

EFFECTIVELY NOWHERE SIMPLE RELATIONS ON COMPUTABLE STRUCTURES

VALENTINA S. HARIZANOV

ABSTRACT. Let \mathcal{A} be a computable structure and let R be an additional relation on its domain. The notion of “quasi-simplicity” of R on \mathcal{A} , first studied by G. Hird, is analogous to the computability-theoretic notion of simplicity, given the definability of various subrelations of $\neg R$. In the present paper, we define corresponding versions of the notions “nowhere simple” and “effectively nowhere simple.” We establish a sufficient condition for existence of noncomputable effectively nowhere simple relations on a restricted class of computable structures.

1. INTRODUCTION AND NOTATION

We will denote structures by script letters, and their domains by the corresponding capital Latin letters. Let \mathcal{A} be a structure for L . For $X \subseteq A$, let L_X be the language $L \cup \{\mathbf{a} : a \in X\}$, L expanded by adding a constant \mathbf{a} for every $a \in X$. Let $\mathcal{A}_X = (\mathcal{A}, a)_{a \in X}$ be the expansion of \mathcal{A} to the language L_X such that for every $a \in X$, \mathbf{a} is interpreted by a . The *atomic diagram* of \mathcal{A} is the set of all atomic and negated atomic sentences of L_A which are true in \mathcal{A}_A . A structure \mathcal{A} is computable if its domain A is a computable subset of ω and its relations and operations are uniformly computable. That is, \mathcal{A} is computable if its atomic diagram is computable.

A structure \mathcal{B} isomorphic to a computable structure \mathcal{A} is not necessarily computable. However, even if \mathcal{B} is computable, it can still lose many of the computable properties of \mathcal{A} . One of the important and interesting questions in computable model theory is how a specific aspect of a computable structure may change if the structure is isomorphically transformed so that it remains computable.

Let \mathcal{A} be a fixed computable structure. A computable property of \mathcal{A} which Ash and Nerode considered is given by a new computable relation R on the domain A of \mathcal{A} . Here, we call R *new* if R is not named in the language of \mathcal{A} . By $Im_{\mathcal{A}}(R)$ we denote the set of images of R under all isomorphisms from \mathcal{A} to other computable structures. For example, Ash and Nerode [2] investigated the conditions under which every relation in $Im_{\mathcal{A}}(R)$ must be c.e. Such R is called *intrinsically*

c.e. on \mathcal{A} . In [7] we found conditions for (\mathcal{A}, R) under which there is a c.e. relation in $Im_{\mathcal{A}}(R)$ of an arbitrary c.e. degree.

A sequence of variables displayed after a formula contains a subsequence of all free variables occurring in the formula. An $L_{\omega_1\omega}$ formula $\alpha(\vec{x})$ is a Σ_1 formula if it is equivalent to a formula of the form

$$\bigvee_{i \in I} \exists \vec{y}_i \theta_i(\vec{x}, \vec{y}_i),$$

where for every $i \in I$, $\theta_i(\vec{x}, \vec{y}_i)$ is a finitary quantifier-free formula. If the index set I is c.e., then $\alpha(\vec{x})$ is a *computable* Σ_1 formula. If I is finite, then $\alpha(\vec{x})$ is a finitary Σ_1 formula.

For simplicity, we shall assume throughout the text that all new relations on computable structures are unary. We fix a new computable relation R on the domain of a computable structure \mathcal{A} for a language L . Ash and Nerode [2] introduced a computable syntactic condition for R on \mathcal{A} , called being a formally c.e. relation. The relation R is *formally c.e.* (on \mathcal{A}) if R can be defined by a computable Σ_1 formula with parameters. That is, there is a sequence $\vec{c} \in A^{<\omega}$ and a computable Σ_1 formula $\alpha(\vec{c}, x)$ such that the following equivalence holds for every $a \in A$:

$$R(a) \Leftrightarrow \mathcal{A}_A \models \alpha(\vec{c}, a).$$

Clearly, if R is formally c.e. on \mathcal{A} , then R is intrinsically c.e. on \mathcal{A} . Ash and Nerode [2] also established the converse for a computable relation R , under an additional decidability condition on (\mathcal{A}, R) .

Let \mathcal{B} be a computable structure for L , and let X be a new relation the domain B . The complement of X with respect to B is \overline{X} . Let \mathbf{R} be a symbol for X . Since we are interested in the case when X is c.e., certain first-order formulae with positive occurrences of \mathbf{R} in the expanded language $L \cup \{\mathbf{R}\}$ play a special role (see [3], [7] and [10]). A Σ_1 formula in $L \cup \{\mathbf{R}\}$, possibly with individual constants (parameters), in which \mathbf{R} occurs only positively is also called a Σ_1^Γ formula. This notation was introduced in [1], where a hierarchy of infinitary formulae was defined in a general setting in which Γ is a function assigning computable ordinals to relation symbols. A unary relation F on B is definable by a computable Σ_1^Γ formula $\alpha(\vec{d}, x)$ (with parameters $\vec{d} \in B^{<\omega}$) if for every $b \in B$:

$$F(b) \Leftrightarrow (\mathcal{B}_B, X) \models \alpha(\vec{d}, b).$$

The subsets of \overline{X} which are definable by computable Σ_1^Γ formulae, and their subsets, play the role of finite sets to some extent.

