, x, \vec{a}_1)$. Then there exists $q \supseteq p$ such that $q(b) = a$ and $q \Vdash$ "$b \in W_e^{\mathcal{B}}$" iff $\mathcal{A} \models \theta_b(a)$. Consequently, $a \in D$ iff $\mathcal{A} \models \bigvee\!\!\!\bigvee_{b \in \mathcal{B} - \{\vec{d}\}} \theta_b(x)$. $\square$

3 Relatively Simple Relations

Let $\mathcal{A}$ be an L-structure, and $\mathbf{R}$ be an additional unary relation symbol. If we are interested in c.e. relations, computable Σ_1 formulae with positive occurrences of $\mathbf{R}$ in the expanded language $L \cup \{\mathbf{R}\}$ play an important role. The importance of this kind of the so-called "positive logic" in the study of c.e. vector subspaces was remarked in [9]. Computable Σ_1 formulae with positive occurrences of $\mathbf{R}$ were first used in [5], and later in [7], [1] and [4].

Assume that there is an infinite set $D \subseteq \neg R$ such that D is definable in $(\mathcal{A}, R)$ by a computable Σ_1 formula with finitely many parameters and with only positive occurrences of $\mathbf{R}$. In any copy $\mathcal{B}$ of $\mathcal{A}$, if the image of R is c.e. relative to $\mathcal{B}$, then so is the image of D. Therefore, under this definability assumption, the image of R cannot be made simple relative to $\mathcal{B}$. It turns out that this is the only obstacle.

Theorem 3.1. *Let $\mathcal{A}$ be an infinite computable structure in a relational language L, and let R be a computable unary infinite and co-infinite relation on A. Then the following are equivalent:*

(*i*) *For all copies $\mathcal{B}$ of $\mathcal{A}$ and all isomorphisms F from $\mathcal{A}$ onto $\mathcal{B}$, $F(R)$ is not simple relative to $\mathcal{B}$.*

(*ii*) *There are an infinite set D and a finite tuple of parameters $\vec{c}$ such that $D \subseteq \neg R$, and D is definable in $(\mathcal{A}, R)$ by a computable Σ_1 formula $\varphi(\vec{c}, x)$ of $L \cup \{\mathbf{R}\}$ with only positive occurrences of $\mathbf{R}$.*

Proof: The rest of this section consists of a proof by contrapositive that $(i) \Rightarrow (ii)$. If R is definable in $\mathcal{A}$ by a computable Σ_1 formula $\varphi(\vec{c}, x)$, then in any copy $\mathcal{B}$ of $\mathcal{A}$, the image of R is c.e. relative to $\mathcal{B}$. If $\mathcal{B}$ is a copy in which the image of $\neg R$ is relatively immune, then the image of R will automatically be relatively simple. Assume that R is not definable this way. If we form a generic copy $\mathcal{B}$ of $\mathcal{A}$ as in the previous section, then the image of R will definitely *not* be c.e. relative to $\mathcal{B}$. (A standard forcing argument shows that if the image of R is c.e. relative to a generic copy $\mathcal{B}$, then R is indeed definable by a computable Σ_1

formula with parameters.) Therefore, we shall first define an expanded language L^* and replace the L-structure $\mathcal{A}$ by a L^*-structure $\mathcal{A}^*$, in which $\mathcal{A}$ sits as a relativized reduct, such that:

(1) the domain A of $\mathcal{A}$ is definable in $\mathcal{A}^*$ by an open formula of L^*;

(2) the relation R is definable in $\mathcal{A}^*$ by a computable Σ_1 formula of L^*;

(3) if a set $D \subseteq A$ is definable in $\mathcal{A}^*$ by a computable Σ_1 formula of L^* with finitely many parameters, then it is definable in $(\mathcal{A}, R)$ by a computable Σ_1 formula in $L \cup \{\mathbf{R}\}$ with finitely many parameters and only positive occurrences of $\mathbf{R}$.

Let $L^* = L \cup \{\mathbf{R}'\} \cup \{\mathbf{Q}\}$, and let $\mathcal{A}^*$ be the result of extending the universe A by another infinite computable set R', and expanding $\mathcal{A}$ to include the unary relation R' and a binary relation Q that is a $1-1$ mapping from R' onto R. In $\mathcal{A}^*$, the formula $\neg\mathbf{R}'(x)$ defines A, so we have (1). The formula $\exists\, y \mathbf{Q}(y,x)$ defines R, so we have (2). The lemma below gives (3).

