TURING DEGREES OF CERTAIN ISOMORPHIC IMAGES OF COMPUTABLE RELATIONS

VALENTINA S. HARIZANOV

This paper is dedicated to Chris Ash, who invented α-systems.

ABSTRACT. A model is computable if its domain is a computable set and its relations and functions are uniformly computable. Let $\mathcal{A}$ be a computable model and let R be an extra relation on the domain of $\mathcal{A}$. That is, R is not named in the language of $\mathcal{A}$. We define $Dg_{\mathcal{A}}(R)$ to be the set of Turing degrees of the images $f(R)$ under all isomorphisms f from $\mathcal{A}$ to computable models. We investigate conditions on $\mathcal{A}$ and R which are sufficient and necessary for $Dg_{\mathcal{A}}(R)$ to contain every Turing degree. These conditions imply that if every Turing degree $\leq \mathbf{0}''$ can be realized in $Dg_{\mathcal{A}}(R)$ via an isomorphism of the same Turing degree as its image of R, then $Dg_{\mathcal{A}}(R)$ contains every Turing degree. We also discuss an example of $\mathcal{A}$ and R whose $Dg_{\mathcal{A}}(R)$ coincides with the Turing degrees which are $\leq \mathbf{0}'$.

1. INTRODUCTION AND NOTATION

We consider only computable first-order languages and only countable models. Models are denoted by script letters, and their domains by the corresponding capital Latin letters. The isomorphism of models is denoted by $\cong$. Let $\mathcal{A}$ be a model. $L(\mathcal{A})$ is the language of $\mathcal{A}$. $L(\mathcal{A})_A$ is the language $L(\mathcal{A}) \cup \{\mathbf{a} : a \in A\}$. $\mathcal{A}_A$ is the expansion of $\mathcal{A}$ to the language $L(\mathcal{A})_A$ such that every $\mathbf{a}$ is interpreted by a. A basic sentence is an atomic sentence or the negation of an atomic sentence. The atomic diagram of $\mathcal{A}$ is the set of all basic sentences of $L(\mathcal{A})_A$ which are true in $\mathcal{A}_A$. Let α be a computable ordinal. Ash [1] has defined computable Σ_α and Π_α formulas of $L_{\omega_1\omega}$, recursively and simultaneously, and together with their Gődel numbers (because the indexing of formulas in infinite disjunctions and conjunctions will be by their Gődel numbers). The computable Σ_0 and Π_0 formulas are the finitary quantifier-free formulas. The computable $\Sigma_{\alpha+1}$ ($\Pi_{\alpha+1}$, respectively) formulas are computably enumerable disjunctions (conjunctions, respectively) of $\exists\Pi_\alpha$ ($\forall\Sigma_\alpha$, respectively) formulas. If α is a limit ordinal, then the Π_α (Σ_α, respectively) formulas are of the form $\bigvee_{n \in W} \theta_n$ ($\bigwedge_{n\in W} \theta_n$, respectively), where W is a computably enumerable set of natural numbers and there is a sequence $(\alpha_n)_{n\in W}$ of ordinals having limit α, given by the ordinal notation for α, such that θ_n is a Σ_{α_n} (Σ_{α_n}, respectively) formula. For a more precise definition of computable Σ_α and Π_α formulas see [1]. A sequence of variables displayed after a formula contains all free variables occurring in the formula.

A model $\mathcal{A}$ is computable if its domain A is a computable set and the relations and functions of $\mathcal{A}$ are uniformly computable. Equivalently, $\mathcal{A}$ is a computable

This paper has been supported by the George Washington University Facilitating Fund. I thank Doug Cenzer for very useful discussions.

model if A is computable and the atomic diagram of $\mathcal{A}$ is computable. That is, A is computable and there is a computable enumeration $(a_i)_{i\in\omega}$ of A and an algorithm which determines for every quantifier-free formula $\theta(x_{i_0},\dots,x_{i_{n-1}})$ in $L(\mathcal{A})$ and for every sequence $(a_{i_0},\dots,a_{i_{n-1}})\in A^n$, whether $\mathcal{A}_A\vDash\theta(\mathbf{a}_{i_0},\dots,\mathbf{a}_{i_{n-1}})$.

