

# Effective Categoricity of Abelian $p$ -Groups

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## Abstract

We investigate effective categoricity of computable Abelian  $p$ -groups  $\mathcal{A}$ . We prove that all computably categorical Abelian  $p$ -groups are relatively computably categorical, that is, have computably enumerable Scott families of existential formulas. We investigate which computable Abelian  $p$ -groups are  $\Delta_2^0$  categorical and relatively  $\Delta_2^0$  categorical.

## 1 Introduction and Preliminaries

In computable model theory we are interested in effective versions of model theoretic notions and constructions. We consider in particular computability theoretic bounds on the complexity of isomorphisms of structures within the same isomorphism type. This paper is a sequel to [6] where we studied equivalence structures. Here we will investigate computable Abelian groups. We

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consider only countable structures for computable languages, and for infinite structures we may assume that their universe is  $\omega$ . We identify sentences with their Gödel codes. The *atomic diagram* of a structure  $\mathcal{A}$  for  $L$  is the set of all quantifier-free sentences in  $L_A$ ,  $L$  expanded by constants for the elements in  $A$ , which are true in  $\mathcal{A}$ . A structure is *computable* if its atomic diagram is computable. In other words, a structure  $\mathcal{A}$  is computable if there is an algorithm that determines for every quantifier-free formula  $\theta(x_0, \dots, x_{n-1})$  and every sequence  $(a_0, \dots, a_{n-1}) \in A^n$ , whether  $\mathcal{A} \models \theta(a_0, \dots, a_{n-1})$ .

A computable structure  $\mathcal{A}$  is *computably categorical* if for every computable isomorphic copy  $\mathcal{B}$  of  $\mathcal{A}$ , there is a *computable* isomorphism from  $\mathcal{A}$  onto  $\mathcal{B}$ . For example, the ordered set of rational numbers is computably categorical, while the ordered set of natural numbers is not. Goncharov and Dzegoev [15], and Remmel [30] proved that a computable linear ordering is computably categorical if and only if it has only finitely many successors. Goncharov and Dzegoev [15], and Remmel [31] established that a computable Boolean algebra is computably categorical if and only if it has finitely many atoms (see also LaRoche [24]). Miller [28] proved that no computable tree of height  $\omega$  is computably categorical. Lempp, McCoy, Miller, and Solomon [25] characterized computable trees of finite height that are computably categorical. Nurtazin [29], and Metakides and Nerode [27] established that a computable algebraically closed field of finite transcendence degree over its prime field is computably categorical. In the recent paper [6], the authors showed that a computable equivalence structure  $\mathcal{A}$  is computably categorical if and only if  $\mathcal{A}$  has at most finitely many finite equivalence classes, or  $\mathcal{A}$  has only finitely many infinite classes and there is a finite bound on the size of the finite classes and there is at most one finite  $k$  such that  $\mathcal{A}$  has infinitely many classes of size  $k$ .

The present paper will be concerned with the categoricity of Abelian  $p$ -groups Goncharov [12] and Smith [33] characterized computably categorical Abelian  $p$ -groups as those that can be written in one of the following forms:  $(\mathbb{Z}(p^\infty))^l \oplus \mathcal{G}$  for  $l \in \omega \cup \{\infty\}$  and  $\mathcal{F}$  is finite, or  $(\mathbb{Z}(p^\infty))^n \oplus \mathcal{F} \oplus (\mathbb{Z}(p^k))^\infty$ , where  $n, k \in \omega$  and  $\mathcal{F}$  is finite. Goncharov, Lempp, and Solomon [18] proved that a computable, ordered, Abelian group is computably categorical if and only if it has finite rank. Similarly, they showed that a computable, ordered, Archimedean group is computably categorical if and only if it has finite rank.

In [6], we characterized the relatively  $\Delta_2^0$  categorical equivalence structures as those with either finitely many infinite equivalence classes, or with an upper bound on the size of the finite equivalence classes. We also consider the complexity of isomorphisms for structures  $\mathcal{A}$  and  $\mathcal{B}$  such that both  $Fin^{\mathcal{A}}$  and  $Fin^{\mathcal{B}}$  are computable, or  $\Delta_2^0$ . Finally, we show that every computable equivalence structure is relatively  $\Delta_3^0$  categorical.

For any computable ordinal  $\alpha$ , we say that a computable structure  $\mathcal{A}$  is  $\Delta_\alpha^0$  *categorical* if for every computable structure  $\mathcal{B}$  isomorphic to  $\mathcal{A}$ , there is a  $\Delta_\alpha^0$  isomorphism from  $\mathcal{A}$  onto  $\mathcal{B}$ . Lempp, McCoy, Miller, and Solomon [25] proved that for every  $n \geq 1$ , there is a computable tree of finite height, which is  $\Delta_{n+1}^0$  categorical but not  $\Delta_n^0$  categorical. We say that  $\mathcal{A}$  is *relatively computably categorical* if for every structure  $\mathcal{B}$  isomorphic to  $\mathcal{A}$ , there is an isomorphism that

is computable relative to the atomic diagram of  $\mathcal{B}$ . Similarly, a computable  $\mathcal{A}$  is *relatively  $\Delta_\alpha^0$  categorical* if for every  $\mathcal{B}$  isomorphic to  $\mathcal{A}$ , there is an isomorphism that is  $\Delta_\alpha^0$  relative to the atomic diagram of  $\mathcal{B}$ . Clearly, a relatively  $\Delta_\alpha^0$  categorical structure is  $\Delta_\alpha^0$  categorical. We are especially interested in the case when  $\alpha = 2$ . McCoy [26] characterized, under certain restrictions,  $\Delta_2^0$  categorical and relatively  $\Delta_2^0$  categorical linear orderings and Boolean algebras. For example, a computable Boolean algebra is relatively  $\Delta_2^0$  categorical if and only if it can be expressed as a finite direct sum  $c_1 \vee \dots \vee c_n$ , where each  $c_i$  is either atomless, an atom, or a 1-atom. Using an enumeration result of Selivanov [32], Goncharov [13] showed that there is a computable structure, which is computably categorical but not relatively computably categorical.

Using a relativized version of Selivanov's enumeration result, Goncharov, Harizanov, Knight, McCoy, Miller, and Solomon [16] showed that for each computable successor ordinal  $\alpha$ , there is a computable structure, which is  $\Delta_\alpha^0$  categorical but not relatively  $\Delta_\alpha^0$  categorical. It was later shown by Chisholm, Fokina, Goncharov, Harizanov, Knight, and Quinn [9] that the same is true for every computable limit ordinal  $\alpha$ .

It is not known whether for any computable successor ordinal  $\alpha$ , there is a rigid computable structure that is  $\Delta_\alpha^0$  categorical but not relatively  $\Delta_\alpha^0$  categorical. Another open question is whether every  $\Delta_1^1$  categorical computable structure must be relatively  $\Delta_1^1$  categorical (see [17]).

There are syntactic conditions that are equivalent to relative  $\Delta_\alpha^0$  categoricity. These conditions involve the existence of certain families of formulas, that is, certain Scott families. Scott families come from Scott's Isomorphism Theorem, which says that for a countable structure  $\mathcal{A}$ , there is an  $L_{\omega_1\omega}$  sentence whose countable models are exactly the isomorphic copies of  $\mathcal{A}$ . A *Scott family* for a structure  $\mathcal{A}$  is a countable family  $\Phi$  of  $L_{\omega_1\omega}$  formulas, possibly with finitely many fixed parameters from  $\mathcal{A}$ , such that:

- (i) Each finite tuple in  $\mathcal{A}$  satisfies some  $\psi \in \Phi$ ;
- (ii) If  $\vec{a}, \vec{b}$  are tuples in  $\mathcal{A}$  of the same length, satisfying the same formula in  $\Phi$ , then they are automorphic; that is, there is an automorphism of  $\mathcal{A}$  that maps  $\vec{a}$  to  $\vec{b}$ .

A *formally c.e. Scott family* is a c.e. Scott family consisting of finitary existential formulas. A *formally  $\Sigma_\alpha^0$  Scott family* is a  $\Sigma_\alpha^0$  Scott family consisting of computable  $\Sigma_\alpha$  formulas. Roughly speaking, computable infinitary formulas are  $L_{\omega_1\omega}$  formulas in which the infinite disjunctions and conjunctions are taken over computably enumerable (c.e.) sets. We can classify computable formulas according to their complexity as follows. A computable  $\Sigma_0$  or  $\Pi_0$  formula is a finitary quantifier-free formula. Let  $\alpha > 0$  be a computable ordinal. A computable  $\Sigma_\alpha$  formula is a c.e. disjunction of formulas  $(\exists \vec{u})\theta(\vec{x}, \vec{u})$ , where  $\theta$  is computable  $\Pi_\beta$  for some  $\beta < \alpha$ . A computable  $\Pi_\alpha$  formula is a c.e. conjunction of formulas  $(\forall \vec{u})\theta(\vec{x}, \vec{u})$ , where  $\theta$  is computable  $\Sigma_\beta$  for some  $\beta < \alpha$ . Precise definition of computable infinitary formulas involves assigning indices to the formulas, based on Kleene's system of ordinal notations (see [2]). The important property of these formulas is given in the following theorem due to Ash.