Lemma 3.2. *Let $D \subseteq A$. If the set D is definable in $\mathcal{A}^*$ by a computable Σ_1 formula $\varphi(\vec{c}, x)$ of L^*, then it is definable in $(\mathcal{A}, R)$ by some computable Σ_1 formula in $L \cup \{\mathbf{R}\}$ with finitely many parameters and only positive occurrences of* $\mathbf{R}$.

Proof of Lemma 3.2. Assume that there is $\vec{c} \in A^*$ and a computable infinitary Σ_1 formula $\varphi(\vec{c}, x)$ of the form $\bigvee\!\!\!\bigvee_{i \in I} \exists \vec{y}_i(\psi_i(\vec{y}_i, \vec{c}, x))$, where each ψ_i is finitary and quantifier-free, so that $a \in D$ iff $\mathcal{A}^* \models \varphi(\vec{c}, a)$. Clearly, $D = \cup_{i \in I} D_i$, where $a \in D_i$ iff $\mathcal{A}^* \models \exists \vec{y}_i \psi_i(\vec{y}_i, \vec{c}, a)$. Consequently, we need only prove the statement in the case where $\varphi(\vec{c}, x)$ is $\exists \vec{y}(\psi(\vec{y}, \vec{c}, x))$, where ψ is finitary quantifier-free. Furthermore, we may suppose that the elements of $\vec{c} = (c_1, \ldots, c_n)$ are all in A. Indeed, if some c_i is in R', we may replace $\exists \vec{y}(\psi(\vec{y}, c_1, \ldots, c_i, \ldots, c_n, x))$ by $\exists z \exists \vec{y}[\mathbf{Q}(z, c') \wedge \psi(\vec{y}, c_1, \ldots, z, \ldots, c_n, x)]$, where c' is the element of R corresponding to c_i.

In addition, using the basic rules of predicate logic, we may rewrite $\varphi(\vec{y}, \vec{c}, x)$ as a finite disjunction of formulas, each of the form $\exists \vec{y}_i(\psi_i(\vec{y}_i, \vec{c}, x))$, where every ψ_i is a finitary conjunction of atomic formulas and the negations of atomic formulas. Consequently, we may assume that ψ itself is of this form. Moreover, we may assume that all existential quantifiers are relativized to either $\mathbf{R}'$ or $\neg\mathbf{R}'$: if $\vec{y} = (y_1, \ldots, y_m)$, then we replace $\varphi(\vec{y}, \vec{c}, x)$ with 2^m disjuncts, each of the form $\exists \vec{y}(\psi(\vec{y}, \vec{c}, x) \wedge \pm\mathbf{R}'(y_1) \wedge \cdots \wedge \pm\mathbf{R}'(y_m))$ (where the symbol $-\mathbf{R}'$ represents $\neg\mathbf{R}'$, and the symbol $+\mathbf{R}'$ represents $\mathbf{R}'$. Also, we may suppose that for each variable u such that the conjunct $\mathbf{R}'(u)$ appears in ψ, there is a corresponding variable v so that the conjuncts $\neg\mathbf{R}'(v)$ and $\mathbf{Q}(u, v)$ appear in ψ.