We say that the relation X is *quasi-simple* on \mathcal{B} if X is c.e., \overline{X} is not definable by a computable Σ_1^I formula, and for every c.e. $W \subseteq \overline{X}$, there is a unary relation F definable by a computable Σ_1^I formula such that

$$W \subseteq F \subseteq \overline{X}.$$

Quasi-simplicity was first investigated by Hird [8], [10]. He proved that, under certain decidability conditions on (\mathcal{A}, R) , $Im_{\mathcal{A}}(R)$ contains a quasi-simple relation. Ash, Knight and Remmel [3] gave conditions on (\mathcal{A}, R) which are sufficient for obtaining a quasi-simple relation of an arbitrary c.e. degree in $Im_{\mathcal{A}}(R)$.

For a formula θ , let $\theta^1 =_{def} \theta$ and $\theta^0 =_{def} \neg\theta$. If f is a partial function, then $dom(f)$ is the domain of f , $rng(f)$ is the range of f , and $f(a) \downarrow$ denotes that $a \in dom(f)$. The length of a sequence \vec{x} is denoted by $lh(\vec{x})$. If $\vec{x} = (x_0, \dots, x_{m-1})$ and f is a function, then $f(\vec{x}) =_{def} (f(x_0), \dots, f(x_{m-1}))$. The concatenation of sequences is denoted by $\hat{}$.

We fix $\langle \cdot, \cdot \rangle$ to be a computable bijection from ω^2 onto ω , which is strictly increasing with respect to both coordinates. Let W_0, W_1, \dots be a standard computable enumeration of all c.e. sets. By \equiv_T we denote Turing equivalence of sets.

2. NOWHERE SIMPLE RELATIONS ON STRUCTURES

Shore [15] introduced the concepts of nowhere simple and effectively nowhere simple sets in computability theory. Let $X \subseteq \omega$. We say that X is *nowhere simple* if X is c.e., and for every c.e. set W such that $W - X$ infinite, there is an infinite c.e. set W' such that $W' \subseteq W - X$. Thus, nowhere simplicity is definable in the lattice of all c.e. sets. Similarly, X is *effectively nowhere simple* if X is c.e., and there is a unary computable function g such that for every $e \in \omega$, $W_{g(e)} \subseteq W_e - X$ and

$$W_e - X \text{ is infinite} \Rightarrow W_{g(e)} \text{ is infinite.}$$

The function g is called a *witness function* for X . Clearly, every computable set is effectively nowhere simple. Miller and Remmel [12] established that X is effectively nowhere simple if there is a c.e. set W such that $W \cap X = \emptyset$ and for every $e \in \omega$,

$$W_e - X \text{ is infinite} \Rightarrow W_e \cap W \text{ is infinite.}$$

The set W is called a *witness set* for X . Hence, effective nowhere simplicity is definable in the lattice of all c.e. sets. Shore [15], and

Miller and Remmel [12] proved that every c.e. Turing degree contains an effectively nowhere simple set.

Let \mathcal{V}_∞ be a computable infinite dimensional vector space over a computable field F . We also assume that \mathcal{V}_∞ has a dependence algorithm. We consider only subspaces of \mathcal{V}_∞ . For any subset $S \subseteq V_\infty$, by \mathcal{S}^* we denote the subspace (with the domain S^*) generated by S . Let $V_e = W_e^*$. Then $\mathcal{V}_0, \mathcal{V}_1, \mathcal{V}_2, \dots$ is a computable list of all c.e. subspaces of \mathcal{V}_∞ . A subspace \mathcal{V} of \mathcal{V}_∞ is *nowhere simple* (see [14], [4] and [5]) if \mathcal{V} is c.e., and for every $e \in \omega$, there is a c.e. subspace \mathcal{V}'_e such that $V'_e \subseteq V_e$, $V'_e \cap V = \{0\}$, and

$$\dim(\mathcal{V}_e/\mathcal{V}) = \infty \Rightarrow \dim(\mathcal{V}'_e) = \infty.$$

If, in addition, there is a unary computable function g such that for every $e \in \omega$, $V'_e = V_{g(e)}$ then \mathcal{V} is called *effectively nowhere simple*. Nerode and Remmel [14] proved that a nowhere simple subset of a computable basis of \mathcal{V}_∞ generates a nowhere simple subspace of \mathcal{V}_∞ . Similarly, an effectively nowhere simple subset of a computable basis generates an effectively nowhere simple subspace. Let \mathbf{c} be a nonzero c.e. Turing degree. Downey and Remmel [5] established that \mathbf{c} contains a simple subset of a given computable basis of \mathcal{V}_∞ , which generates an effectively nowhere simple subspace \mathcal{V} of \mathcal{V}_∞ . Moreover, \mathcal{V} has a *witness space* \mathcal{W} , where \mathcal{W} is c.e., $W \cap V = \{0\}$ and for every $e \in \omega$,

$$\dim(\mathcal{V}_e/\mathcal{V}) = \infty \Rightarrow \dim([\mathcal{V}_e \cap (\mathcal{W} \cup \mathcal{V})^*]/\mathcal{V}) = \infty.$$

Downey and Remmel [5] proved that the existence of a witness space guarantees effective nowhere simplicity. It is not known whether every effectively nowhere simple subspace has a witness space. It is not even known for an effectively nowhere simple subspace \mathcal{V} and $v \in V_\infty - V$, whether $V \cup \{v\}$ generates an effectively nowhere simple subspace.