**Theorem 1.1** *For a structure  $\mathcal{A}$ , if  $\theta(\vec{x})$  is a computable  $\Sigma_\alpha$  formula, then the set  $\{\vec{a} : \mathcal{A} \models \theta(\vec{a})\}$  is  $\Sigma_\alpha^0$  relative to the atomic diagram of  $\mathcal{A}$ .*

An analogous result holds for computable  $\Pi_\alpha$  formulas.

It is easy to see that if  $\mathcal{A}$  has a formally c.e. Scott family, then  $\mathcal{A}$  is relatively computably categorical. In general, if  $\mathcal{A}$  has a formally  $\Sigma_\alpha^0$  Scott family, then  $\mathcal{A}$  is relatively  $\Delta_\alpha^0$  categorical. Goncharov [13] showed that if  $\mathcal{A}$  is 2-decidable and computably categorical, then it has a formally c.e. Scott family. Ash [1] showed that, under certain decidability conditions on  $\mathcal{A}$ , if  $\mathcal{A}$  is  $\Delta_\alpha^0$  categorical, then it has a formally  $\Sigma_\alpha^0$  Scott family. For the relative notions, the decidability conditions are not needed. Moreover, Ash, Knight, Manasse, and Slaman [3], and independently Chisholm [8] established the following result.

**Theorem 1.2** *Let  $\mathcal{A}$  be a computable structure. Then the following are equivalent:*

- (a)  *$\mathcal{A}$  is relatively  $\Delta_\alpha^0$  categorical;*
- (b)  *$\mathcal{A}$  has a formally  $\Sigma_\alpha^0$  Scott family;*
- (c)  *$\mathcal{A}$  has a c.e. Scott family consisting of computable  $\Sigma_\alpha$  formulas.*

A structure is *rigid* if it does not have nontrivial automorphisms. A computable structure is  $\Delta_\alpha^0$  *stable* if every isomorphism from  $\mathcal{A}$  onto a computable structure is  $\Delta_\alpha^0$ . If a computable structure is rigid and  $\Delta_\alpha^0$  categorical, then it is  $\Delta_\alpha^0$  stable. A *defining family* for a structure  $\mathcal{A}$  is a set  $\Phi$  of formulas with one free variable and a fixed finite tuple of parameters from  $A$  such that:

- (i) Every element of  $A$  satisfies some formula  $\psi \in \Phi$ ;
- (ii) No formula of  $\Phi$  is satisfied by more than one element of  $A$ .

A defining family  $\Phi$  is *formally  $\Sigma_\alpha^0$*  if it is a  $\Sigma_\alpha^0$  set of computable  $\Sigma_\alpha$  formulas. In particular, a defining family  $\Phi$  is *formally c.e.* if it is a c.e. set of finitary existential formulas. For a rigid computable structure  $\mathcal{A}$ , there is a formally  $\Sigma_\alpha^0$  Scott family iff there is a formally  $\Sigma_\alpha^0$  defining family.

In [13], Goncharov obtained a rigid structure that is computably stable but not relatively computably stable. It is not known for any computable ordinal  $\alpha > 1$  whether there is a computable structure that is  $\Delta_\alpha^0$  stable but not relatively  $\Delta_\alpha^0$  stable.

In Section 2, we investigate algorithmic properties of Abelian groups and their characters, and we provide a connection between equivalence structures and Abelian  $p$ -groups. In Section 3, we examine effective categoricity of Abelian  $p$ -groups. We show that every computably categorical Abelian  $p$ -group is also relatively computably categorical.

The notions and notation of computability theory are standard and as in Soare [34]. We fix  $\langle \cdot, \cdot \rangle$  to be a computable bijection from  $\omega^2$  onto  $\omega$ . Let  $(W_e)_{e \in \omega}$  be an effective enumeration of all c.e. sets.

## 2 Computable Abelian $p$ -Groups and Equivalence Structures

Let  $\mathcal{G} = (G, +^{\mathcal{A}}, 0)$  be a computable Abelian  $p$ -group, and assume that  $G = \omega$  and that  $0$  is the identity for the operation  $+^{\mathcal{G}}$ . It is immediate that the subtraction function  $-^{\mathcal{G}}$  as well as the inverse function are also computable. In this section, we will focus on direct sums of cyclic and quasicyclic groups and their connection with equivalence structures.

First, we need some definitions. Let  $p$  be a prime number. The group  $\mathcal{G}$  is said to be a  $p$ -group if, for all  $g \in G$ , the order of  $g$  is a power of  $p$ .  $\mathbb{Z}(p^n)$  is the cyclic group of order  $p^n$ .  $\mathbb{Z}(p^\infty)$  denotes the quasicyclic  $p$ -group, the direct limit of the sequence  $\mathbb{Z}(p^n)$  and also the set of rationals in  $[0, 1)$  of the form  $\frac{i}{p^n}$  with addition modulo 1.

**Definition 2.1** *The period of  $G$  is  $\max\{|g| : g \in G\}$  if this quantity is finite, and  $\infty$  otherwise.*

The subgroups  $p^\alpha G$ , where  $\alpha$  is an ordinal, are defined recursively as follows:

$$p^0 G = G, \quad pG = \{px : x \in G\},$$

$$p^{\alpha+1} G = p(p^\alpha G), \text{ and}$$

$$p^\lambda G = \bigcap_{\alpha < \lambda} p^\alpha G \text{ for limit } \lambda.$$

The *length* of  $\mathcal{G}$ ,  $lh(\mathcal{G})$ , is the least ordinal  $\alpha$  such that  $p^{\alpha+1} G = p^\alpha G$ . The *divisible part* of  $\mathcal{G}$  is  $D(\mathcal{G}) = p^{lh(\mathcal{G})} G$  and is a subgroup of  $\mathcal{G}$ .  $\mathcal{G}$  is said to be *reduced* if  $D(\mathcal{G}) = \{0\}$ .

For an element  $g \in G$ , the height  $ht(g)$  is  $\infty$  if  $g \in D(G)$  and is otherwise the least  $\alpha$  such that  $g \notin p^{\alpha+1} G$ . For a computable group  $G$ ,  $ht(g)$  can be an arbitrary computable ordinal. The height of  $G$  is the supremum of  $\{ht(g) : g \in G\}$ .

Here are some classic results about Abelian  $p$ -groups which we will need. The reader is referred to Fuchs [11] for a full development of the theory of infinite Abelian groups.

**Theorem 2.2** 1. **(Baer)** *For any  $p$ -group  $\mathcal{G}$ , there exists a subgroup  $\mathcal{A}$  such that  $G = \mathcal{A} \oplus D(\mathcal{G})$ .*

2. **(Prüfer)** *If  $G$  is a countable Abelian  $p$ -group, then  $G$  is a direct sum of cyclic groups if and only if all nonzero elements have finite height.*

**Definition 2.3** *Let  $\mathcal{A}$  be a subgroup of  $\mathcal{G}$ .*

1.  $\mathcal{A}$  is a direct summand of  $\mathcal{G}$  if there exists a subgroup  $\mathcal{B}$  of  $\mathcal{G}$  such that  $\mathcal{G} = \mathcal{A} \oplus \mathcal{B}$ .

2.  $\mathcal{A}$  is a pure subgroup of  $\mathcal{G}$  if  $A \cap p^n G = p^n A$  for all  $n$ , that is the height of an element  $a \in A$  is the same in  $\mathcal{A}$  as it is in  $\mathcal{G}$ .

We need some results from group theory on direct summands. See [11] for more details.

**Theorem 2.4** 1. **(Kulikov)** *If  $\mathcal{A}$  has finite period and is a pure subgroup of  $\mathcal{G}$ , then  $\mathcal{A}$  is a direct summand of  $\mathcal{G}$ .*

2. **(Baer)** *Any divisible subgroup  $\mathcal{D}$  of a group  $\mathcal{A}$  is a direct summand of  $\mathcal{A}$ .*

The *Ulm subgroups*  $G^\alpha$  are defined by  $G^\alpha = p^{\omega\alpha}G$ . The  $\alpha$ th *Ulm factor*  $G_\alpha$  of  $G$  is  $G^\alpha/G^{\alpha+1}$ , and the *Ulm length*  $\lambda(A)$  of  $G$  is the least  $\alpha$  such that  $G^\alpha = G^{\alpha+1}$ .