Next, recall that the language L is relational, and in $\mathcal{A}^*$ elements of R' satisfy no relations of L among themselves or with other elements of $\mathcal{A}^*$. We claim that we may assume that, except for those of the form $\mathbf{Q}(u, v)$, $\mathbf{R}'(u)$, or $\neg\mathbf{R}'(u)$, all conjuncts involve only variables relativized to $\mathbf{R}'$ and symbols from L. First, we show that we can assume no conjunct is of the form $\neg\mathbf{Q}(u_1, u_2)$. If $\mathbf{R}'(u_1)$ and $\mathbf{R}'(u_2)$ both appear as conjuncts, then $\neg\mathbf{Q}(u_1, u_2)$ is true automatically, and so we need not include it in ψ. The same is true if $\neg\mathbf{R}'(u_1)$ appears as a conjunct. Finally, if $\mathbf{R}'(u_1)$ and $\neg\mathbf{R}'(u_2)$ both appear as conjuncts, then $\neg\mathbf{Q}(u_1, u_2)$ is equivalent in $\mathcal{A}^*$ to the formula $\exists z(z \not= u_2 \wedge \neg\mathbf{R}'(z) \wedge \mathbf{Q}(u_1, z))$. Second, a conjunct of the form $\mathbf{S}(y_{k_1}, \ldots, y_{k_l})$, where at least one y_{k_i} is in R' and $\mathbf{S}$ is a relational symbol from L, is automatically false; and one of the form $\neg\mathbf{S}(y_{k_1}, \ldots, y_{k_l})$, where at least one y_{k_i} is in R', is automatically true. Finally, we can assume that ψ is not an "obviously false" formula. For instance, we assume that it does not contain a conjunct α and a conjunct $\neg\alpha$. Similarly, we assume that if ψ contains a conjunct of the form $\mathbf{Q}(u_1, u_2)$, then it also contains $\mathbf{R}'(u_1)$ and $\neg\mathbf{R}'(u_2)$.

Having argued that we can make all of the above assumptions about φ, we now can produce a formula of $L \cup \{\mathbf{R}\}$, satisfied in $(\mathcal{A}, R)$ by the same elements as the formula $\varphi(\vec{c}, x)$. Notice that for all v in A, $\mathcal{A}^* \models \exists u(\mathbf{Q}(u, v))$ iff $(\mathcal{A}, R) \models \mathbf{R}(v)$. Consequently, we delete each quantifier relativized to $\mathbf{R}'$ and each conjunct mentioning the variable u corresponding to this quantifier; thus, we rid the formula of all occurrences of $\mathbf{Q}$. We add a conjunct $\mathbf{R}(v)$ for each variable v corresponding to such a u, and we no longer relativize the remaining quantifiers to $\neg\mathbf{R}'$. We are left with the desired formula in $L \cup \{\mathbf{R}\}$. It is satisfied in $(\mathcal{A}, R)$ by the same elements as the formula $\varphi(\vec{c}, x)$.

Having completed the proof of Lemma 3.2, we now have $\mathcal{A}^*$ satisfying (1), (2), and (3). From (3) and the hypothesis of the implication we are attempting to prove, it follows that there is no infinite set $D \subseteq \neg(R \cup R')$ such that D is definable in $\mathcal{A}^*$ by a computable Σ_1 formula of L^* with finitely many parameters.

If we apply the result from the previous section to the structure $\mathcal{A}^*$ and the relation $R \cup R'$, we get an isomorphism F from $\mathcal{A}^*$ onto a copy $\mathcal{B}^*$ of $\mathcal{A}^*$, with $\mathcal{B}$ corresponding to $\mathcal{A}$ under F, such that the following are true:

(i) $\mathcal{B} \leq_T \mathcal{B}^*$;

(ii) the relation $F(R)$ is c.e. relative to $\mathcal{B}^*$;

(iii) $\neg F(R \cup R')$ is immune relative to $\mathcal{B}^*$.

Note that $(B \cap \neg F(R)) = (B^* - F(R \cup R')) = \neg F(R \cup R')$, and any set c.e. relative to $\mathcal{B}$ is c.e. relative to $\mathcal{B}^*$ by (i). Consequently, there is no infinite subset of the universe B which is contained in $\neg F(R)$ and is c.e. relative to $\mathcal{B}$. In other words, $B \cap \neg F(R)$ is immune relative to $\mathcal{B}$. However, we are not done, because $F(R)$ is not necessarily c.e. relative to $\mathcal{B}$, and so not necessarily simple relative to $\mathcal{B}$. To prove the theorem, we need the following lemma from [8]. We call a structure $\mathcal{A}$ *trivial* if there is a finite tuple $\vec{c}$ of the elements in its universe such that the automorphism group of $\mathcal{A}$ includes all permutations of the elements in its universe A that fix $\vec{c}$ pointwise.

Lemma 3.3. *Let $\mathcal{A}$ be any structure, and let $X \subseteq \omega$.*

(i) If $\mathcal{A}$ is trivial, then all copies of $\mathcal{A}$ have the same Turing degree.