We consider the following general definition of nowhere simplicity and effective nowhere simplicity of relations on computable structures.

Definition 2.1. *Let \mathcal{B} be a computable structure and let X be a new unary relation on \mathcal{B} . Then X is nowhere simple on \mathcal{B} if X is c.e., and for every c.e. set W , there is a c.e. set $W' \subseteq W - X$ such that if $W - X$ is not contained in any subset F of \overline{X} definable by a computable Σ_1^Γ formula, then the same is true of W' .*

The relation X is effectively nowhere simple on \mathcal{B} if there is a computable unary function g such that for every $e \in \omega$, $W_{g(e)} \subseteq W_e - X$ and if $W_e - X$ is not contained in any subset F of \overline{X} definable by a computable Σ_1^Γ formula, then the same is true of $W_{g(e)}$.

3. MAIN RESULT

We fix a computable structure \mathcal{A} for a language L , and a new computable unary relation R on A . Let $\vec{c} \in A^{<\omega}$ and $a \in A$. The property of \vec{c} and a , termed a is free over \vec{c} , played an essential role in the results in [7] and [3]. We say that a is *free over* \vec{c} if $a \in \overline{R}$ ($\overline{R} = A - R$) and for every finitary Σ_1^Γ formula $\alpha(\vec{c}, x)$, if

$$(\mathcal{A}_A, R) \models \alpha(\vec{c}, \mathbf{a})$$

then

$$(\exists a' \in R)[(\mathcal{A}_A, R) \models \alpha(\vec{c}, \mathbf{a}')].$$

Let the set of all free elements over \vec{c} be denoted by $fr(\vec{c})$. Note that $fr(\vec{c}) \subseteq \overline{R}$. Clearly, if $a \in fr(\vec{c})$ and \vec{d} is a subsequence of \vec{c} , then $a \in fr(\vec{d})$. As in [3], let

$$bd(\vec{c}) =_{def} \{a \in \overline{R}: a \text{ is not free over } \vec{c}\}.$$

Thus, if $a \in bd(\vec{c})$ and \vec{c} is a subsequence of \vec{d} , then $a \in bd(\vec{d})$. A maximal relation on \overline{R} (with respect to the set-theoretic inclusion) which is definable by a computable Σ_1^Γ formula with parameters \vec{c} is of the form $bd(\vec{c})$. Conversely, if $bd(\vec{c})$ is definable by a computable Σ_1^Γ formula with parameters \vec{c} , then $bd(\vec{c})$ is a maximal relation on \overline{R} definable by such a formula.

Theorem 3.1. *Let the following conditions hold for a computable structure \mathcal{A} and a new computable unary relation R on A :*

- (1) *for every $\vec{c} \in A^{<\omega}$, there is an element a such that $a \in fr(\vec{c})$,*
- (2) *for every $\vec{c} \in A^{<\omega}$, $bd(\vec{c})$ is definable by a computable Σ_1^Γ formula with parameters \vec{c} ,*
- (3) *there is an algorithm which for a given $\vec{c} \in A^{<\omega}$ and $a \in A$, decides whether $a \in fr(\vec{c})$ (equivalently, whether $a \in bd(\vec{c})$),*
- (4) *for every $\vec{c} \in A^{<\omega}$ and $a, v \in \overline{R}$, if $a \in fr(\vec{c})$, and $v \in bd(\vec{c})$, then $a \in fr(\vec{c} \hat{\ } v)$,*
- (5) *for every $\vec{c} \in A^{<\omega}$, $\vec{u} \in \overline{R}^{<\omega}$ and $a, v \in \overline{R}$, if $a \in fr(\vec{c} \hat{\ } \vec{u})$, $v \in fr(\vec{c} \hat{\ } a)$, and $v \in fr(\vec{u})$, then $a \in fr(\vec{c} \hat{\ } \vec{u} \hat{\ } v)$.*

Then there is a computable structure \mathcal{B} isomorphic to \mathcal{A} such that the image of R on B is noncomputable and effectively nowhere simple on \mathcal{B} .

Proof. For simplicity, we assume that $A = \omega$. We will construct a computable structure \mathcal{B} with domain $B = \omega$ and an isomorphism f from \mathcal{B} to \mathcal{A} such that $f^{-1}(R)$ is noncomputable and effectively nowhere simple on \mathcal{B} . Let s be an arbitrary stage of the construction. We will