It follows from Theorem 2.2 that each Ulm factor is a direct sum of cyclic groups. Thus each Ulm factor  $G_\alpha$  is a direct sum of cyclic groups. Now consider the sequence of

$$P_\alpha(G) = G_\alpha \cap \{x \in G : px = 0\}.$$

Let  $u_\alpha(G) = \dim_{\mathbb{Z}_p} P_\alpha(G)/P_{\alpha+1}(G)$ .

**Theorem 2.5 (Ulm)** *Two Abelian  $p$ -groups  $G$  and  $H$  are isomorphic if and only if they have the same Ulm sequence, that is, if and only if  $\lambda(G) = \lambda(H)$  and  $u_\alpha(G) = u_\alpha(H)$  for all  $\alpha$ .*

**Definition 2.6** 1.  $\oplus_\alpha \mathcal{H}$  denotes the direct sum of  $\alpha$  copies of  $\mathcal{H}$  where  $\alpha \leq \omega$ .

2. If  $\mathcal{A} = \oplus_{i < \omega} Z(p^{n_i})$ , then the character of  $\mathcal{A}$  is

$$\chi(\mathcal{A}) = \{(n, k) : \text{card}(\{i : n_i = n\}) \geq k\}.$$

3. If  $\mathcal{G} = \mathcal{A} \oplus \oplus_\alpha Z(p^\infty)$  for some  $\alpha \leq \omega$  and some  $\mathcal{A}$  as above, then  $\chi(\mathcal{G}) = \chi(\mathcal{A})$ .

4. We say that  $\mathcal{G}$  has bounded character if for some finite  $b$  and all  $(n, k) \in \chi(\mathcal{G})$ ,  $n \leq b$ , and is said to have unbounded character otherwise.

In the previous paper [6], we studied a similar notion for equivalence structures, and constructed structures of various characters. We will show that for a general class of such structures, a corresponding  $p$ -group *with the same character* may be constructed from a given equivalence structure. Here are the basic definitions for computable equivalence structures.

An equivalence structure  $\mathcal{A} = (A, E^{\mathcal{A}})$  consists of a set  $A$  with a binary relation  $E^{\mathcal{A}}$  that is reflexive, symmetric, and transitive. An equivalence structure  $\mathcal{A}$  is *computable* if  $A$  is a computable subset of  $\omega$  and  $E^{\mathcal{A}}$  is a computable relation. If  $A$  is an infinite set (which is usual), we may assume, without loss of generality, that  $A = \omega$ . The  $\mathcal{A}$ -equivalence class of  $a \in A$  is

$$[a]^{\mathcal{A}} = \{x \in A : xE^{\mathcal{A}}a\}.$$

We generally omit the superscript  $^{\mathcal{A}}$  when it can be inferred from the context.

**Definition 2.7** (i) Let  $\mathcal{A}$  be an equivalence relation. The character of  $\mathcal{A}$  is the set

$$\chi(\mathcal{A}) = \{\langle n, k \rangle : n, k > 0 \text{ and } \mathcal{A} \text{ has at least } k \text{ equivalence classes of size } n\}.$$

(ii) We say that  $\mathcal{A}$  has bounded character if there is some finite  $n$  such that all finite equivalence classes of  $\mathcal{A}$  have size at most  $n$ .

For both groups and equivalence relations, we may define the notion of a *character* as a subset  $K$  of  $(\omega - \{0\}) \times \omega$  such that for all  $k$  and  $n$ ,  $(n, k + 1) \in K \Rightarrow (n, k) \in K$ . Let  $o_{\mathcal{G}}(g)$  be the order of  $g$  in  $\mathcal{G}$ . The  $\mathcal{G}$  may be omitted when it is clear.

**Theorem 2.8 (Khisamiev [20])** Suppose that  $\mathcal{G}$  is a computable Abelian  $p$ -group and isomorphic to  $\bigoplus_{\alpha} Z(p^{\infty}) \oplus \bigoplus_{i < \omega} Z(p^{n_i})$ . Then:

1.  $\{\langle g, n \rangle : o(g) = p^n\}$  is computable,
2.  $\{\langle g, n \rangle : ht(g) \geq n\}$  is  $\Sigma_1^0$ ,
3.  $D(\mathcal{G})$  is a  $\Pi_2^0$  set (recall that  $D(\mathcal{G})$  is the divisible part of  $\mathcal{G}$ ), and
4. the character  $\chi(\mathcal{G})$  is a  $\Sigma_2^0$  set.

**Proof:** (1)  $o(g) = p^n \iff p^n \cdot g = 0 \ \& \ p^{n-1} \cdot g \neq 0$ .

(2)  $ht(g) \geq p^n \iff (\exists h)(p^n \cdot h = g)$ .

(3) Under the hypothesis,  $ht(g) \geq \omega$  implies that  $g \in D(\mathcal{G})$ , so that  $g \in D(\mathcal{G}) \iff (\forall n)(ht(g) \geq n)$ .

(4) We have the following characterization of  $\chi(\mathcal{G})$  by Theorem 2.4:  $\langle n, k \rangle \in \chi(\mathcal{G})$  if and only if there exist  $g_0, \dots, g_{k-1}$  such that for all  $i < k$ ,  $o(g_i) = p^n$  and  $ht(g_i) = 0$  and

$$(*) (\forall c_0, c_1, \dots, c_{k-1} < p^n) [c_0 \cdot g_0 + \dots + c_{k-1} \cdot g_{k-1} = 0 \Rightarrow (\forall i < k)(c_i = 0)].$$

That is, if  $\langle n, k \rangle \in \chi(\mathcal{G})$ , then  $\mathcal{G}$  has at least  $k$  summands  $\langle g_1 \rangle, \dots, \langle g_{k-1} \rangle$  isomorphic to  $Z(p^n)$  and the sequence  $g_1, \dots, g_{k-1}$  satisfies (\*). On the other hand, if  $g_0, \dots, g_{k-1}$  satisfy (\*), then they generate a bounded, pure subgroup  $\mathcal{A}$  of  $\mathcal{G}$ , which must be a summand by Theorem 2.4. Hence  $\mathcal{G}$  has at least  $k$  summands of the form  $Z(p^n)$ .  $\square$

Here is a connection between equivalence structures and Abelian  $p$ -groups.

**Theorem 2.9** Let  $p$  be a prime number and let  $\mathcal{A}$  be a computable equivalence structure with character  $K$  and with  $\alpha$  infinite equivalence classes. We write  $\text{Inf}(\mathcal{A})$  to denote the set of elements in  $\mathcal{A}$  whose equivalence classes are infinite. Then there exists a computable  $p$ -group  $\mathcal{G}$  isomorphic to

$$\mathcal{H} \oplus \bigoplus_{\alpha} Z(p^{\infty}),$$

where  $\mathcal{H}$  is a direct sum of cyclic  $p$ -groups with character  $K$ . Furthermore, if  $\text{Inf}(\mathcal{A})$  is  $\Sigma_1^0$ , then  $D(\mathcal{G})$  is also  $\Sigma_1^0$ , and if  $\text{Inf}(\mathcal{A})$  is computable, then  $D(\mathcal{G})$  is also computable.

**Proof:** Let  $\mathcal{A}$ ,  $\alpha$  and  $K$  be given as stated and let  $\equiv$  denote  $\equiv^{\mathcal{A}}$ . First define the computable set  $B$  of basic elements of  $\mathcal{G}$  to consist of  $\{a \in A : (\forall m < a) \neg(m \equiv a)\}$ . Then enumerate  $B$  in numerical order as  $b_0, b_1, \dots$ . The group  $\mathcal{G}$  will be the direct sum of  $\mathcal{G}_i$ , where the groups  $\mathcal{G}_i$  and  $\mathcal{G}$  are constructed in stages  $\mathcal{G}_i^s$ . Initially,  $\mathcal{G}_0^0 = \{0, 1, \dots, p-1\}$  is a copy of  $Z_p$  and in general,  $\mathcal{G}_i^s$  is a copy of  $Z(p^k)$ , where  $k = \text{card}(\{j < s : j \equiv b_i\})$ . Now  $\mathcal{G}^s$  is isomorphic to the direct sum  $\mathcal{G}_0^s \oplus \mathcal{G}_1^s \oplus \dots \oplus \mathcal{G}_s^s$ .