(ii) If $\mathcal{A}$ is not trivial, and $\mathcal{A} \leq_T X$, then there is an isomorphism G from $\mathcal{A}$ onto a copy $\mathcal{B}$ such that $X \leq_T \mathcal{B} \leq_T G \oplus \mathcal{A} \leq_T X$.

Using the facts we noted about $\mathcal{B}^*$ and Lemma 3.3, we complete the proof of Theorem 3.1. We consider two cases.

Case 1: Suppose $\mathcal{A}$ is trivial.

Modulo a finite tuple $\vec{c}$, we have complete freedom in defining an automorphism of $\mathcal{A}$. Moreover, if X and Y have a finite symmetric difference, then X is simple iff Y is simple. Consequently, it is clear that there is an automorphism G of $\mathcal{A}$ in which $G(R)$ is simple.

Case 2: Suppose $\mathcal{A}$ is not trivial.

Let X be the atomic diagram of the structure $\mathcal{B}^*$ above, and let F be the isomorphism from $\mathcal{A}^*$ onto $\mathcal{B}^*$. If F_1 is the restriction of F to the domain A, then F_1 is an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$. Throughout the rest of this argument, if H is some function with range Y, then $\neg H(R)$ denotes the complement with respect to the universe Y. Therefore, $\neg F_1(R) = B - F_1(R) = B^* - F(R \cup R')$. By the facts above, $\mathcal{B} \leq_T X$, $F(R) = F_1(R)$ is c.e. relative to X, and there is no infinite $W \subseteq \neg F_1(R)$ such that W is c.e. relative to X. Applying Lemma 3.3 to the structure $\mathcal{B}$, we obtain an isomorphism G from $\mathcal{B}$ onto a copy $\mathcal{C}$ such that $X \leq_T \mathcal{C} \leq_T G \oplus \mathcal{B} \leq_T X$.

Claim 1. The relation $G(F_1(R))$ is c.e. relative to $\mathcal{C}$.

This is clear from the fact that $F_1(R)$ is c.e. relative to X, and G and X are both computable in $\mathcal{C}$.

Claim 2. There is no infinite subset $W \subset C$ such that W is c.e. relative to $\mathcal{C}$ and $W \subseteq \neg G(F_1(R)) = G(\neg F_1(R))$.

If there were such a set W, then $G^{-1}(W)$ would be an infinite subset of $\neg F_1(R)$, and it would be c.e. relative to X, since G^{-1} is computable relative to X. This is a contradiction.

Therefore, $G \circ F_1 : \mathcal{A} \cong \mathcal{C}$, and from Claims 1 and 2, it follows that $G(F_1(R))$ is simple relative to $\mathcal{C}$. □

4 Immune and Simple Relations on Computable Structures

Here are our results on Problem 1 and Problem 2. They involve extra decidability conditions which imply that both $\mathcal{A}$ and R are computable.

Theorem 4.1. *Let $\mathcal{A}$ be an infinite (computable) L-structure, and let R be a unary (computable) infinite and co-infinite relation on A. Assume that we have an effective procedure for deciding whether*

$$(\mathcal{A}_A, R) \models (\exists x \in \mathbf{R})\, \theta(\vec{c}, x),$$

where $\theta(\vec{c}, x)$ is a finitary existential formula of L with finitely many parameters. If there is no infinite set D such that $D \subseteq \neg R$ and D is definable in $\mathcal{A}$ by a computable Σ_1 formula of L with finitely many parameters, then there is an isomorphism F from $\mathcal{A}$ onto a computable copy $\mathcal{B}$ such that the relation $\neg F(R)$ is immune.

Proof: We use the finite injury priority method. Let $B = \{b_0, b_1, b_2, \ldots\}$ be an infinite computable set of constants for the universe of $\mathcal{B}$. The construction has the following requirements:
$P_n^0 : a_n \in dom(F)$;
$P_n^1 : b_n \in ran(F)$;
N_e: W_e is infinite $\Rightarrow F(R) \cap W_e \not= \emptyset$.