define a finite set Ψ^s of formulae of the open diagram of \mathcal{B} and certain formulae of the form $\mathbf{R}(\mathbf{b})$ for $b \in B$. We will also define a finite partial isomorphism f_s from \mathcal{B} to \mathcal{A} . A finite isomorphism from \mathcal{B} to \mathcal{A} at stage s , is an injective function p with a finite domain such that for every $\theta \in \Psi^s$, if $\theta = \theta(\mathbf{b}_0, \dots, \mathbf{b}_{l-1})$ for some $b_0, \dots, b_{l-1} \in B$, then $p(b_0) \downarrow, \dots, p(b_{l-1}) \downarrow$ and $\mathcal{A} \models \theta[p(b_0), \dots, p(b_{l-1})]$, and if $\mathbf{R}(\mathbf{b}) \in \Psi^s$ then $p(b) \in R$. Let $X_s = f_s^{-1}(R)$. Define $\Psi^{-1} = \emptyset$, $f_{-1} = \emptyset$ and $X_0 = \emptyset$. At the end of the construction, we will have that $f = \lim_{s \rightarrow \infty} f_s$ exists. Let $X =_{def} f^{-1}(R)$ and $\Psi = \bigcup_{s \geq 0} \Psi^s$. The construction will ensure that X is c.e.

Let $(\theta_e)_{e \in \omega}$ be an effective list of all atomic formulae in L_A . The construction will meet the following requirements for every $e \geq 0$,

$$P_e^0 : (\theta_e \in \Psi \text{ or } \neg \theta_e \in \Psi), \text{ and } (b \in X \Rightarrow \mathbf{R}(\mathbf{b}) \in \Psi);$$

$$P_e^1 : e \in \text{dom}(f);$$

$$P_e^2 : e \in \text{rng}(f);$$

$$Q_e : \overline{X} \neq W_e;$$

$N_{\langle e, k \rangle} : \text{If } W_e - X \text{ is not contained in any subset of } \overline{X} \text{ definable by a computable } \Sigma_1^{\Gamma} \text{ formula, we define } u_{e, k} \text{ such that } u_{e, k} \in W_e - X \text{ and}$

$$f(u_{e, k}) \in \text{fr}(f(u_{e, 0}, \hat{\ } \dots \hat{\ } u_{e, k-1})),$$

where for $i \in \{0, \dots, k-1\}$, $u_{e, i}$ is previously defined by $N_{\langle e, i \rangle}$.

The priority ordering of the requirements is:

$$P_0^0, P_0^1, P_0^2, Q_0, N_0, P_1^0, P_1^1, P_1^2, Q_1, N_1, \dots$$

Let $e, i \in \omega$. We set that at stage -1 , $u_{e, i}$ is undefined. To assure effective nowhere simplicity, we will have that the element $u_{e, i}$ once defined will remain unchanged, and $\{u_{e, j} : j \in \omega \wedge (u_{e, j} \text{ is defined})\}$ will be a required c.e. subset W'_e of $W_e - X$. Since the construction will be effective uniformly in e , it will follow from the $s - m - n$ theorem that there is a unary computable function g such that $W_{g(e)} = W'_e$.

The strategy for meeting a single requirement Q_e is to chose an element d_e such that

$$d_e \in W_e \Leftrightarrow d_e \in X.$$

We will have $d_e = \lim_{s \rightarrow \infty} d_e^s$, where d_e^s is the approximation to d_e at stage s , and d_e^s may be undefined for finitely many s . If d_e^s is defined, then d_0^s, \dots, d_{e-1}^s are also defined. We assume that for every $e \in \omega$, d_e^{-1} is undefined. For $n \in \omega$, let $\overrightarrow{d_{<n}^s} =_{def} (d_0^s, \dots, d_e^s)$, where e is the largest number such that $e < n$ and d_e^s is defined. The construction will ensure that all elements of $\overrightarrow{d_{<n}^s}$ are in $\text{dom}(f_s)$.

Let $\overrightarrow{u^s}$ be the vector consisting of all elements $u_{e,i}$, $e, i \in \omega$, which are defined at stage s (listed in the order in which they are first defined). The construction will ensure that all elements of $\overrightarrow{u^s}$ are contained in $\text{dom}(f_s)$. For $k \in \omega$, if $u_{e,k}$ is defined at some stage, then $u_{e,0}, \dots, u_{e,k-1}$ are also defined at that stage. Let $\overrightarrow{u_{e,<k}} =_{\text{def}} (u_{e,0}, \dots, u_{e,k-1})$. We have the following definitions which will be used in the construction.

Requirement P_e^0 requires attention and can be attacked at s if $\theta_e \notin \Psi^{s-1}$, $\neg\theta_e \notin \Psi^{s-1}$ and all elements of B occurring in θ_e are in the domain of f_{s-1} . Requirement P_e^0 also requires attention if θ is of the form $\mathbf{R}(\mathbf{b})$, $\theta \notin \Psi^{s-1}$, $f_{s-1}(b) \downarrow$ and $f_{s-1}(b) \in R$.

Requirement P_e^1 requires attention and can be attacked at s if $e \notin \text{dom}(f_{s-1})$.

Requirement P_e^2 requires attention and can be attacked at s if $e \notin \text{rng}(f_{s-1})$.

Requirement Q_e requires attention and can be attacked at s if d_e^{s-1} is undefined at $s-1$, or $d_e^{s-1} \in W_{e,s} - W_{e,s-1}$.