At stage  $s+1$ , we initiate the component group  $\mathcal{G}_{s+1}^{s+1}$  and for each  $i \leq s$ , we check whether  $s+1 \neq b_i$  and  $s+1 \equiv b_i$ . If not, just let  $\mathcal{G}_i^{s+1} = \mathcal{G}_i^s$ . If so, then extend  $\mathcal{G}_i^s$  to a copy of  $Z(p^{k+1})$ , where  $\mathcal{G}_i^s$  is a copy of  $Z(p^k)$ . That is, given that  $\mathcal{G}_i^s$  is a cyclic group of order  $p^k$  with generator  $a$ , we put a new element  $b$  into  $\mathcal{G}_i^{s+1}$  so that  $p \cdot b = a$ , and also add elements to represent  $i \cdot b + g$  for  $i = 1, 2, \dots, p-1$  and  $g \in \mathcal{G}_i^s$ . Then we also add elements to  $\mathcal{G}$  to represent the new elements of  $\mathcal{G}_0^{s+1} \oplus \dots \oplus \mathcal{G}_{s+1}^{s+1}$ . This is done so that the elements of  $\mathcal{G}^{s+1}$  are an initial segment of  $\omega$ .

$\mathcal{G}$  will be a computable group, since for each  $a \in \omega$ ,  $a \in \mathcal{G}^a$  and for any two elements  $a \leq b$ ,  $a +^{\mathcal{G}} b$  is defined by stage  $b$ .

It is clear that if  $\text{card}([b_i]) = n$  in  $\mathcal{A}$ , then for some  $s$  and all  $t \geq s$ ,  $\mathcal{G}_i^t$  is isomorphic to  $Z(p^n)$ , and if  $\text{card}([b_i]) = \omega$  in  $\mathcal{A}$ , then the inverse limit  $\mathcal{G}_i$  of  $\langle \mathcal{G}_i^s : s < \omega \rangle$  will be a copy of  $Z(p^\infty)$ . Thus  $\mathcal{G}$  has character  $K$  and has  $\alpha$  components of  $Z(p^\infty)$ , as desired.

For the furthermore clause, suppose that  $\text{Inf}(\mathcal{A})$  is a  $\Sigma_1^0$  (respectively, computable) set. Then  $\{i : b_i \in \text{Inf}(\mathcal{A})\}$  is also  $\Sigma_1^0$  (respectively, computable). Now a sequence  $\langle c_0, \dots, c_{k-1} \rangle \in \mathcal{G}$  is in  $D(\mathcal{G})$  if and only if for all  $i < k$ , if  $c_i \neq 0$ , then  $i \in \text{Inf}(\mathcal{A})$ .  $\square$

There are several corollaries to results from [6].

**Corollary 2.10** *For any  $\Sigma_2^0$  character  $K$ , there is a computable Abelian  $p$ -group  $\mathcal{G}$  with character  $K$  and with  $D(\mathcal{G})$  isomorphic to  $\oplus_\omega Z(p^\infty)$ . Furthermore, the domain of  $D(\mathcal{G})$  is a  $\Sigma_1^0$  set.*

**Proof:** By Lemma 2.3 of [6], there is a computable equivalence structure  $\mathcal{A}$  with character  $K$  such that  $\text{Inf}(\mathcal{A})$  is  $\Sigma_1^0$ . The result now follows immediately from Theorem 2.9.  $\square$

**Corollary 2.11** *For any  $r \leq \omega$  and any bounded character  $K$ , there is a computable Abelian  $p$ -group  $\mathcal{G}$  with character  $K$  and with  $D(\mathcal{G})$  isomorphic to  $\oplus_r Z(p^\infty)$ . Furthermore, the domain of  $D(\mathcal{G})$  is a computable set.*

**Proof:** By Lemma 2.4 of [6], there exists a computable equivalence structure  $\mathcal{A}$  with character  $K$ , with exactly  $r$  equivalence classes, and with  $\text{Inf}(\mathcal{A})$  computable. The result now follows from Theorem 2.9.  $\square$

If the character is not bounded, then the notions of an  $s$ -function and an  $s_1$ -function are important. These functions were introduced by Khisamiev in [19]. The  $s$ -functions are called *limitwise monotonic* in [21].

**Definition 2.12** *Let  $f : \omega^2 \rightarrow \omega$ . The function  $f$  is an  $s$ -function if the following hold:*

1. For every  $i$  and  $s$ ,  $f(i, s) \leq f(i, s + 1)$ ;
2. For every  $i$ , the limit  $m_i = \lim_s f(i, s)$  exists.

*We say that  $f$  is an  $s_1$ -function if, in addition:*

3. For every  $i$ ,  $m_i < m_{i+1}$ .

The following result about characters,  $s$ -functions and  $s_1$ -functions is immediate from Lemma 2.6 of [6].

**Lemma 2.13** *Let  $\mathcal{G}$  be a computable Abelian  $p$ -group with infinite character and with  $D(\mathcal{G})$  isomorphic to  $\oplus_r Z(p^\infty)$  where  $r$  is finite. Then*

1. *There exists a computable  $s$ -function  $f$  with corresponding limits  $m_i = \lim_s f(i, s)$  such that  $\langle n, k \rangle \in \chi(\mathcal{G})$  if and only if  $\text{card}(\{i : n = m_i\}) \geq k$ .*
2. *If the character is unbounded, then there is a computable  $s_1$ -function  $f$  such that  $\langle m_i, 1 \rangle \in \chi(\mathcal{G})$  for all  $i$ .*

**Corollary 2.14** *Let  $K$  be a  $\Sigma_2^0$  character, and let  $r$  be finite.*

1. *Let  $f$  be a computable  $s$ -function with the corresponding limits  $m_i = \lim_s f(i, s)$  such that*

$$\langle k, n \rangle \in K \iff \text{card}(\{i : n = m_i\}) \geq k.$$

*Then there is a computable Abelian  $p$ -group  $\mathcal{G}$  with  $\chi(\mathcal{G}) = K$  and with  $D(\mathcal{G})$  isomorphic to  $\oplus_r Z(p^\infty)$ .*

2. *Let  $f$  be a computable  $s_1$ -function with corresponding limits  $m_i = \lim_s f(i, s)$  such that  $\langle m_i, 1 \rangle \in K$  for all  $i$ . Then there is a computable Abelian  $p$ -group  $\mathcal{G}$  with  $\chi(\mathcal{G}) = K$  and  $D(\mathcal{G})$  isomorphic to  $\oplus_r Z(p^\infty)$ .*

**Proof:** These results follow from Theorem 2.9 and from Lemma 2.8 of [6] where corresponding equivalence structures are constructed.  $\square$

### 3 Categoricity of Abelian $p$ -Groups

The computably categorical Abelian  $p$ -groups were characterized by Goncharov [12] and Smith [33] as follows.

**Theorem 3.1 (Goncharov, Smith)** *A computable Abelian  $p$ -group  $\mathcal{G}$  is computably categorical if and only if either*

1.  $\mathcal{G} \approx \oplus_{\alpha} Z(p^{\infty}) \oplus \mathcal{F}$ , where  $\alpha \leq \omega$ , or
2.  $\mathcal{G} \approx \oplus_r Z(p^{\infty}) \oplus \oplus_{\omega} Z(p^m) \oplus \mathcal{F}$ , where  $\mathcal{F}$  is a finite Abelian  $p$ -group and  $r, m \in \omega$ .

**Lemma 3.2** 1. *If computable groups  $\mathcal{G}$  and  $\mathcal{H}$  are relatively  $\Delta_{\alpha}^0$  categorical, and  $\mathcal{G}$  and  $\mathcal{H}$  are  $\Sigma_1^0$  definable in  $\mathcal{G} \oplus \mathcal{H}$ , then  $\mathcal{G} \oplus \mathcal{H}$  is relatively  $\Delta_{\alpha}^0$  categorical.*

2. *If computable groups  $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_k$  are relatively computably categorical and each  $\mathcal{G}_i$  is  $\Sigma_1^0$  definable in  $\mathcal{G} = \mathcal{G}_1 \oplus \dots \oplus \mathcal{G}_n$ , then  $\mathcal{G}$  is relatively  $\Delta_2^0$  categorical.*

**Proof:** (1) The Scott formulas for  $\mathcal{G}$  and  $\mathcal{H}$  may be modified for  $\mathcal{G} \oplus \mathcal{H}$  to quantify only over  $\mathcal{G}$  and  $\mathcal{H}$ . Then for an element  $a = g + h \in \mathcal{G} \oplus \mathcal{H}$ , the Scott formula is

$$(\exists y \in G)(\exists z \in H)[x = y + z \ \& \ \phi^G(y) \ \& \ \psi^H(z)],$$

where  $\phi^G$  is the Scott formula for  $g$ , relativized to  $\mathcal{G}$ , and  $\psi^H$  is the Scott formula for  $h$ , relativized to  $\mathcal{H}$ . It can be checked that these formulas will be  $\Sigma_{\alpha}^0$ . For tuples  $\langle a_1, \dots, a_n \rangle$ , the method is the same. If  $\langle a_1, \dots, a_n \rangle$  and  $\langle a'_1, \dots, a'_n \rangle$  satisfy the same Scott formula, then we have  $a_i = g_i + h_i$  and  $a'_i = g'_i + h'_i$  where  $g_i$  and  $g'_i$  satisfy the same Scott formula in  $\mathcal{G}$  so that there is an automorphism  $\Phi$  of  $\mathcal{G}$  taking  $g_i$  to  $g'_i$  and similarly there is automorphism  $\Psi$  of  $\mathcal{H}$  taking  $h_i$  to  $h'_i$ . Then the mapping taking  $x + y$  to  $\Phi(x) + \Psi(y)$  will be an automorphism of  $\mathcal{G} \oplus \mathcal{H}$  taking each  $a_i$  to  $a'_i$ . Therefore, these formulas make up a Scott family as desired, so that  $\mathcal{G} \oplus \mathcal{H}$  is relatively  $\Delta_{\alpha}^0$  categorical.