The construction proceeds in stages. At stage $s+1$, we inherit form stage s a finite chain $(p_0, \ldots, p_{k_s})$ of partial $1-1$ functions from B to A, so that $\cup_{i \leq k_s} p_i$ is also a partial 1-1 function from B to A. Each p_i has worked on the i^{th} requirement according to stage s information. Thus, for instance, if no action on behalf of requirement R_i was taken or preserved at stage s, then $p_i = \emptyset$.

We also have a finite set δ_s of sentences in L_B such that $\delta_s \subseteq D(\mathcal{B})$. When information changes at stage $s+1$, we may backup and change some p_m, dropping the later ones. However, we must retain δ_s to ensure that the copy $\mathcal{B}$ we construct is computable. As we shall see below, our construction ensures that every sentence of δ_{s+1} is determined by the partial function $\cup_{i \leq k_{s+1}} p_i$.

A requirement of the form P_n^0 or P_n^1 needs attention at stage $s+1$ for the obvious reason. The way in which we satisfy such a requirement is equally obvious. If a requirement of the form N_e is the m^{th} requirement in our list, then it needs attention at stage $s+1$ if the following are true:

i) $p_m(R) \cap W_{e,s+1} = \emptyset$;

ii) $b \in W_{e,s+1} - W_{e,s}$, and for all $n < m$, $b \not\in ran(p_n)$;

Assume that N_e is the highest priority requirement which needs attention and that b is the least element satisfying *ii)*. Then the strategy for satisfying N_e at stage $s+1$ is to put b into $F(R)$, if possible. Assume $\theta_s(\vec{d}, b, \vec{b}_1) =_{def} \bigwedge\!\!\bigwedge \delta_s$, where the image of $\vec{d}$ is fixed for the sake of higher priority requirements, and

$\cup_{i\leq k_s} p_i$ maps $\vec{d}, b, \vec{b_1}$ to $\vec{c}, a, \vec{a_1}$, where $a \notin R$. We effectively check whether there is $a' \in R$ satisfying $\exists \vec{u}(\theta_s(\vec{c}, x, \vec{u}) \wedge (x \not\in \vec{c}) \wedge (x \not\in \vec{u}) \wedge (\vec{c} \cap \vec{u} = \emptyset))$. If that is the case, then we change p_m to take care of the requirement in such a way that b and $\vec{b_1}$ are in the domain of p_m. We let the chain at the end of stage $s+1$ be $(p_0, p_1, \ldots, p_m)$. Otherwise, we add the pair (b, a) to the partial function p_m, and we let the chain at the nd of stage $s+1$ be $(p_0, p_1, \ldots, p_{k_s})$.

In defining δ_{s+1}, we consider the first atomic sentence $\psi(\vec{b})$ from L_B so that neither ψ nor $\neg\psi$ is included in δ_s. If $\vec{b} \subseteq dom(\cup_{i\leq k_{s+1}} p_i)$ and is mapped to $\vec{a}$, then $\psi(\vec{b})$ is added if $\mathcal{A} \models \psi(\vec{a})$, and $\neg\psi(\vec{b})$ is added if $\mathcal{A} \models \neg\psi(\vec{a})$. Otherwise, if $\vec{b} \not\subseteq dom(\cup_{i\leq k_{s+1}} p_i)$, then $\delta_{s+1} = \delta_s$.

If N_e is the least requirement which is never satisfied, then we obtain a *single* tuple of parameters $\vec{c}$ so that there are an infinite sequence of steps s, each with a *different* corresponding b and $a \notin R$, and a formula $\exists \vec{u}\, \theta_s(\vec{c}, x, \vec{u})$ satisfied by a and not by any element of R. (Note that it is important to protect p_m from lower priority requirements even when p_m fails to satisfy N_e. This guarantees that the element a is different for each stage s.) Then the disjunction of these formulas $\exists \vec{u} \theta_s(\vec{c}, x, \vec{u})$ is a computable Σ_1 formula with parameters $\vec{c}$ defining an infinite subset of $\neg R$, contradicting the assumption. □

Theorem 4.2. *Let $\mathcal{A}$ be an infinite (computable) L-structure, and let R be a unary (computable) infinite and co-infinite relation on A. Assume that we have an effective procedure for deciding whether*

$$(\mathcal{A}_A, R) \models (\exists x \in \mathbf{R})\, \varphi(\vec{c}, x),$$

where φ is a finitary existential formula in $L \cup \{\mathbf{R}\}$ with finitely many parameters and with positive occurrences of $\mathbf{R}$. If there is no infinite $D \subseteq \neg R$ definable by such a formula, then there is an isomorphism F from $\mathcal{A}$ onto a computable copy $\mathcal{B}$ such that $F(R)$ is simple.