Let R be an additional relation on the domain of a computable model $\mathcal{A}$. That is, R is not named in $L(\mathcal{A})$. For simplicity, we assume that R is unary. (However, all definitions introduced and results established can be easily extended to relations of arbitrary arity.) For various computability-theoretic complexity classes $\mathcal{P}$, Ash and Nerode and others have investigated syntactic conditions on $\mathcal{A}$ and R under which for every isomorphism f from $\mathcal{A}$ onto a computable model $\mathcal{B}$, $f(R)\in\mathcal{P}$. Such relations R are called *intrinsically* $\mathcal{P}$ on $\mathcal{A}$. For example, Ash and Nerode [5] have established that, under some extra decidability condition on $\mathcal{A}$ (which involves R), R is intrinsically c.e. if and only if R is definable by a computable Σ_1 formula with finitely many parameters. Barker [6] has extended this result to every computable ordinal $\alpha\geq 2$. He has established that, under certain extra decidability conditions on $\mathcal{A}$, R is intrinsically Σ^0_α on $\mathcal{A}$ if and only if R is definable by a computable Σ_α formula with finitely many parameters. In the previous results, the extra decidability conditions are only needed to show that the corresponding syntactic conditions are necessary. We [8] have defined the (*Turing*) *degree spectrum* of R on $\mathcal{A}$, in symbols $Dg_{\mathcal{A}}(R)$, to be the set of all Turing degrees of the images of R under all isomorphisms from $\mathcal{A}$ onto computable models. For a computable model $\mathcal{B}$ such that $\mathcal{B}\cong\mathcal{A}$, the (Turing) degree spectrum of R on $\mathcal{A}$ with respect to $\mathcal{B}$, in symbols $Dg_{\mathcal{A},\mathcal{B}}(R)$, is the set of all Turing degrees of the images $f(R)\subseteq B$ under all isomorphisms f from $\mathcal{A}$ to $\mathcal{B}$. In [8] we have studied uncountable degree spectra, and have established conditions which are sufficient for $Dg_{\mathcal{A}}(R)$ to contain all Turing degrees. Here we prove that these conditions are necessary. For another, independent proof, see [2].

The computability-theoretic notation is standard and as in [12]. We review some of it. By D_x we denote the finite set of natural numbers whose canonical index is x. Thus, $D_0=\emptyset$. If φ is a partial function, then $dom(\varphi)$ is the domain of φ, $rng(\varphi)$ is the range of φ, and $\varphi(a)\downarrow$ denotes that $a\in dom(\varphi)$. The concatenation of sequences is denoted by $\hat{}$. We often identify a set X with its characteristic function χ_X. We fix $\langle\cdot,\cdot\rangle$ to be a computable bijection from ω^2 onto ω. Let $X\subseteq\omega$. Then $\varphi^X_0,\varphi^X_1,\varphi^X_2,\dots$ is a fixed effective enumeration of all unary X-computable functions. φ^X_e is also denoted by $\{e\}^X$. We write $\varphi^X_{e,s}(n)=m$ if $e,n,m<s$, only numbers $z<s$ are used in the computation, and $\varphi^X_e(n)=m$ in fewer than s steps. Let $p\in 2^{<\omega}$. We write $\varphi^p_{e,s}(n)=m$ if $\varphi^X_{e,s}(n)=m$ for some $X\supset p$ and only elements in $dom(p)$ are used in the computation. Let $Y\subseteq\omega$. The join $X\oplus Y$ is $\{2n:n\in X\}\cup\{2n+1:n\in Y\}$. By $X\leq_T Y$ ($X\equiv_T Y$, respectively) we denote that X is Turing reducible to Y (X is Turing equivalent to Y, respectively). $X<_T Y$ denotes that $X\leq_T Y$ but $Y\not\leq_T X$. $\mathbf{x}=\deg(X)$ is the Turing degree of X. Hence $\mathbf{0}=\deg(\emptyset)$ and $\mathbf{x}^{(n)}=\deg(X^{(n)})$, where $X^{(n)}$ is the n-th jump of X. A Turing degree is c.e. (Δ^0_2, respectively) if it contains a c.e. (Δ^0_2, respectively) set. The set of all Turing degrees is denoted by $\mathcal{D}$. A binary function $f:\omega^2\to\omega$ is called selective if for every $x,y\in\omega$, $f(x,y)\in\{x,y\}$. X is a *semirecursive* set if there is a selective computable function such that if exactly one of x,y belongs to X, then $f(x,y)$ selects the element in X. An example of a semirecursive set is the deficiency set of a non-computable c.e. set for a 1-1 computable enumeration.

2. Realizing every Turing degree in a degree spectrum

Let $\mathcal{A}$ be a computable model and let R be an extra relation on the domain A of $\mathcal{A}$. As mentioned before, we will assume, without loss of generality, that R is unary. Let a computable model $\mathcal{B}$ be such that $\mathcal{A} \cong \mathcal{B}$. By $\mathcal{I}(\mathcal{A}, \mathcal{B})$ we denote the set of all isomorphisms from $\mathcal{A}$ to $\mathcal{B}$. We say that a partial function p from A to B is a *finite isomorphism* from $\mathcal{A}$ to $\mathcal{B}$ if p is $1-1$, $dom(p)$ is finite and for every atomic formula $\alpha = \alpha(x_0, \dots, x_{n-1})$ in $L(\mathcal{A})$, and every $a_0, \dots, a_{n-1} \in dom(p)$, we have

$$\mathcal{A}_A \models \alpha(\mathbf{a}_0, \dots, \mathbf{a}_{n-1}) \Leftrightarrow \mathcal{B}_B \models \alpha(\mathbf{b}_0, \dots, \mathbf{b}_{n-1}).$$

where $b_0 = (a_0), \dots, b_{n-1} = p(a_{n-1})$. By $\mathcal{I}_{fin}(\mathcal{A}, \mathcal{B})$ we denote the set of all finite isomorphisms from $\mathcal{A}$ to $\mathcal{B}$. In [8] we have defined the R-equivalence relation $\sim_R$ on $\mathcal{I}_{fin}(\mathcal{A}, \mathcal{B})$ as follows:

$$q \sim_R r \Longleftrightarrow (\forall b \in ran(q) \cap ran(r))[q^{-1}(b) \in R \Leftrightarrow r^{-1}(b) \in R].$$