Requirement $N_{\langle e,k \rangle}$ requires attention at s if $u_{e,k}$ is undefined at $s-1$. Such a requirement $N_{\langle e,k \rangle}$ can be attacked at s if $u_{e,0}, \dots, u_{e,k-1}$ are defined at $s-1$, all elements of \overrightarrow{c} , where

$$\overrightarrow{c} = (0, \dots, \langle e, k \rangle) \wedge (f_{s-1}^{-1}(0), \dots, f_{s-1}^{-1}(\langle e, k \rangle)) \wedge \overrightarrow{d_{\langle e,k \rangle}^{s-1} + 1},$$

are in $\text{dom}(f_{s-1})$, and there is $u \in W_{e,s}$ such that $u \in \text{dom}(f_{s-1})$, and

$$f_{s-1}(u) \in \text{fr}(f_{s-1}(\overrightarrow{c} \wedge \overrightarrow{u_{e,<k}})).$$

Notice that $f_{s-1}(u) \in \overline{R}$.

Requirement P_e^0 is never injured.

Requirement P_e^1 is injured at s if $f_s(e) \neq f_{s-1}(e)$.

Requirement P_e^2 is injured at s if $f_s^{-1}(e) \neq f_{s-1}^{-1}(e)$.

Requirement Q_e is injured at s if d_e^{s-1} is defined, and d_e^s is undefined; or if d_e^s is defined, $d_e^s \in \overline{R}$, and

$$f_s(d_e^s) \in \text{bd}(f_s(0, \dots, e) \wedge (0, \dots, e) \wedge f_s(\overrightarrow{d_{<e}^s}) \wedge f_s(\overrightarrow{u^s})).$$

Requirement $N_{\langle e,k \rangle}$ is never injured.

Construction

Stage $s \geq 0$

Let *Req* be the highest priority requirement which requires attention and which can be attacked at stage s .

(i) $\text{Req} = P_e^0$

Let $\theta_e = \theta_e(\mathbf{b}_0, \dots, \mathbf{b}_{l-1})$ for some $b_0, \dots, b_{l-1} \in \omega$. Define $\Psi^s = \Psi^{s-1} \cup \{\theta_e^k\}$, where $k \in \{0, 1\}$ is such that $\mathcal{A} \models \theta_e^k[f_{s-1}(b_0), \dots, f_{s-1}(b_{l-1})]$.

Let $f_s =_{\text{def}} f_{s-1}$. The constants associated with Q -requirements are as at stage $s - 1$.

(ii) $Req = P_e^1$

We define $f_s = f_{s-1} \cup \{(e, a)\}$, where a is the least unused element in A . If $a \in R$, we set $\Psi^s = \Psi^{s-1} \cup \{\mathbf{R}(\mathbf{e})\}$; otherwise, we set $\Psi^s = \Psi^{s-1}$. The constants associated with Q -requirements are as at stage $s - 1$.

(iii) $Req = P_e^2$

We define $f_s = f_{s-1} \cup \{(b, e)\}$, where b is the least unused element in B . If $e \in R$, we set $\Psi^s = \Psi^{s-1} \cup \{\mathbf{R}(\mathbf{b})\}$; otherwise, we set $\Psi^s = \Psi^{s-1}$. The constants associated with Q -requirements are as at stage $s - 1$.

(iv) $Req = Q_e$

Let $\vec{c} = (0, \dots, e) \wedge (f_{s-1}^{-1}(0), \dots, f_{s-1}^{-1}(e)) \wedge \overrightarrow{d_{<e}^{s-1}} \wedge \overrightarrow{u^{s-1}}$. It follows by construction that all elements of \vec{c} are in $\text{dom}(f_{s-1})$.

Case (a): d_e^{s-1} is undefined.

Let a be the least element in $A - f_{s-1}(W_{e,s})$ such that a is free over $f_{s-1}(\vec{c})$. This a exists by condition (1) of the theorem. If $a \in \text{ran}(f_{s-1})$, we set $d_e^s =_{\text{def}} f_{s-1}^{-1}(a)$ and $f_s =_{\text{def}} f_{s-1}$. If $a \notin \text{ran}(f_{s-1})$, we choose b to be the least new element in $B - W_{e,s}$, and set $d_e^s =_{\text{def}} b$ and $f_s =_{\text{def}} f_{s-1} \cup \{(b, a)\}$. Notice that in both subcases, $d_e^s \notin X_s$ and $d_e^s \notin W_{e,s}$. For every $e' \neq e$, let $d_{e'}^s =_{\text{def}} d_{e'}^{s-1}$. Let $\Psi^s =_{\text{def}} \Psi^{s-1}$.

Case (b): $d_e^{s-1} \in W_{e,s} - W_{e,s-1}$.