The proof is similar to that of the previous theorem. □

We now present some examples on simplicity and immunity.

Example 1. Let $\mathcal{A} = (\omega, <_\omega)$ and let R be the set of all even numbers. First, we show that no infinite subset of the odds is definable by a computable Σ_1 formula (in the language $\{<, \mathbf{R}\}$) with finitely many parameters $\vec{c}$ and positive occurrences of $\mathbf{R}$. Otherwise, we can assume, without loss of generality, that a disjunct of such a formula is a finitary formula $\exists \vec{u} \psi(\vec{c}, x, \vec{u})$ so that the following are true:

i) the formula $\psi(\vec{c}, x, \vec{u})$ is a conjunct which gives the complete ordering of $\vec{c}, x, \vec{u}$ and expresses that certain elements of $\vec{c}, \vec{u}$ are in R;

ii) there is a tuple $\vec{d}$, and an odd number a bigger than every element in $\vec{c}$ so that $(\mathcal{A}_A, R) \models \psi(\vec{c}, a, \vec{d})$.

Define a' and a tuple $\vec{d'}$ as follows:

i) $a' = a + 1$;

ii) if $d_i \in \vec{d}$ and d_i is less than a, set $d'_i =_{def} d_i$;

iii) if $d_i \in \vec{d}$ and d_i is greater than a , set $d'_i =_{def} d_i + 2$.

Clearly, $(\mathcal{A}_A, R) \models \psi(\vec{c}, a', \vec{d'})$. Hence $(\mathcal{A}_A, R) \models \exists\vec{u}\psi(\vec{c}, a', \vec{u})$, but a' is even, which is a contradiction. Next, the structure $(\mathcal{A}, R)$ satisfies the decidability condition of Theorem 4.2. Therefore, there is a computable copy $\mathcal{B}$ of $\mathcal{A}$ and $F : \mathcal{A} \cong \mathcal{B}$ so that $F(R)$ is simple. (In [5] it was shown that for any c.e. set C, there is a computable copy $\mathcal{B}$ and $F : \mathcal{A} \cong \mathcal{B}$ so that $F(R)$ is a c.e. set and $F(R) \equiv_T F \equiv_T C$.)

Example 2. Let $\mathcal{A}$ be an equivalence structure with infinitely many equivalence classes, all of size 2. Let R be a relation containing exactly one element from each class so that the pair $(\mathcal{A}, R)$ satisfies the decidability condition of Theorem 4.1. No infinite subset of $\neg R$ is definable by a computable Σ_1 formula (in the language $\{E\}$) with only finitely many parameters: if an element a and its equivalent are both outside the parameters, then any formula satisfied by a is also satisfied by its equivalent element. Therefore, there is a computable copy $\mathcal{B}$ and $F : \mathcal{A} \cong \mathcal{B}$ so that $\neg F(R)$ is immune.

However, $\neg R$ is definable by a computable Σ_1 formula $\varphi(x)$ in $\{E, \mathbf{R}\}$ with only positive occurrences of $\mathbf{R}$. Namely, $\varphi(x)$ is the following finitary formula $\exists y(\mathbf{R}(y) \wedge yEx \wedge y \neq x)$. Therefore, in any copy $\mathcal{B}$ in which $F(R)$ is c.e. relative to $\mathcal{B}$, $F(R)$ is, in fact, computable relative to $\mathcal{B}$.

Example 3. Let $\mathcal{A}$ be a computable equivalence structure as in Example 2. Let R be a relation such that the following are satisfied:

i) there are infinitely many equivalence classes from which R contains exactly one element;

ii) there are no equivalence classes from which R contains both elements;

iii) there are infinitely many equivalence classes from which R contains neither element;

iv) the pair $(\mathcal{A}, R)$ satisfies the decidability condition of Theorem 4.1.