Equivalently,

$$q \sim_R r \Longleftrightarrow (\forall b \in ran(q) \cap ran(r))[b \in q(R) \Leftrightarrow b \in r(R)].$$

Since for every Turing degree $\mathbf{x}$, there are at most countably many Turing degrees which are $\leq \mathbf{x}$, and since every countable set of Turing degrees has an upper bound, a set of Turing degrees is uncountable if and only if it is unbounded.

Theorem 2.1. (*Harizanov* [8]) (*i*) *The following are equivalent:*
(0) *$Dg_{\mathcal{A}}(R)$ is uncountable.*
(1) *$Dg_{\mathcal{A},\mathcal{B}}(R)$ is uncountable.*
(2) *$Dg_{\mathcal{A},\mathcal{B}}(R)$ has cardinality 2^ω.*
(3) *There is a nonempty set $\mathbb{S} \subseteq \mathcal{I}_{fin}(\mathcal{A}, \mathcal{B})$ such that the following two conditions are satisfied:*

$$(A)\ (\forall p \in \mathbb{S})(\forall a \in A)(\forall b \in B)(\exists q \in \mathbb{S})[q \supseteq p \wedge a \in dom(q) \wedge b \in ran(q)];$$

$$(B)\ (\forall p \in \mathbb{S})(\exists q, r \in \mathbb{S})[q \supseteq p \wedge r \supseteq p \wedge \neg(q \sim_R r)].$$

(*ii*) *Let $\mathbb{S}$ be as in* (3). *Then for every set $C \geq_T \mathbb{S}$, there is an isomorphism f from $\mathcal{A}$ to $\mathcal{B}$ such that*

$$C \equiv_T f(R) \oplus \mathbb{S} \equiv_T f \oplus \mathbb{S}.$$

In particular, if $\mathbb{S}$ is computable (*or c.e.*), *then $Dg_{\mathcal{A},\mathcal{B}}(R) = \mathcal{D}$ and, moreover, for every set $C \subseteq \omega$, there is an isomorphism f from $\mathcal{A}$ to $\mathcal{B}$ such that*

$$C \equiv_T f(R) \equiv_T f.$$

In [8], we have also given examples of uncountable degree spectra $Dg_{\mathcal{A},\mathcal{B}}(R)$ such that $Dg_{\mathcal{A},\mathcal{B}}(R) \neq \mathcal{D}$. Now we further investigate degree spectra which coincide with $\mathcal{D}$. The following example motivates the theorem that follows it.

Clearly, $\mathcal{Q} = (Q, \leq)$, where Q is the set of all rational numbers, is a computable model. $X \subseteq Q$ is an initial segment of $\mathcal{Q}$ if

$$\forall a, b \in Q[(a \in X \wedge b \leq a) \Rightarrow b \in X].$$

Example 2.1. *Every Turing degree contains an initial segment of $\mathcal{Q}$. That is, if $R = \{q \in Q : q < \sqrt{2}\}$, then $Dg_{\mathcal{Q},\mathcal{Q}}(R) = \mathcal{D}$.*

Proof. Let C be an arbitrary infinite coinfinite set of natural numbers. We will show that there is an initial segment X of $\mathcal{Q}$ of the same Turing degree as C. We define a real number r_C by

$$r_C = \sum_{n \in C} \frac{1}{2^n}.$$

Let X be the initial segment of $\mathcal{Q}$ determined by r_C. That is, $X = \{q \in Q : q < r_C\}$.

First, let us prove that $C \leq_T X$. By transfinite induction on k, we will show that we can X-computably determine whether $k \in C$. Assume that we can determine, computably in X, $C \cap \{0, \dots, k-1\}$. Then we can find, computably in X, $\sum\limits_{n \in C \cap \{0,\dots,k-1\}} \frac{1}{2^n}$. If $k \in C$, then, since C is infinite, $(\sum\limits_{n \in C \cap \{0,\dots,k-1\}} \frac{1}{2^n}) + \frac{1}{2^k} < r_C$. Conversely, if $(\sum\limits_{n \in C \cap \{0,\dots,k-1\}} \frac{1}{2^n}) + \frac{1}{2^k} < r_C$, then, since C is coinfinite and $\frac{1}{2^k} = \frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \dots$, we conclude that $k \in C$. Hence

$$k \in C \Leftrightarrow (\sum_{n \in C \cap \{0,\dots,k-1\}} \frac{1}{2^n}) + \frac{1}{2^k} \in X.$$

Thus, we can determine, computably in X, whether $k \in C$.