(2) The relativized Scott formulas will now be  $\Sigma_2^0$  and the proof follows as in part (1).  $\square$

We can now investigate the relative computable categoricity of computable Abelian  $p$ -groups.

**Theorem 3.3** *If  $\mathcal{G}$  is a computably categorical Abelian  $p$ -group, then  $\mathcal{G}$  is relatively computably categorical.*

**Proof:** By Theorem 3.1, we have an expression for the form of  $\mathcal{G}$ . Any finite structure is certainly relatively computably categorical, so we may ignore the  $\mathcal{F}$  by Lemma 3.2.

- (1) If all summands are of the form  $Z(p^{\infty})$ , then the Scott sentence for a tuple  $\langle g_1, \dots, g_n \rangle$  simply tells the order  $o_i$  of each  $g_i$  and tells whether  $c_1 \cdot g_1 + \dots + c_n \cdot g_n = 0$  for all  $c_1 < o_1, \dots, c_n < o_n$ . Suppose that single elements  $g$

and  $g'$  have the same order  $p^k$ . Then there are divisible subgroups  $\mathcal{D}$  and  $\mathcal{D}'$  of  $\mathcal{G}$  containing  $g$  and  $g'$  (respectively), each isomorphic to  $Z(p^\infty)$ . Since  $g$  and  $g'$  have the same order, there is an isomorphism taking  $\mathcal{D}$  to  $\mathcal{D}'$ , which maps  $g$  to  $g'$  and, since  $\mathcal{D}$  and  $\mathcal{D}'$  are direct summands of  $\mathcal{G}$  by Theorem 2.4, this can be extended to an isomorphism of  $\mathcal{G}$  taking  $g$  to  $g'$ . This shows that groups of type (1) are relatively computably categorical.

(2) We may assume that  $\mathcal{F} = 0$  and  $\mathcal{G} = \mathcal{D} \oplus \mathcal{H}$ , where  $\mathcal{D} = \bigoplus_r Z(p^\infty)$  and  $\mathcal{H} = \bigoplus_\omega Z(p^m)$ . We claim that  $\mathcal{D}$  is  $\Delta_1^0$  definable, by the following. First, note that  $g \in \mathcal{D}$  if and only if  $g$  is divisible by  $p^m$ , so that  $\mathcal{D}$  is  $\Sigma_1^0$ . Second, note that there are exactly  $p^{kr}$  elements in  $\mathcal{D}$  of order  $\leq p^k$ . Now given  $g \in \mathcal{G}$ , it follows that  $g \in \mathcal{D}$  if and only if for any  $p^{kr}$  distinct elements of  $\mathcal{D}$  with order at most  $p^{kr}$ , the element  $g$  equals one of those elements. This gives a  $\Pi_1^0$  formula for  $\mathcal{D}$ . The Scott formula for a single element  $g \in \mathcal{G}$  says whether  $c \cdot g \in \mathcal{D}$  for  $c = 1, p, \dots, p^n = o(g)$ . Now suppose that  $g_1$  and  $g_2$  have the same Scott formula. If both are divisible, they are automorphic as in part (1). Now suppose that  $g_1 \notin \mathcal{D}$  and  $p^k \cdot g_1$  is not divisible for any  $p^k < o(g_1)$ . Then  $g_i = d_i + h_i$  where  $d_i \in \mathcal{D}$  and  $h_i \in \mathcal{H}$  with  $o(h_i) = p^m$ . By the previous argument, we may assume that  $d_1 = d_2$ . Now for each  $i$ ,  $h_i$  generates a pure subgroup  $\mathcal{H}_i$  of  $\mathcal{H}$  of order  $p^m$ , so by Theorem 2.4, we have  $\mathcal{H} = \mathcal{H}_i \oplus \mathcal{C}_i$  for some (isomorphic) subgroups  $\mathcal{C}_i$  of  $\mathcal{H}$ . There is certainly an isomorphism of  $\mathcal{H}_1$  onto  $\mathcal{H}_2$  taking  $h_1$  to  $h_2$  and this isomorphism may be extended to an automorphism of  $\mathcal{G}$  taking  $g_1$  to  $g_2$ . Now suppose that  $p^k \cdot g_1$  is divisible for some  $k$  with  $p^k < o(g_1)$ . Then  $g_1 = d_1 + h_1$ , where  $p^k \cdot h_1 = 0$  and hence  $h_1$  is divisible by  $p^{m-k}$ . Thus we can find  $d'_1$  and  $h'_1$  with  $g'_1 = d'_1 + h'_1$  such that  $g_1 = p^{m-k}(d'_1 + h'_1)$  and similarly for  $g_2$  and  $g'_2$ . It follows from the previous argument that  $g'_1$  and  $g'_2$  are automorphic and the same automorphism takes  $g_1$  to  $g_2$ .

For a sequence  $\langle g_1, \dots, g_n \rangle$  from  $\mathcal{G}$ , the Scott formula includes the Scott formulas for each element and also says which linear combinations  $c_1 \cdot g_1 + \dots + c_n \cdot g_n = 0$  and which are divisible, where each  $c_i < o(g_i)$ . We prove by induction on  $n$  that if  $\langle g_1, g_2, \dots, g_n \rangle$  and  $\langle g'_1, g'_2, \dots, g'_n \rangle$  satisfy the same Scott formula, then they are automorphic. The case  $n = 1$  is given above. For  $n > 1$ , suppose that  $\langle g_1, g_2, \dots, g_n \rangle$  and  $\langle g'_1, g'_2, \dots, g'_n \rangle$  satisfy the same Scott formula; it follows that  $\langle g_1, g_2, \dots, g_{n-1} \rangle$  and  $\langle g'_1, g'_2, \dots, g'_{n-1} \rangle$  also satisfy the same Scott formula, and are therefore automorphic by induction. There are two cases. If some constant  $c_i$  in a true equation is not divisible by  $p$ , then, without loss of generality, we can solve the equation for  $g_n$  and use the observation above that  $\langle g_1, g_2, \dots, g_{n-1} \rangle$  and  $\langle g'_1, g'_2, \dots, g'_{n-1} \rangle$  are automorphic. If all constants of all true equations are divisible by  $p$ , then we may find  $a_i$  and  $a'_i$  with  $g_i = pa_i$  and  $g'_i = pa'_i$ , and it suffices to show that  $\langle a_1, \dots, a_n \rangle$  and  $\langle a'_1, \dots, a'_n \rangle$  are automorphic. After some finite number of divisions, we will eventually get coefficients not divisible by  $p$ .  $\square$

Note that this argument depends on the fact that in  $\bigoplus_\infty Z(p^m)$  an element has order  $\leq p^k$  if and only if it is divisible by  $p^{m-k}$ . This is not true in the group  $\bigoplus_\infty Z(p^m) \oplus \bigoplus_\infty Z(p^n)$  where  $m \neq n$ . Of course any group which is not computably categorical cannot be relatively computably categorical, so Theorem

3.3 characterizes the relatively computably categorical Abelian  $p$ -groups.

Next we consider  $\Delta_2^0$  categoricity. The first case is when the reduced part of  $\mathcal{G}$  has finite period.