No infinite subset of $\neg R$ is definable by a computable Σ_1 formula (in the language $\{E\}$) with only finitely many parameters, so there is a computable copy $\mathcal{B}$ and $F : \mathcal{A} \cong \mathcal{B}$ in which $\neg F(R)$ is immune.

Furthermore, there is a computable copy $\mathcal{B}$ in which the image of R is c.e., but not computable. However, the formula $\varphi(x)$ in the language $\{E, \mathbf{R}\}$ given in Example 2 defines an infinite subset of $\neg R$. Consequently, there is no $F : \mathcal{A} \cong \mathcal{B}$ such that $F(R)$ is simple relative to $\mathcal{B}$.

Example 4. Let $\mathcal{A}$ be the structure $(\mathcal{Q}, <_Q)$, and let R be the set of all rationals less than π. There is no computable formula (in the language $\{<\}$) with finitely many parameters which defines $\neg R$. However, the formula "$5 < x$" does define an infinite subset of $\neg R$. Consequently, there is no $F : \mathcal{A} \cong \mathcal{B}$ in which $\neg F(R)$ is immune relative to $\mathcal{B}$.

5 Open Problems

We now recall some fundamental definitions from computability theory (for more information, see [12] and [10]). Let $X \subseteq \omega$. The set X is *cohesive* if it is infinite and for any infinite c.e. set W, only one of W, $\neg W$ has infinite intersection with X. A set is *maximal* if it is c.e. and its complement is cohesive. The set X is *hh-immune* if there is no computable function $f : \omega \to \omega$ such that $(W_{f(n)})_{n\in\omega}$ is a sequence of pairwise disjoint finite c.e. sets, each having non-empty intersection with X. A set is *hh-simple* if it is c.e. and its complement is *hh* -immune.

Problem 5 *Under what syntactic conditions is there an isomorphism F from $\mathcal{A}$ onto a computable copy such that $\neg F(R)$ is cohesive?*

Problem 6 *Under what syntactic conditions is there an isomorphism F from $\mathcal{A}$ onto a computable copy such that $F(R)$ is maximal?*

Problem 7 *Under what syntactic conditions is there an isomorphism F from $\mathcal{A}$ onto a computable copy such that $\neg F(R)$ is hh-immune?*

Problem 8 *Under what syntactic conditions is there an isomorphism F from $\mathcal{A}$ onto a computable copy such that $F(R)$ is hh-simple?*

As in Definition 1, we may define what it means for a new relation on the domain B of a countable structure $\mathcal{B}$ to be *cohesive relative to $\mathcal{B}$*, *maximal relative to $\mathcal{B}$*, *hh-immune relative to $\mathcal{B}$*, or *hh-simple relative to $\mathcal{B}$*. Thus, we have the following relative analogues of the above problems.

Problem 9 *Under what syntactic conditions is there an isomorphism F from $\mathcal{A}$ onto a copy $\mathcal{B}$ such that $\neg F(R)$ is cohesive relative to $\mathcal{B}$?*

Problem 10 *Under what syntactic conditions is there an isomorphism F from $\mathcal{A}$ onto a copy $\mathcal{B}$ such that $F(R)$ is maximal relative to $\mathcal{B}$?*

Problem 11 *Under what syntactic conditions is there an isomorphism F from $\mathcal{A}$ onto a copy $\mathcal{B}$ such that $F(R)$ is hh-immune relative to $\mathcal{B}$?*

Problem 12 *Under what syntactic conditions is there an isomorphism F from $\mathcal{A}$ onto a copy $\mathcal{B}$ such that $F(R)$ is hh-simple relative to $\mathcal{B}$?*

There are natural definability conditions necessary for the image of $\neg R$ to be cohesive. There should be no computable Σ_1 formula $\varphi(\vec{c}, x)$, in the language L, either defining $\neg R$, or true of infinitely many elements of $\neg R$ without being true of almost all of them.

There are also natural definability conditions necessary for the image of $\neg R$ to be maximal. They are the same as above except that the Σ_1 formula is in the language $L \cup \{\mathbf{R}\}$ with only positive occurrences of $\mathbf{R}$.

It turns out, as shown by the following examples, that these conditions are not sufficient.