Now, let us prove that $X \leq_T C$. We will establish the following equivalence

$$q \in X \Leftrightarrow \exists n_0[\sum_{n \in C \cap \{0,\dots,n_0\}} \frac{1}{2^n} \geq q].$$

The implication $\Leftarrow$ is clear. Conversely, if $\forall n_0[\sum\limits_{n \in C \cap \{0,\dots,n_0\}} \frac{1}{2^n} < q]$, then $r_C \leq q$, so $q \notin X$.

If $q > r_C$, then $\exists n_0[q - r_C > \frac{1}{2^{n_0}}]$, hence $[q - \sum\limits_{n \in C \cap \{0,\dots,n_0\}} \frac{1}{2^n}] > \frac{1}{2^{n_0}}$. Conversely, if $[q - \sum\limits_{n \in C \cap \{0,\dots,n_0\}} \frac{1}{2^n}] > \frac{1}{2^{n_0}}$, then, since C is coinfinite, we conclude that $q - r_C > 0$. Therefore, for $q \neq r_C$,

$$q \notin X \Leftrightarrow \exists n_0[q - \sum_{n \in C \cap \{0,\dots,n_0\}} \frac{1}{2^n} > \frac{1}{2^{n_0}}].$$

Hence, to decide for a given $q \in Q$, computably in C, whether $q \in X$, we search for n_0 such that either

$$\sum_{n \in C \cap \{0,\dots,n_0\}} \frac{1}{2^n} \geq q$$

or

$$[q - \sum_{n \in C \cap \{0,\dots,n_0\}} \frac{1}{2^n}] > \frac{1}{2^{n_0}}.$$

∎

Theorem 2.2. *The following are equivalent:*
(1) $Dg_{\mathcal{A},\mathcal{B}}(R)$=$\mathcal{D}$ *and, moreover, for every set* $C \subseteq \omega$*, there is an isomorphism* f

from $\mathcal{A}$ *to* $\mathcal{B}$ *such that* $C \equiv_T f(R) \equiv_T f$.
(2) *There is* $e \in \omega$ *and* $p \in 2^{<\omega}$ *such that the set*

$$\mathbb{S}_{e,p} =_{def} \{\varphi_e^q : q \in 2^{<\omega} \wedge q \supseteq p\}$$

has the following properties:

$$\mathbb{S}_{e,p} \subseteq \mathcal{I}_{fin}(\mathcal{A}, \mathcal{B}),$$

$$(A) \text{ from Theorem 2.1 is satisfied for } \mathbb{S} = \mathbb{S}_{e,p}, \text{ and}$$

$$(\exists i \in \omega)(\forall q \supseteq p)(\forall a \in dom(q))[\varphi_i^{\varphi_e^q(R)}(a) \downarrow = q(a)].$$

(3) *There is a nonempty computable (or c.e.) set* $\mathbb{S} \subseteq \mathcal{I}_{fin}(\mathcal{A}, \mathcal{B})$ *such that the conditions* (A) *and* (B) *from Theorem 2.1 are satisfied.*

Proof. $\neg(2) \Rightarrow \neg(1)$ Assume the negation of (2). That is, for every $\langle e, i\rangle$ and every $p \in 2^{<\omega}$, there is $q \in 2^{<\omega}$ such that $q \supseteq p$ and

$$\begin{array}{rcl} (i)\ \varphi_e^q & \notin & \mathcal{I}_{fin}(\mathcal{A}, \mathcal{B}) \text{ or} \\ (ii)\ (\exists a & \in & A)(\forall r \supseteq q)[a \notin dom(\varphi_e^r)] \text{ or} \\ (iii)\ (\exists b & \in & B)(\forall r \supseteq q)[b \notin ran(\varphi_e^r)] \text{ or} \\ (iv)\ (\exists a & \in & dom(q))[\varphi_i^{\varphi_e^q(R)}(a) \downarrow \neq q(a)]. \end{array}$$

We will now use a finite extension argument to construct the characteristic function of a set $C \subseteq \omega$ which satisfies the following requirement for every $\langle e, i\rangle$:

$$R_{\langle e,i\rangle} : \varphi_e^C \in \mathcal{I}(\mathcal{A}, \mathcal{B}) \Rightarrow \varphi_i^{\varphi_e^C(R)} \neq C.$$

Construction

Let $p_{-1} =_{def} \emptyset$.

Stage s. Let $s = \langle e, i\rangle$. We have already constructed $p_{s-1} \in 2^{<\omega}$. Let q be the least binary sequence such that $q \supseteq p_{s-1}$ and one of the conditions (i)-(iv) is satisfied. Let $p_s =_{def} q$. End of construction.