**Theorem 3.4** *Suppose that  $\mathcal{G}$  is isomorphic to  $\oplus_\alpha Z(p^\infty) \oplus \mathcal{H}$ , where  $\mathcal{H}$  has finite period and  $\alpha \leq \omega$ . Then  $\mathcal{G}$  is relatively  $\Delta_2^0$  categorical.*

**Proof:** Since the period  $p^r$  is finite,  $\mathcal{G}$  is a direct sum of computably categorical groups of the form  $\oplus_\alpha Z(p^\infty)$  and  $\oplus_\omega Z(p^m)$ , together with some finite  $F$ . The Scott formulas are similar to those given above for the computably categorical groups, except that when  $g$  is not divisible, we need to ask whether it is divisible by  $p^k$  for each  $k < r$ , or is not divisible by  $p^k$ . The latter question is  $\Pi_1^0$ , so the Scott formulas are a conjunction of  $\Sigma_1^0$  and  $\Pi_1^0$ . Likewise for a sequence of elements, we need to ask whether each linear combination is divisible by  $p^k$ . After that, the argument is essentially the same as in Theorem 3.3.  $\square$

There is a special case when  $\mathcal{G}$  has no divisible part.

**Theorem 3.5** *Suppose that  $\mathcal{G}$  is a computable Abelian  $p$ -group with all elements of finite height. Then  $\mathcal{G}$  is relatively  $\Delta_2^0$  categorical. [Note: These are exactly the reduced Abelian  $p$ -groups of length at most  $\omega$ .]*

**Proof:** For any finite subgroup  $\mathcal{F}$  of  $\mathcal{G}$  and any finite sequence  $\vec{g}$  of elements of  $\mathcal{F}$ , the formula  $\phi_{\vec{g}, \mathcal{F}}(\vec{x})$  gives the atomic diagram of  $\mathcal{F}[\vec{g}]$  and also states that  $\mathcal{F}$  is a pure subgroup. The latter question is a  $\Pi_1$  condition,

$$(\forall g \in F)(\forall n)(\forall x)[p^n \cdot x = g \Rightarrow (\exists y \in F)p^n \cdot y = g].$$

The Scott formula for  $\vec{g}$  states that there exists a finite set  $F = \{a_1, \dots, a_t\}$  so that  $\phi_{\vec{g}, \mathcal{F}}(\vec{g})$  and furthermore, no subgroup of  $\mathcal{F}$  is pure. If  $\vec{g}$  and  $\vec{g}'$  satisfy the same Scott formula, then there are isomorphic pure subgroups  $\mathcal{F}$  containing  $\vec{g}$  and  $\mathcal{F}'$  containing  $\vec{g}'$ . Since  $\mathcal{F}$  and  $\mathcal{F}'$  are pure, there exist isomorphic summands  $\mathcal{H}$  and  $\mathcal{H}'$  so that  $\mathcal{G} = \mathcal{F} \oplus \mathcal{H} = \mathcal{F}' \oplus \mathcal{H}'$  so that the isomorphism between  $\mathcal{F}$  and  $\mathcal{F}'$  may be extended to an automorphism of  $\mathcal{G}$ .

Now  $\mathcal{G} = \oplus_{n < \omega} Z(p^{i_n})$  where each  $i_n$  is finite, so that for each  $k$ ,  $\oplus_{n < k} Z(p^{i_n})$  is a pure subgroup of  $\mathcal{G}$  and any finite sequence  $\vec{g}$  will be included in one of these pure subgroups. Thus every  $\vec{g}$  satisfies some Scott formula.  $\square$

We claim that no other Abelian  $p$ -groups are relatively  $\Delta_2^0$  categorical. We first show this for groups which are products of cyclic and quasicyclic groups. It turns out that even for a group  $\mathcal{G}$  of infinite period with only finitely many  $Z(p^\infty)$  components,  $\mathcal{G}$  is not relatively  $\Delta_2^0$  categorical. This differs from equivalence structures, where any structure with only a finite number of infinite equivalence classes is relatively  $\Delta_2^0$  categorical. For equivalence structures, each class is necessarily computable but  $D(\mathcal{G})$  need not be computable even when there is just one copy of  $Z(p^\infty)$ .

**Theorem 3.6** *Suppose that a computable group  $\mathcal{G}$  is isomorphic to  $\oplus_\alpha Z(p^\infty) \oplus \mathcal{H}$  for some group  $\mathcal{H}$  with infinite period and all elements of finite height, where  $\alpha \neq 0$ . Then  $\mathcal{G}$  is not relatively  $\Delta_2^0$  categorical.*

**Proof:** Let  $\mathcal{G} = \mathcal{D} \oplus \mathcal{H}$ , where  $\mathcal{D}$  is divisible and  $\mathcal{H}$  is a product of cyclic groups of unbounded order. Suppose  $\mathcal{G}$  had a  $\Sigma_2^0$  family of Scott sentences. We will show that there is an element of the divisible part  $\mathcal{D}$  whose Scott formula is satisfied by some element of  $\mathcal{H}$ . But there can be no automorphism of  $\mathcal{G}$  mapping a divisible element to a non-divisible element. This contradiction will show that there is no such Scott sentence.

We first assume that  $\alpha = \omega$ .

Let  $a$  be an element of  $\mathcal{D}$  of order  $p$  which satisfies a  $\Sigma_2^0$  Scott formula  $\Psi(x, \vec{d})$ . We observe first that we may assume that the parameters  $\vec{d}$  are independent, and in fact belong to different components in the product decomposition of  $\mathcal{G}$ . For the finite summands, we can assume the parameter is a generator, and for the quasicyclic summands, we can take the parameter to have maximal order (and therefore generate any other possible parameters).

Of course any other divisible element of order  $p$  must satisfy the same formula, so we may assume that  $a$  belongs to a subgroup  $\mathcal{A}$  isomorphic to  $Z(p^\infty)$  which does not contain any of the parameters. Then, by choosing witnesses  $\vec{c}$  to instantiate the existentially quantified variables in  $\Psi$ , we have a computable  $\Pi_1^0$  formula  $\theta(x, \vec{d}, \vec{c})$  satisfied by  $a$ .

We can now use the fact that this  $\Pi_1^0$  sentence is true in  $\mathcal{G}$  if and only if it is true in all finite subgroups of  $\mathcal{G}$  containing  $a, \vec{c}, \vec{d}$ .

Let  $a, \vec{c}, \vec{d}$  generate a finite subgroup  $\mathcal{F}_1$  of  $\mathcal{G}$  and let  $\mathcal{A}_1 = \mathcal{A} \cap \mathcal{F}_1$  be a finite subgroup of  $\mathcal{A}$  of order  $p^m$ , and  $\mathcal{F}_1 = \mathcal{A}_1 \oplus \mathcal{B}_1$  for some group  $\mathcal{B}_1$ . Now find a factor group  $\mathcal{H}_1 \subset \mathcal{H}$  of  $\mathcal{G}$  of order type  $\geq p^m$  and independent of  $\mathcal{F}_1$ ; this exists since  $\mathcal{H}$  has infinite period. We may assume without loss of generality that  $|\mathcal{H}_1| = p^m$  and that each of  $\vec{d}$  is in  $\mathcal{H}_1$ . Let  $\phi$  be an isomorphism from  $\mathcal{A}_1 \oplus \mathcal{F}_1$  to  $\mathcal{H}_1 \oplus \mathcal{F}_1$  which is the identity on  $\mathcal{H}_1$ , and  $a' = \phi(a)$  and let  $\vec{b}$  be the image of  $\vec{c}$  under this mapping.

We claim that  $\theta(a', \vec{d}, \vec{b})$  holds. Now let  $\mathcal{H}'$  be any finite subgroup of  $\mathcal{G}$  containing  $a', \vec{d}, \vec{b}$ ; we may assume that  $\mathcal{H}' = \mathcal{H}_1 \oplus \mathcal{F}_2$  where  $\mathcal{F}_1 \subseteq \mathcal{F}_2$ . Furthermore, we may assume (by taking an automorphism of  $\mathcal{G}$  if necessary) that  $\mathcal{F}_2 \cap \mathcal{A}_1 = \emptyset$ . Then  $\phi^{-1}$  may be extended to an isomorphism from  $\mathcal{H}'$  to a finite subgroup  $\mathcal{A}_1 \oplus \mathcal{F}_2$ . Since  $\theta$  is  $\Pi_1^0$ ,  $\mathcal{A}_1 \oplus \mathcal{F}_2 \models \theta(a, \vec{c}, \vec{d})$ . Thus by the isomorphism,  $\mathcal{H}' \models \theta(a', \vec{b}, \vec{d})$ . Since this is true for any finite subgroup of  $\mathcal{G}$ , it follows that  $\mathcal{G} \models \theta(a', \vec{b}, \vec{d})$ . Therefore  $\Psi(a', \vec{d})$  holds for the Scott formula  $\Psi$ .

But  $a'$  is not divisible, so it is not automorphic with  $a$ . This contradiction proves the theorem in the first case.