It will follow by construction that $a = f_{s-1}(d_e^{s-1})$ is free over $f_{s-1}(\vec{c})$. Hence $a \in \bar{R}$. Let $\psi(\vec{c}, x)$ be the Σ_1^Γ formula with parameters \vec{c} which captures the part of the diagram of (\mathcal{B}, X) , determined by stage $s - 1$. That is, $\psi(\vec{c}, x) = (\exists \vec{y}) \delta(\vec{c}, x, \vec{y})$, where $\delta(\vec{c}, d_e^{s-1}, \vec{b})$ is the conjunction of all formulas of Ψ^{s-1} , and $\text{lh}(\vec{y}) = \text{lh}(\vec{b})$. Clearly, $(\mathcal{B}, X) \models \psi[\vec{c}, d_e^{s-1}]$, so $(\mathcal{A}, R) \models \psi[f_{s-1}(\vec{c}), a]$. We choose a' to be the least element such that $a' \in R$ and $(\mathcal{A}, R) \models \psi[f_{s-1}(\vec{c}), a']$. Let $\vec{a} \in A^{\text{lh}(\vec{b})}$ be the least sequence such that $(\mathcal{A}, R) \models \delta[f_{s-1}(\vec{c}), a', \vec{a}]$. We set

$$f_s =_{\text{def}} (\vec{c} \wedge d_e^{s-1} \wedge \vec{b}, f_{s-1}(\vec{c}) \wedge a' \wedge \vec{a}).$$

For every $e' \leq e$, let $d_{e'}^s =_{\text{def}} d_{e'}^{s-1}$, and for every $e' > e$, let $d_{e'}^s$ be undefined. Let $\Psi^s =_{\text{def}} \Psi^{s-1} \cup \{\mathbf{R}(\mathbf{b}) : b \in X_s - X_{s-1}\}$.

(v) $Req = N_j$, where $j = \langle e, k \rangle$.

Let $\vec{c}_j = (0, \dots, j) \wedge (f_{s-1}^{-1}(0), \dots, f_{s-1}^{-1}(j)) \wedge \overrightarrow{d_{<j+1}^{s-1}} \wedge \overrightarrow{u_{e,<k}}$. It follows by construction that all elements of \vec{c}_j are in $\text{dom}(f_{s-1})$. We choose

the least element $u \in W_{e,s}$ such that $u \in \text{dom}(f_{s-1})$ and $f_{s-1}(u) \in \text{fr}(f_{s-1}(\overrightarrow{c_j}))$. Hence $u \notin X_{s-1}$. We set

$$u_{e,k} =_{\text{def}} u.$$

For every $e' \leq j$, let $d_{e'}^s =_{\text{def}} d_{e'}^{s-1}$, and for every $e' > j$, let $d_{e'}^s$ be undefined. Let $f_s =_{\text{def}} f_{s-1}$ and $\Psi^s =_{\text{def}} \Psi^{s-1}$.

End of the construction.

Lemma 3.2. *No Q -requirement is injured by an action of a lower priority N -requirement.*

Proof. Let us consider N_j for some $j = \langle e, k \rangle$, and $Q_{e'}$ for some $e' \leq j$. Let s be a stage at which the requirement N_j is being satisfied. We will use the same notation as in (v) of the construction. We assume that $f_{s-1}(d_{e'}^{s-1})$ is free over

$$f_{s-1}(0, \dots, e') \wedge (0, \dots, e') \wedge f_{s-1}(\overrightarrow{d_{<e'}^{s-1}}) \wedge f_{s-1}(\overrightarrow{u^{s-1}}).$$

We want to show that $f_s(d_{e'}^s)$ remains free over

$$f_s(0, \dots, e') \wedge (0, \dots, e') \wedge f_s(\overrightarrow{d_{<e'}^s}) \wedge f_s(\overrightarrow{u^s}).$$

Since $d_{e'}^s = d_{e'}^{s-1}$, $f_s(d_{e'}^s) = f_{s-1}(d_{e'}^{s-1})$, $f_s(0, \dots, e') = f_{s-1}(0, \dots, e')$, $\overrightarrow{d_{<e'}^s} = \overrightarrow{d_{<e'}^{s-1}}$, $f_s(\overrightarrow{d_{<e'}^s}) = f_{s-1}(\overrightarrow{d_{<e'}^{s-1}})$, $\overrightarrow{u^s} = \overrightarrow{u^{s-1}} \wedge u_{e,k}$, $f_s(\overrightarrow{u^s}) = f_{s-1}(\overrightarrow{u^{s-1}})$, and $f_s(u_{e,k}) = f_{s-1}(u_{e,k})$, it is sufficient to conclude that $f_{s-1}(d_{e'}^{s-1})$ is free over

$$f_{s-1}(0, \dots, e') \wedge (0, \dots, e') \wedge f_{s-1}(\overrightarrow{d_{<e'}^{s-1}}) \wedge f_{s-1}(\overrightarrow{u^{s-1}}) \wedge f_{s-1}(u_{e,k}).$$

First, we assume that $f_{s-1}(u_{e,k}) \in \text{bd}(f_{s-1}(\overrightarrow{u^{s-1}}))$. Then the required conclusion follows from condition (4) of the theorem if we set

$$\overrightarrow{c} = f_{s-1}(0, \dots, e') \wedge (0, \dots, e') \wedge f_{s-1}(\overrightarrow{d_{<e'}^{s-1}}) \wedge f_{s-1}(\overrightarrow{u^{s-1}}),$$

$$a = f_{s-1}(d_{e'}^{s-1}) \text{ and } v = f_{s-1}(u_{e,k}).$$

That is, $a \in \text{fr}(\overrightarrow{c})$ by our first assumption, and $v \in \text{bd}(\overrightarrow{c})$ since, by our second assumption, v is bounded over a subsequence of \overrightarrow{c} .