Let $C \subseteq \omega$ be such that $\chi_C = \bigcup_{s \geq -1} p_s$. Hence, for $f \in \mathcal{I}(\mathcal{A}, \mathcal{B})$, if $f \leq_T C$, that is, if $f = \varphi_e^C$ for some $e \in \omega$, then $\neg(C \leq_T f(R))$. Let $\mathbf{c} = \deg(C)$. Thus, $\mathbf{c}$ can not be realized in $Dg_{\mathcal{A},\mathcal{B}}(R)$ via an isomorphism of degree $\mathbf{c}$.

(2) $\Rightarrow$ (3) Fix the corresponding e and p. By assumption, $\mathbb{S}_{e,p} \subseteq \mathcal{I}_{fin}(\mathcal{A}, \mathcal{B})$ and (A) is satisfied for $\mathbb{S} = \mathbb{S}_{e,p}$. Let us show that (B) is also satisfied for $\mathbb{S} = \mathbb{S}_{e,p}$. Fix the corresponding $i \in \omega$. Let $p_1 \in 2^{<\omega}$ be such that $p_1 \supseteq p$. Now, choose binary sequences q and r such that $q \supseteq p_1$, $r \supseteq p_1$, and

$$(\exists a \in dom(q) \cap dom(r))[q(a) \neq r(a)].$$

Then

$$\varphi_i^{\varphi_e^q(R)}(a) \downarrow \neq \varphi_i^{\varphi_e^r(R)}(a) \downarrow .$$

Hence

$$\exists b[b \in \varphi_e^q(R) \Leftrightarrow b \notin \varphi_e^r(R)].$$

Thus, $\neg(\varphi_k^q \sim_R \varphi_k^r)$.

(3) $\Rightarrow$ (1) This is already proven in [8] (see (ii) of Theorem 2.1). ∎

The equivalence of (1) and (3) in Theorem 2.2 has also been established independently by Ash, Cholak and Knight in [2]. Their proof uses the forcing method.

Remark 2.1. *In the proof of* $\neg(2) \Rightarrow \neg(1)$ *for Theorem 2.2, the construction of* C *can be done computably in* $\emptyset''$*. Hence* $C \in \Delta^0_3$*. Thus, if not every Turing degree is obtained in a degree spectrum* $Dg_{\mathcal{A},\mathcal{B}}(R)$ *via an isomorphism of the same Turing degree, then there is such a* Δ^0_3 *degree. This conclusion also follows from the proof in* [2] *since there is a generic* Δ^0_3 *set.*

3. Realizing Δ^0_2 degrees in a degree spectrum

In [9] we have given a general condition for $\mathcal{A}$ and R which is sufficient for every c.e. degree to be realized in $Dg_{\mathcal{A}}(R)$ via a c.e. set of the same Turing degree as the corresponding isomorphism. This condition is satisfied by the following model $\mathcal{A}_0$ and relation R_0.

Let $\mathcal{A}_0 = (\omega, \prec)$ be the following computable linear order of order type $\omega + \omega^*$:

$$0 \prec 2 \prec 4 \prec \ldots \prec 5 \prec 3 \prec 1.$$

A computable relation R_0 is the initial segment of type ω; that is, $R_0 = 2\omega$.

Hence every c.e. degree can be realized in $Dg_{\mathcal{A}_0}(R_0)$ via a c.e. set of the same Turing degree as the corresponding isomorphism. It is easy to see that R_0 is intrinsically Δ^0_2 on $\mathcal{A}$, because it satisfies the syntactic condition in [6]. Namely,

$$x \in R_0 \Leftrightarrow \bigvee_{n\in\omega} \exists x_0 \ldots \exists x_n [x_0 \prec x_1 \prec \ldots \prec x_n \wedge x = x_n \wedge$$
$$\forall y[\neg(y \prec x_0) \wedge \neg(x_0 \prec y \prec x_1) \wedge \ldots \wedge \neg(x_{n-1} \prec y \prec x_n)],$$

and

$$x \notin R_0 \Leftrightarrow \bigvee_{n\in\omega} \exists x_0 \ldots \exists x_n [x_0 \succ x_1 \succ \ldots \succ x_n \wedge x = x_n \wedge$$
$$\forall y[\neg(y \succ x_0) \wedge \neg(x_0 \succ y \succ x_1) \wedge \ldots \wedge \neg(x_{n-1} \succ y \succ x_n)].$$

Ash, Cholak and Knight [2] have extended the sufficient condition in [9] to the α-th level in Ershov's classification of Δ^0_2 degrees, where α is any fixed computable ordinal. A Turing degree is α-c.e. if it contains an α-c.e. set. A set $C \subseteq \omega$ is α-c.e. if there is a computable function $f : \omega^2 \to \{0, 1\}$ and a computable function $o : \omega^2 \to \{\beta : \beta \text{ is an ordinal} \wedge \beta \leq \alpha\}$ with the following properties:

$$(\forall x)[\lim_{s\to\infty} f(x, s) = C(x) \wedge\ f(x, 0) = 0],$$

$$(\forall x)(\forall s)[o(x, s + 1) \leq o(x, s) \wedge o(x, 0) = \alpha], \text{ and}$$

$$(\forall x)(\forall s)[f(x, s + 1) \neq f(x, s) \Rightarrow o(x, s + 1) < o(x, s)].$$