Suppose now that  $\alpha$  is finite; we will assume for simplicity that  $\alpha = 1$ . Let  $d_i$  be the parameter of greatest order in any quasicyclic summand, and let  $a$  be an element of that summand with  $p \cdot a = d$ . Let  $p^m$  be the order of  $a$  and let  $\mathcal{F}_1$  be the cyclic subgroup generated by  $a$ ; note that any other parameter in  $\mathcal{F}_1$  is a multiple of  $d_i$ . Now choose an element  $g$  of order  $p^m$ , generating a subgroup  $\mathcal{F}_2$  so that any element from  $\vec{d}$  in  $\mathcal{F}_1 \oplus \mathcal{F}_2$  is already in  $\mathcal{F}_1$ . This can be done since  $\mathcal{H}$  has infinite period. Now let  $a' = a + p^{m-1} \cdot g$ , so that  $p \cdot a' = d_i$ . Then there

is an automorphism of  $\mathcal{F}_1 \oplus \mathcal{F}_2$  taking  $a$  to  $a'$  and preserving the parameters, defined by  $\psi(j \cdot a + k \cdot g) = j \cdot a' + k \cdot g$ . This automorphism may be extended to an automorphism of the finite subgroup  $\mathcal{H}_1$  generated by  $a, \vec{d}, \vec{b}$ . Let  $\vec{c}$  be the image of  $\vec{b}$  under this extended automorphism.

We claim that  $\theta(a', \vec{d}, \vec{c})$  holds. Let  $\mathcal{H}'$  be any finite subgroup of  $\mathcal{G}$  containing  $a', \vec{d}, \vec{c}$ ; we may assume that  $\mathcal{H}' = \mathcal{H}_1 \oplus \mathcal{F}$  for some  $\mathcal{F}$ , so that there is an automorphism of  $\mathcal{H}'$  taking  $a, \vec{d}, \vec{b}$  to  $a', \vec{d}, \vec{c}$ . Since  $\mathcal{H}'$  is a finite subgroup of  $\mathcal{G}$ , we have  $\mathcal{H}' \models \theta(a, \vec{d}, \vec{b})$  and hence, by the automorphism,  $\mathcal{H}' \models \theta(a', \vec{d}, \vec{c})$ . Since this holds for any finite subgroup  $\mathcal{H}'$ , it follows that  $\theta(a', \vec{d}, \vec{c})$  and hence  $\Psi(a', \vec{d})$ . But  $a'$  is not divisible, so cannot be automorphic with  $a$ .  $\square$

In the paper [6], we defined a uniformly  $\Sigma_2^0$  enumeration  $K_e$  of the  $\Sigma_2^0$  characters and an enumeration  $\mathcal{C}_e$  of the computable equivalence structures. For a total computable function  $\phi_e : \omega \times \omega \rightarrow \omega$ , let  $\mathcal{G}_e$  be the structure with universe  $\omega$  and with group operation  $\phi_e$ .

**Lemma 3.7 ([6])** *For any fixed infinite  $\Sigma_2^0$  character  $K$ ,  $\{e : K_e = K\}$  is  $\Pi_3^0$  complete.*

**Theorem 3.8 ([6])** *Let  $\mathcal{A}$  be a computable equivalence structure with unbounded character  $K$  and with infinitely many infinite equivalence classes. Suppose also that there exists a structure  $\mathcal{B}$  with character  $K$  and with no infinite equivalence classes. Then  $\{e : \mathcal{C}_e \simeq \mathcal{A}\}$  is  $\Pi_4^0$  complete.*

We can apply this analysis to  $p$ -groups for a similar result, using Theorem 2.9.

**Theorem 3.9** *Let  $\mathcal{G}$  be isomorphic to  $\oplus_\omega Z(p^\infty) \oplus \mathcal{H}$ , with  $\mathcal{H}$  having infinite period and all elements of finite height. Suppose also that there is a computable copy of  $\mathcal{H}$ . Then  $\{e : \mathcal{C}_e \simeq \mathcal{G}\}$  is  $\Pi_4^0$  complete.*

**Proof:** Fix such a group  $\mathcal{G}$  with character  $K$ , and let  $\mathcal{C}$  be an equivalence structure with character  $K$ . It can be checked that  $\{e : \mathcal{G}_e \simeq \mathcal{G}\}$  is a  $\Pi_4^0$  set. For the completeness, we observe that the uniformity of the proof of Theorem 2.9 provides a computable function  $f$  such that  $\mathcal{C}_a$  is isomorphic to  $\mathcal{C}_b$  if and only if  $\mathcal{G}_{f(a)}$  is isomorphic to  $\mathcal{G}_{f(b)}$ . Then  $\mathcal{C}_e \simeq \mathcal{C}$  if and only if  $\mathcal{G}_{f(e)} \simeq \mathcal{G}$  and the completeness follows from Theorem 3.8.  $\square$

This gives the following result for categoricity.

**Theorem 3.10** *Suppose that a computable group  $\mathcal{G}$  is isomorphic to*

$$\oplus_\omega Z(p^\infty) \oplus \mathcal{H}$$

*for some group  $\mathcal{H}$  with infinite period and all elements of finite height, and suppose, in addition, that there is a computable group isomorphic to  $\mathcal{H}$ . Then  $\mathcal{G}$  is not  $\Delta_2^0$  categorical.*

**Proof:** If  $\mathcal{G}$  were  $\Delta_2^0$  categorical, then  $\{e : \mathcal{G}_e \simeq \mathcal{G}\}$  has a  $\Sigma_4^0$  definition. That is, let  $M$  be a complete c.e. set, let  $+$  be  $+^{\mathcal{G}}$  and let  $+_e$  be  $+^{\mathcal{G}_e}$ . Then  $\mathcal{G}_e \simeq \mathcal{G}$  if and only if

$$(\exists a)[a \in \text{Tot}^M \wedge (\forall m)(\forall n)(\phi_a^M(m+n) = \phi_a^m(m) +_e \phi_a^M(n))].$$

But this contradicts the  $\Pi_4^0$  completeness from Theorem 3.9.  $\square$

Finally, all of the groups discussed above are certainly relatively  $\Delta_3^0$  categorical.

**Theorem 3.11** *Let  $\mathcal{G}$  be a computable group isomorphic to  $\oplus_{\alpha} Z(p^\infty) \oplus \mathcal{H}$ , where  $\mathcal{H}$  has all elements of finite height. Then  $\mathcal{G}$  is relatively  $\Delta_3^0$  categorical.*

**Proof:** The divisible part  $D(\mathcal{G})$  can be defined by a  $\Pi_2^0$  sentence. The definition of the Scott sentences builds on that of Theorem 3.5. Given a finite pure subgroup  $\mathcal{F}$  and a finite subgroup  $\mathcal{D}$  of  $D(\mathcal{G})$ , we define the formula  $\phi_{\vec{g}, \mathcal{F}, \mathcal{D}}$  as before to give the atomic diagram of  $\mathcal{D} \oplus \mathcal{F}[\vec{g}]$ . Any such formula satisfied by  $\vec{g}$  will be a Scott formula.  $\square$

There is a stronger result for groups  $\mathcal{G}$  with  $D(\mathcal{G})$  computable.

**Theorem 3.12** *For any two isomorphic computable Abelian  $p$ -groups  $\mathcal{G}_1$  and  $\mathcal{G}_2$  of length  $\leq \omega$  such that  $D(\mathcal{G}_1)$  and  $D(\mathcal{G}_2)$  are both computable,  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are  $\Delta_2^0$  isomorphic.*

**Proof:** We will construct computable subgroups  $\mathcal{H}_1$  and  $\mathcal{H}_2$  such that  $\mathcal{G}_i = D(\mathcal{G}_i) \oplus \mathcal{H}_i$ . The subgroup  $\mathcal{H}_i$  will be defined as the union of a computable sequence  $\mathcal{A}_s$  of pure finite subgroups.  $\mathcal{A}_0 = \{0\}$ . Given  $\mathcal{A}_s$ , find the least element  $g \notin \mathcal{A}_s$  such that  $\langle \mathcal{A}_s \cup \{g\} \rangle \cap D(\mathcal{G}_i) = \{0\}$  and let  $\mathcal{A}_{s+1} = \langle \mathcal{A}_s \cup \{g\} \rangle$ . Then for each  $s$ ,  $D(\mathcal{G}_i) \cap \mathcal{A}_s = \{0\}$  and therefore  $D(\mathcal{G}) \cap \mathcal{H}_i = \{0\}$ . The factor group  $\mathcal{H}_i$  is computable since  $s \in \mathcal{H}_i$  if and only if  $s \in \mathcal{A}_{s+1}$ . We argue by induction on the order of  $g$  that any element  $g \in \mathcal{G}$  belongs to  $D(\mathcal{G}) \oplus \mathcal{H}_i$ . For the initial case, suppose that  $p \cdot g = 0$ . Then either  $g \in \mathcal{A}_{g+1}$  or else  $a+g = d \neq 0$  for some  $d \in D(\mathcal{G})$ . But in the latter case,  $g = a - d \in D(\mathcal{G}) \oplus \mathcal{H}_i$  as desired. Now suppose all elements of order  $< p^m$  belong to  $D(\mathcal{G}) \oplus \mathcal{H}_i$  and let  $p^m g = 0$ . Then  $h = p \cdot g = a + d$  for some  $a \in \mathcal{H}_i$  and  $d \in D(\mathcal{G})$ . Since  $d$  is divisible, we can choose  $d'$  so  $d = pd'$ . Then we have  $a = p \cdot (g - d')$  so that, since  $\mathcal{H}_i$  is pure,  $a = p \cdot a'$  for some  $a' \in \mathcal{H}_i$ . Now  $p \cdot (g - d' - a') = 0$ , so that  $g - d' - a' \in D(\mathcal{G}) \oplus \mathcal{H}_i$  by the initial case. Therefore,  $g \in D(\mathcal{G}) \oplus \mathcal{H}_i$  as desired.