Now, we assume that $f_{s-1}(u_{e,k}) \in \text{fr}(f_{s-1}(\overrightarrow{u^{s-1}}))$. Then the required conclusion follows from condition (5) of the theorem if we set

$$\overrightarrow{c} = f_{s-1}(0, \dots, e') \wedge (0, \dots, e') \wedge f_{s-1}(\overrightarrow{d_{<e'}^{s-1}}),$$

$$\overrightarrow{u} = f_{s-1}(\overrightarrow{u^{s-1}}),$$

$$a = f_{s-1}(d_{e'}^{s-1}) \text{ and } v = f_{s-1}(u_{e,k}).$$

That is, $a \in fr(\vec{c} \hat{\ } \vec{u})$ by our first assumption, $v \in fr(\vec{c} \hat{\ } a)$ by a condition in (v) of the construction, and $v \in fr(\vec{u})$ by our second assumption. ■

It follows from the construction that a Q -requirement can be injured at most once by every higher priority N -requirement. Since P^1 -requirements, P^2 -requirements and Q -requirements may be injured only by higher priority Q -requirements, it is easy to show that each requirement is attacked and injured only finitely often. Thus, all P -requirements are met. Hence there is an isomorphism f from \mathcal{B} to \mathcal{A} such that $f = \lim_{s \rightarrow \infty} f_s$. The construction is effective because of the condition (3) of the theorem. Therefore, \mathcal{B} is a computable structure. Let $X = \bigcup_{s \geq 0} X_s$. Clearly, X is c.e. It follows by construction that $X = f^{-1}(R)$. Since every Q -requirement is satisfied, X is noncomputable.

Lemma 3.3. *Every N -requirement is satisfied.*

Proof. We fix $e \in \omega$. If $u_{e,k}$ is defined for every $k \in \omega$, then every $N_{\langle e,k \rangle}$ is satisfied. Therefore, let us assume that k is the least number such that $u_{e,k}$ is not defined. Let s_0 be a stage by which all requirements of higher priority than $N_{\langle e,k \rangle}$ have been attacked for the last time. Hence, at s_0 , the sequence of numbers in B coming from the higher priority P -requirements and Q -requirements has reached its final value \vec{b} . Then for $s > s_0$, for every $u \in W_{e,s} - X_s$ such that $u \in dom(f_{s-1})$, we have that $f_{s-1}(u) \in bd(f(\vec{b}) \hat{\ } f(u_{e,<k}))$. It then follows from condition (2) of the theorem that $W_e - X$ is contained in a subset of \overline{X} which is definable by a computable Σ_1^Γ formula. Hence $N_{\langle e,k \rangle}$ is satisfied for every k . ■

Finally, we will prove that X is effectively nowhere simple on \mathcal{B} . Let $e \in \omega$. If for some k , $u_{e,k}$ is not defined, then $W_e - X$ is contained in some $F \subseteq \overline{X}$ such that F is definable by a computable Σ_1^Γ formula. We now assume that $u_{e,k}$ is defined for every k . We set $W'_e = \{u_{e,0}, u_{e,1}, u_{e,2}, \dots\}$. It is enough to show that W'_e is not contained in any subset of \overline{X} which is definable by a computable Σ_1^Γ formula. Let us assume otherwise. That is, let W'_e be contained in some subset of \overline{X} which is definable by a computable Σ_1^Γ formula with parameters \vec{d} . Then $f(u_{e,k}) \in bd(f(\vec{d}))$ for every k . This is not possible since, by (v) of the construction, for every k such that $\langle e, k \rangle \geq \max(rng(\vec{d}))$, we have that $f(u_{e,k}) \in fr(f(\vec{d}))$. ■

Theorem 3.1 can be modified, by using coding and permitting, as described in Theorem 2.5 in [7], to obtain an effectively nowhere simple set in $Im_{\mathcal{A}}(R)$ of an arbitrary c.e. degree.

Theorem 3.4. *Let conditions (1)-(5) from Theorem 3.1 hold for a unary computable relation R on the domain of a computable structure \mathcal{A} . Then, for every c.e. degree \mathbf{c} , there is a computable structure \mathcal{B} isomorphic to \mathcal{A} such that the image X of R in B is of degree \mathbf{c} and effectively nowhere simple on \mathcal{B} .*

Proof. We just sketch the main idea. Let $C \subseteq \omega$ be an arbitrary noncomputable c.e. set such that at every stage s , C receives at most one new element c , and $c \leq s$. We replace the Q -requirements of Theorem 3.1 by the requirement $X \equiv_T C$. At every stage s of the construction, we have a sequence of movable markers on the elements in \overline{X}_s , where $\overline{X}_s = \{d_0^s < d_1^s < d_2^s < \dots\}$. More precisely, we effectively define a partial binary function γ with the intention that if $c \in C_s - C_{s-1}$, then $\gamma(c, s-1) \downarrow$ and $d_{\gamma(c, s-1)}^{s-1} \in X_s$. ■

Example 3.1. *Let \mathcal{A} be $(\omega, =)$ and let $R \subseteq A$ be a computable infinite co-infinite set. It is shown in [3] that \mathcal{A} and R satisfy conditions (1)-(4). The condition (5) is also satisfied because for $\vec{c} \in A^{<\omega}$ and $a \in \overline{R}$, we have*

$$a \in fr(\vec{c}) \Leftrightarrow a \notin rng(\vec{c}).$$

Hence, we obtain Shore's and Miller-Remmel's theorem that every c.e. Turing degree contains an effectively nowhere simple set.