In particular, 1-c.e. sets are c.e. sets, and 2-c.e. sets are d-c.e. sets. For other equivalent definitions of α-c.e. sets, see [7] and [4]. Epstein, Haas and Kramer [7] have shown that some levels in Ershov's hierarchy are notation-dependent, and that for every Δ^0_2 set X, there is an ordinal notation in which X is ω^2- c.e. Ash and Knight [4] have given a syntactic condition which is, under appropriate decidability conditions, sufficient and necessary for R to be intrinsically α-c.e. on $\mathcal{A}$. As a corollary, they have shown that for every computable ordinal α, R_0 is not intrinsically

α-c.e. on $\mathcal{A}_0$. This result also follows from the following proposition because for a fixed ordinal notation, the α-c.e. degrees form a proper hierarchy (see Theorem 9 in [7]).

Proposition 3.1. *$Dg_{\mathcal{A}_0}(R_0)$ consists of all Δ^0_2 degrees.*

Proof. (1) Jockusch (Theorem 5.2 in [11]), has established that every nonzero Turing degree computable in $\mathbf{0}'$ contains a semirecursive set which is both immune and coimmune. However, a set of natural numbers is semirecursive if and only if it is an initial segment of a computable linear ordering on ω (see Theorem 4.1 in [11]). Let $\mathbf{c}$ be an arbitrary nonzero Δ^0_2 degree. Hence there is a computable linear ordering $\mathcal{B} = (\omega, \prec_{\mathcal{B}})$ and an initial segment X on $\mathcal{B}$ such that $deg(X) = \mathbf{c}$ and X is immune and coimmune. Since X is immune, no element of X can have infinitely many predecessors. Similarly, no element of $\omega - X$ can have infinitely many successors. Thus, the order type of $\mathcal{B}$ is $\omega + \omega^*$, and X is the ω-part of $\mathcal{B}$. In other words, there is an isomorphism f from $\mathcal{A}_0$ to $\mathcal{B}$ such that $f(R_0) = X$. Therefore, we conclude that $DgSp_{\mathcal{A}_0}(R_0)$ is the set of all Δ^0_2 degrees. ∎

We will also give a direct proof by constructing a computable model $\mathcal{B}$ isomorphic to $\mathcal{A}_0$ and a corresponding isomorphism. In the proof, we will consider binary trees. Such trees can be viewed as growing downward from the top node $\emptyset$. Let $\nu, \mu \in 2^{<\omega}$. As usual, we say that ν is to the left of μ, in symbols $\nu <_L \mu$, if

$$\exists \gamma \in 2^{<\omega}[\gamma\hat{\ }0 \subseteq \nu \wedge \gamma\hat{\ }1 \subseteq \mu].$$

We have the following partial ordering on $2^{<\omega}$:

$$\nu < \mu \Leftrightarrow_{def} (\nu <_L \mu \vee \nu \subsetneq \mu).$$

Let $C \subseteq \omega$. We write $\nu <_L C$ if for $\gamma = C(0)\hat{\ }C(1)\hat{\ }\ldots\hat{\ }C(lh(\nu)-1)$, we have $\nu <_L \gamma$. We similarly define $C <_L \nu$ and $\nu < C$. Let r_C be defined as in Example 2.1. Notice that if C is infinite and coinfinite then $(\forall x \in \omega)[\sum_{n \in D_x} \frac{1}{2^n} \neq r_C]$. Jockusch [11] has defined an infinite and coinfinite set $C \subseteq \omega$ to be *strongly non-c.e.* if neither the set $\{x \in \omega : \sum_{n \in D_x} \frac{1}{2^n} < r_C\}$ is c.e. nor the set $\{x \in \omega : \sum_{n \in D_x} \frac{1}{2^n} > r_C\}$ is c.e. Jockusch [11] has established that every nonzero Turing degree contains a strongly non-c.e. set.