Example 5. Let $\mathcal{A}$ be an $\aleph_0$-dimensional vector space over a finite field, say over a field with 3 elements. Let R be the domain of its subspace of infinite dimension and infinite co-dimension. There is a computable copy of $\mathcal{A}$ in which the image of R is immune, since the only sets definable in $\mathcal{A}$ are finite and co-finite, and there is a copy also satisfying the effectiveness condition of Theorem 4.1.

For $a \notin R$, the formula $\varphi(a, x) \equiv [(\exists y)[\, x = a + y]]$ defines an infinite subset of $\neg R$ that is c.e. (relative to $\mathcal{B}$) if the image of R is. It follows that the image of R can never be relatively simple, or relatively maximal.

We show that the image of $\neg R$ cannot be made relatively cohesive. In any copy $\mathcal{B}$, we consider the set W of elements a such that a is first (in the ordering of ω) in the subspace generated by a, excluding 0. The set W is computable relative to $\mathcal{B}$, and both W and $\neg W$ have infinite intersections with the image of R.

Example 6. Let $\mathcal{A}$ be an equivalence structure as in Examples 2 and 3. Let R consist of infinitely many equivalence classes, such that $\neg R$ also consists of infinitely many equivalence classes. There is a computable copy of $\mathcal{A}$ in which the image of $\neg R$ is immune. In fact, we can make the image of R simple. As in the previous example, the image of $\neg R$ cannot be made relatively cohesive, hence the image of R cannot be made relatively maximal. In a copy $\mathcal{B}$ of $\mathcal{A}$, let W be the set of elements that are first in their equivalence classes. Then W is computable relative to $\mathcal{B}$, and both W and $\neg W$ have infinite intersections with the image of $\neg R$.

Example 7. Let $\mathcal{A}$ and R be as in Example 2. We show that the image of $\neg R$ cannot be made relatively cohesive. For any copy $\mathcal{B}$ of $\mathcal{A}$, the set W of elements that are first in their equivalence classes is computable in $\mathcal{B}$. If the image of $\neg R$ were cohesive, then it would be almost equal to W or to $\neg W$, so it would be computable in $\mathcal{B}$.

References

[1] C. J. Ash, J. F. Knight, and J. B. Remmel, Quasi-simple relations in copies of a given recursive structure, *Annals of Pure and Applied Logic* 86 (1997), 203-218.

[2] C. Ash, J. Knight, M. Manasse and T. Slaman, Generic copies of countable structures, *Annals of Pure and Applied Logic* 42(1989), 195-205.

[3] J. Chisholm, Effective model theory vs. recursive model theory, *Journal of Symbolic Logic* 55 (1990), 1168-1191.

[4] V. S. Harizanov, Effectively nowhere simple relations on computable structures, in: M. M. Arslanov and S. Lempp, eds., *Recursion Theory and Complexity* (de Gruyter, Berlin, 1999), 59-70.

[5] V. S. Harizanov, Some effects of Ash-Nerode and other decidability conditions on degree spectra, *Annals of Pure and Applied Logic* 55 (1991), 51-65.

[6] G. Hird, Recursive properties of intervals of recursive linear orders, in: J. N. Crossley, J. B. Remmel, R. A. Shore, and M. E. Sweedler, eds., *Logical Methods* (Birkhäuser, Boston, 1993),422-437.

[7] G. R. Hird, Recursive properties of relations on models, *Annals of Pure and Applied Logic* 63 (1993), 241-269.

[8] J. F. Knight, "Degrees coded in jumps of orderings", *Journal of Symbolic Logic* 51 (1986), 1034-1042.

[9] G. Metakides and A. Nerode, Recursively enumerable vector spaces, *Annals of Mathematical Logic* 11 (1977), 147-171.

[10] A. Nerode and J. B. Remmel, A survey of lattices of r.e. substructures, in: A. Nerode and R. Shore, eds., *Recursion Theory,* Proceedings of Symposia in Pure Mathematics of the American Mathematical Society 42 (American Mathematical Society, Providence, 1985), 323-375.

[11] J. B. Remmel, Recursive isomorphism types of recursive Boolean algebras, *Journal for Symbolic Logic* 46 (1981), 572-594.

[12] R. I. Soare, *Recursively Enumerable sets and Degrees. A Study of Computable Functions and Computably Generated Sets*, Springer-Verlag, Berlin, 1987.