Example 3.2. *Let \mathcal{A} be a computable linear order of type η and let $R \subseteq A$ be a computable dense co-dense set. It is shown in [3] that \mathcal{A} and R satisfy conditions (1)-(4). The condition (5) is also satisfied because for $\vec{c} \in A^{<\omega}$ and $a \in \overline{R}$, we have*

$$a \in fr(\vec{c}) \Leftrightarrow a \notin rng(\vec{c}).$$

Thus, we have the following corollary.

Corollary 3.5. *For every c.e. set C , there is a computable linear order of type η with a dense co-dense effectively nowhere simple set X such that $X \equiv_T C$.*

Example 3.3. *Let \mathcal{B} be a computable linear order of type $\omega + \omega^*$ and let X be the ω -part of B . Then it is easy to see that X can not be nowhere simple unless it is computable. That is, if X is c.e. and if W is an infinite c.e. set such that $W \subseteq \overline{X}$, then X is computable. On the other hand, Ash, Knight and Remmel [3] have shown that for any*

noncomputable c.e. set C , there is a computable linear order of type $\omega + \omega^*$ with the ω -part X such that X is a simple set and $X \equiv_T C$. Indeed, if $\mathcal{A} = (\omega, <)$ is a computable linear order of type $\omega + \omega^*$ with the computable ω -part R , it is shown in [3] that conditions (1)-(4) are satisfied. Condition (5) is not satisfied because for $\vec{c} \in A^{<\omega}$ and $a \in \overline{R}$, we have

$$a \in fr(\vec{c}) \Leftrightarrow a < c_0,$$

where $ran(\vec{c}) \cap \overline{R} = \{c_0 < \dots < c_l\}$.

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REFERENCES

- [1] C.J. Ash and J.F. Knight, Ramified systems, *Annals of Pure and Applied Logic* 70 (1994), 205-221.
- [2] C.J. Ash and A. Nerode, Intrinsically recursive relations, in: J.N. Crossley, ed., *Aspects of Effective Algebra* (U.D.A. Book Co., Steel's Creek, Australia, 1981), 26-41.
- [3] C.J. Ash, J.F. Knight, and J.B. Remmel, Quasi-simple relations in copies of a given recursive structure, *Annals of Pure and Applied Logic* 86 (1997), 203-218.
- [4] R. Downey, Nowhere simplicity in matroids, *J. Austral. Math. Soc. (Series A)* 35 (1983), 28-45.
- [5] R.G. Downey and J.B. Remmel, Effectively and noneffectively nowhere simple subspaces, in: J.N. Crossley, J.B. Remmel, R.A. Shore, and M.E. Sweedler, eds., *Logical Methods* (Birkhäuser, Boston, 1993), 314-351.
- [6] V.S. Harizanov, Effectively and noneffectively nowhere simple sets, *Mathematical Logic Quarterly* 42 (1996), 241-248.
- [7] V.S. Harizanov, Some effects of Ash-Nerode and other decidability conditions on degree spectra, *Annals of Pure and Applied Logic* 55 (1991), 51-65.
- [8] G.R. Hird, *Recursive Properties of Relations on Models*, Ph.D. Thesis, Monash University, Melbourne, Australia (1984).
- [9] G. Hird, Recursive properties of intervals of recursive linear orders, in: J.N. Crossley, J.B. Remmel, R.A. Shore, and M.E. Sweedler, eds., *Logical Methods* (Birkhäuser, Boston, 1993), 422-437.
- [10] G.R. Hird, Recursive properties of relations on models, *Annals of Pure and Applied Logic* 63 (1993), 241-269.
- [11] G. Metakides and A. Nerode, Recursion theory on fields and abstract dependence, *Journal of Algebra* 65 (1980), 36-59.
- [12] D. Miller and J.B. Remmel, Effectively nowhere simple sets, *The Journal of Symbolic Logic* 49 (1984), 129-136.
- [13] A. Nerode and J.B. Remmel, A survey of lattices of r.e. substructures, in: A. Nerode and R. Shore, eds., *Recursion Theory*, Proceedings of Symposia in Pure Mathematics of the American Mathematical Society 42 (American Mathematical Society, Providence, 1985), 323-375.

- [14] A. Nerode and J. Remmel, Recursion theory on matroids, in: G. Metakides, ed., *Patras Logic Symposium*, Studies in Logic and the Foundations of Mathematics 109 (North-Holland, Amsterdam, 1982), 41-65.
- [15] R.A. Shore, Nowhere simple sets and the lattice of recursively enumerable sets, *The Journal of Symbolic Logic* 43 (1978), 322-330.
- [16] R.I. Soare, *Recursively Enumerable Sets and Degrees. A Study of Computable Functions and Computably Generated Sets* (Springer-Verlag, Berlin, 1987).

DEPARTMENT OF MATHEMATICS, 2201 G STREET, N.W., THE GEORGE WASHINGTON UNIVERSITY, WASHINGTON, D.C. 20052, U.S.A.