Let $p \in A^m$ for some $m \in \omega$, and let $\alpha = \alpha(x_0, \ldots, x_{m-1})$ be a formula. We say that p satisfies α in $\mathcal{A}$ if

$$\mathcal{A} \models \alpha(x_0, \ldots, x_{m-1})[p(0), \ldots, p(m-1)].$$

Proof. (2) We will construct a computable model $\mathcal{B}$ isomorphic to $\mathcal{A}_0$. Let the domain B be ω. Let $\mathbf{c}$ be a nonzero Δ^0_2 degree. We choose a strongly non-c.e. set $C \subseteq \omega$ such that $deg(C) = \mathbf{c}$. Let $h : \omega^2 \to \{0, 1\}$ be a computable function which approximates C, that is,

$$(\forall n \in \omega)[C(n) = \lim_{s \to \infty} h(n, s)].$$

Now we define the following computable binary tree

$$T = \{h(0, s)\hat{\ }h(1, s)\hat{\ }\ldots\hat{\ }h(n, s) : n \leq s \wedge s \in \omega\} \cup \{\emptyset\}.$$

For every $s \in \omega$, T has exactly one maximal branch of length $s + 1$:

$$\nu_s = h(0, s)\hat{\ }h(1, s)\hat{\ }\ldots\hat{\ }h(s, s).$$

At every stage s of the construction, we define a finite isomorphism $p_s : \{0, 1, ..., s\} \to A_0$. The function p_s has the following properties (*):

$$(\forall n \in \omega)[(2n+2 \in ran(p_s) \Rightarrow 2n \in ran(p_s)) \wedge$$
$$(2n+1 \in ran(p_s) \Rightarrow 2n-1 \in ran(p_s))]$$

$$(\forall n, m \in \{0, 1, \dots, s-1\})[\nu_n < \nu_m \Rightarrow p_s(n) \prec p_s(m)], \text{ and}$$

$$(\forall n \in \{0, 1, \dots, s-1\})[(\nu_n < \nu_s \Rightarrow p_s(n) \in R_0) \wedge (\nu_s <_L \nu_n \Rightarrow p_s(n) \in \overline{R}_0)].$$

Construction

Stage 0. Let $p_0 =_{def} \{(0, a)\}$, where a is the least element in R_0 if $\nu_0 < \nu_1$, and the least element in $\overline{R}_0$ if $\nu_1 <_L \nu_0$.

Stage $s > 0$. We have $p_{s-1} : \{0, 1, ..., s-1\} \to A_0$, satisfying the above properties (*), and a finite part $\mathcal{B}_{s-1}$ of the atomic diagram of $\mathcal{B}$, which involves constants $0, 1, ..., s-1$ and is determined by p_{s-1} and $\mathcal{A}_0$.

Let $n < s-1$ be the least number (if it exists, otherwise let $q =_{def} p_{s-1}$) such that $\nu_s <_L \nu_n < \nu_{s-1}$ or $\nu_{s-1} <_L \nu_n < \nu_s$. We change p_{s-1} into the corresponding q with the same domain as p_{s-1} such that $(\forall m < n)[q(m) = p_{s-1}(m)]$, q preserves $\mathcal{B}_{s-1}$, and satisfies conditions (*). Let

$$p_s = q \cup \{(s, a)\},$$

where a is the least element in $R_0 - ran(q)$ if $\nu_{s-1} < \nu_s$, and a the least element in $\overline{R}_0 - ran(q)$ if $\nu_s <_L \nu_{s-1}$.

Let $\mathcal{B}_s$ be the set of all basic sentences with Gődel number $\leq s$, involving constants $0, 1, ..., s$, which is satisfied by p_s in $\mathcal{A}_0$. Note that $\mathcal{B}_{s-1} \subseteq \mathcal{B}_s$. End of the construction.

Let the atomic diagram of $\mathcal{B}$ be $\bigcup_{s \geq 0} \mathcal{B}_s$. Thus, $\mathcal{B}$ is a computable model. Fix $n \in \omega$. Let s_n be the least number such that $s_n \geq n$ and

$$(\forall m \leq n)(\forall s \geq s_n)[h(m, s) = h(m, s_n) = C(m)].$$

Hence

$$(\forall s \geq s_n)[p_s(n) = p_{s_n}(n)].$$

We define

$$f(n) = p_{s_n}(n).$$

f is a 1-1 function from B to A_0.

Lemma 3.2. *f is onto A_0.*

Proof. Assume inductively that $0, 1, \dots, j-1 \in ran(f)$. We will prove that $j \in ran(f)$. Let $f(n_i) = i$ for $i < j$. Let $n = \max\{n_0, n_1, \dots, n_{j-1}\}$ and let $t_0 = s_n$. Hence for every $s \geq t_0$, ν_s extends $C(0)\hat{}C(1)\hat{}\dots\hat{}C(n)$.

Case: $j \in R$.

We claim that there exists $s' \geq t_0$ such that $(\forall s > s')[\nu_{s'} < \nu_s]$. Otherwise, we can effectively enumerate an infinite sequence of stages $t_0 < t_1 < t_2 < \dots$ such that for every $i \in \omega$, $\nu_{t_{i+1}} <_L \nu_{t_i}$. Since h approximates C, we conclude that $(\forall i \in \omega)[C <_L \nu_{t_i}]$. Hence for every $x \in \omega$,

$$(\sum_{n \in D_x} \frac{1}{2^n} > r_C) \Leftrightarrow (\exists i \in \omega)[\chi_{D_x} \geq \nu_{t_i}].$$

Thus, the set $\{x \in \omega : \sum_{n \in D_x} \frac{1}{2^n} > r_C\}$ is c.e., contradicting the fact that C is strongly non-c.e.

We now choose the least stage s' with the property described above. It follows from the construction that $j \in ran(p_{s'+1})$ and that

$$(\forall s > s'+1)[p_s^{-1}(j) = p_{s'+1}^{-1}(j)].$$

Hence $a_j \in ran(f)$.

Case: $a_j \in \overline{R}_0$.

As in the previous case, we prove that there exists $s' \geq t_0$ such that $(\forall s \geq s')[\nu_s <_L \nu_{s'}]$. For the least such s', it follows from the construction that

$$(\forall s > s'+1)[p_s^{-1}(j) = p_{s'+1}^{-1}(j)].$$

Hence $j \in ran(f)$. ∎

Lemma 3.3. $f^{-1}(R_0) \equiv_T C$

Proof. Let $X = f^{-1}(R_0)$. It follows by construction that

$$X = \{n \in \omega : \nu_n < C\}.$$

Hence

$$X \leq_T C.$$

We now prove, by induction, that $C \leq_T X$. To determine whether $k \in C$, we assume that we can find σ using oracle X, where

$$\sigma = C(0)\hat{}C(1)\hat{}\ldots\hat{}C(k-1).$$

Then

$$k \in C \Leftrightarrow (\exists n \in X)[\sigma\hat{}(1) \subseteq \nu_n].$$

Equivalently,

$$k \notin C \Leftrightarrow (\exists n \in \overline{X})[\sigma\hat{}(0) \subseteq \nu_n].$$

∎

Hird [10] has shown that there is a computable copy of $\mathcal{A}_0$ in which the initial segment of type ω is h-simple. However, Jim Owings (unpublished) has observed that every deficiency set of a non-computable c.e. set for a 1-1 computable enumeration is the initial segment of type ω of some computable linear order isomorphic to $\mathcal{A}_0$. That is because every such deficiency set is semirecursive, immune and coimmune. Hence for every c.e. non-computable set C, there is a computable copy of $\mathcal{A}_0$ in which the initial segment of type ω is h-simple and Turing equivalent to C. This conclusion has also been obtained for simple initial segments by Ash, Knight and Remmel in [3], as an example of their general result for the so-called quasi-simple relations on computable models. These simple sets are automatically h-simple because semirecursive immune sets are h-immune. On the other hand, such sets cannot be hh-simple because no semirecursive set can be hh-immune (see [11]). Hird [10] has also established that no interval of a computable linear order is hh-immune.

References

[1] C. J. Ash, Recursive labelling systems and stability of recursive structures in hyperarithmetical degrees, Transactions of the American Mathematical Society 298 (1996), 497-514.

[2] C. J. Ash, P. Cholak and J. F. Knight, Permitting, forcing, and copies of a given recursive relation, to appear in Annals of Pure and Applied Logic.

[3] C. J. Ash, J. F. Knight and J. B. Remmel, Quasi-simple relations in copies of a given recursive structure, to appear in Annals of Pure and Applied Logic.

[4] C. J. Ash and J. F. Knight, Recursive structures and Ershov's hierarchy, Logic Paper No. 82 (1995), Monash University.

[5] C. J. Ash and A. Nerode, Intrinsically recursive relations, in: J. N. Crossley, editor, Aspects of Effective Algebra (U.D.A. Book Co., Steel's Creek, Australia, 1981), 26-41.

[6] E. Barker, Intrinsically Σ^0_α relations, Annals of Pure and Applied Logic 39 (1988), 105-130.

[7] R. L. Epstein, R. Haas and R. L. Kramer, Hierarchies of sets and degrees below $\mathbf{0}'$, in: J. H. Schmerl, M. Lerman and R. I. Soare, editors, Logic Year 1979-1980: University of Connecticut (Lecture Notes in Mathematics 859, Springer-Verlag, Berlin-Heidelberg-New York, 1987), 32-48.

[8] V. S. Harizanov, Uncountable degree spectra, Annals of Pure and Applied Logic 54 (1991), 255-263.

[9] V. S. Harizanov, Some effects of Ash-Nerode and other decidability conditions on degree spectra, Annals of Pure and Applied Logic 55 (1991), 51-65.

[10] G. Hird, Recursive properties of intervals of recursive linear orders, in: J. N. Crossley, J. B. Remmel, R. A. Shore, and M. E. Sweedler, editors, Logical Methods (Birkhāuser Boston, 1993), 422-437.

[11] C. G. Jockusch, Jr., Semirecursive sets and positive reducibility, Transactions of the American Mathematical Society 131 (1968), 420-536.

[12] R. I. Soare, Recursively Enumerable Sets and Degrees (Springer-Verlag, Berlin, 1987).

Department of Mathematics, The George Washington University, Washington, D. C. 20052

E-mail address: val@@math.gwu.edu