It follows from Proposition 3.1 that  $D(\mathcal{G}_1)$  and  $D(\mathcal{G}_2)$  are computably isomorphic, and it follows from Theorem 3.5 that  $\mathcal{H}_1$  and  $\mathcal{H}_2$  are  $\Delta_2^0$  isomorphic. Now, the two corresponding isomorphisms may be combined into a  $\Delta_2^0$  isomorphism between  $\mathcal{G}_1$  and  $\mathcal{G}_2$ .  $\square$

## 4 Groups of length $> \omega$

Index set results can tell us something about many of the groups of greater length. By using the calculations in Theorems 5.6, 5.15, and 5.16 of [7], we can prove the following.

**Theorem 4.1** *Let  $\mathcal{G}$  be an Abelian  $p$ -group.*

1. *If  $\lambda(\mathcal{G}) = \omega \cdot n$  and  $m \leq 2n - 1$ , then  $\mathcal{G}$  is not  $\Delta_m^0$ -categorical.*
2. *If  $\lambda(\mathcal{G}) > \omega \cdot n$  and  $m \leq 2n - 2$ , then  $\mathcal{G}$  is not  $\Delta_m^0$ -categorical.*
3. *If  $\lambda(\mathcal{G}) = \omega \cdot n + k$  where  $k \in \omega$ , and  $\mathcal{H}$  is the reduced part of  $\mathcal{G}$ , the following hold:*
  - (a) *If  $\mathcal{H}_{\omega n}$  is isomorphic to  $\mathbb{Z}_{p^k}$  and  $m \leq 2n - 1$ , then  $\mathcal{G}$  is not  $\Delta_m^0$ -categorical.*
  - (b) *If  $\mathcal{H}_{\omega n}$  is finite but not isomorphic to  $\mathbb{Z}_{p^k}$  and  $m \leq 2n - 1$ , then  $\mathcal{G}$  is not  $\Delta_m^0$ -categorical.*
  - (c) *If there is a unique  $j < k$  such that  $u_{\omega n + j}(\mathcal{H}) = \infty$ , and  $m \leq 2n$ , then  $\mathcal{G}$  is not  $\Delta_m^0$ -categorical.*
  - (d) *If there are distinct  $i, j < k$  such that  $u_{\omega n + i}(\mathcal{H}) = u_{\omega n + j}(\mathcal{H}) = \infty$  and  $m \leq 2n + 1$ , then  $\mathcal{G}$  is not  $\Delta_m^0$ -categorical.*

**Corollary 4.2** *Let  $\mathcal{G}$  be a computable Abelian  $p$ -group whose reduced part has a computable copy, and suppose that  $\mathcal{H}$  is the reduced part of  $\mathcal{G}$ . Then if  $\mathcal{H}$  has infinitely many elements of height at least  $\omega$ , then  $\mathcal{G}$  is not  $\Delta_2^0$ -categorical.*

Results of Barker [5] give the following additional information.

**Theorem 4.3 (Barker)** *Let  $\mathcal{G}$  be a countable reduced Abelian  $p$ -group with computable Ulm invariants such that  $\lambda(\mathcal{G}) = \omega\alpha + \omega + n$  and  $\mathcal{G}_{\omega\alpha + \omega}$  is finite. Then  $\mathcal{G}$  is relatively  $\Delta_{2\alpha+2}^0$ -categorical but not  $\Delta_{2\alpha+1}^0$ -categorical.*

Finally, we can prove the following result on relative categoricity:

**Theorem 4.4** *Let  $\mathcal{G}$  be a computable Abelian  $p$ -group with  $\lambda(\mathcal{G}) > \omega$  whose reduced part has no computable copy. Then  $\mathcal{G}$  is not relatively  $\Delta_2^0$ -categorical.*

**Proof:** The proof is the same as that of Theorem 3.6. □

## 5 Open Problems

The present paper does not completely characterize the relatively  $\Delta_2^0$ -categorical or  $\Delta_2^0$ -categorical Abelian  $p$ -groups. We give below an exhaustive list of the open cases.

**Problem 5.1** *Let  $\mathcal{G}$  be a computable Abelian  $p$ -group isomorphic to  $\mathcal{D} \oplus \mathcal{H}$ , where  $\mathcal{D}$  is a direct sum of finitely many copies of the Prüfer group, and  $\mathcal{H}$  is reduced, with infinite period but all elements of finite height. Can  $\mathcal{G}$  be  $\Delta_2^0$ -categorical?*

**Problem 5.2** *Let  $\mathcal{G}$  be a computable Abelian  $p$ -group whose reduced part has no computable copy. We have shown that  $\mathcal{G}$  cannot be relatively  $\Delta_2^0$ -categorical. Can it be  $\Delta_2^0$ -categorical?*

## References

- [1] C.J. Ash, Categoricity in hyperarithmetical degrees, *Annals of Pure and Applied Logic* 34 (1987), pp. 1–14.
- [2] C.J. Ash and J.F. Knight, *Computable Structures and the Hyperarithmetical Hierarchy* (Elsevier, Amsterdam, 2000).
- [3] C.J. Ash, J. Knight, M. Manasse, and T. Slaman, Generic copies of countable structures, *Annals of Pure and Applied Logic* 42 (1989), pp. 195–205.
- [4] C.J. Ash and A. Nerode, Intrinsically recursive relations, in *Aspects of Effective Algebra*, ed. by J.N. Crossley, Upside Down A Book Co., Steel’s Creek, Australia, 1981, pp. 26–41.
- [5] E.J. Barker, Back and forth relations for reduced Abelian  $p$ -groups, *Annals of Pure and Applied Logic* 75 (1995), pp. 223–249.
- [6] W. Calvert, D. Cenzer, V.S. Harizanov and A. Morozov, Effective categoricity of equivalence structures, *Annals of Pure and Applied Logic* 141 (2006), pp. 61–78.
- [7] W. Calvert, V.S. Harizanov, J.F. Knight, and S. Miller, Index sets of computable structures, *Algebra and Logic* 45 (2006), pp. 306–325.
- [8] J. Chisholm, Effective model theory vs. recursive model theory, *Journal Symbolic Logic* 55 (1990), pp. 1168–1191.
- [9] J. Chisholm, E. Fokina, S.S. Goncharov, V.S. Harizanov, J.F. Knight, and S. Quinn, Intrinsic bounds on complexity at limit levels, submitted.
- [10] R.G. Downey, Computability theory and linear orderings, in: Yu.L. Ershov, S.S. Goncharov, A. Nerode, and J.B. Remmel, editors, *Handbook of Recursive Mathematics*, vol. 2 (North-Holland, Amsterdam, 1998), pp. 823–976.
- [11] L. Fuchs, *Infinite Abelian Groups* (Academic Press, New York, 1970).
- [12] S.S. Goncharov, Autostability of models and Abelian groups, *Algebra and Logic* 19 (1980), pp. 23–44 (Russian), pp. 13–27 (English translation).
- [13] S.S. Goncharov, The quantity of nonautoequivalent constructivizations, *Algebra and Logic* 16 (1977), pp. 257–282 (Russian), pp. 169–185 (English translation).
- [14] S.S. Goncharov, Autostability and computable families of constructivizations, *Algebra and Logic* 14 (1975), pp. 647–680 (Russian), pp. 392–409 (English translation).
- [15] S.S. Goncharov and V.D. Dzgoev, Autostability of models, *Algebra and Logic* 19 (1980), pp. 45–58 (Russian), pp. 28–37 (English translation).

- [16] S.S. Goncharov, V.S. Harizanov, J.F. Knight, C.F.D. McCoy, R.G. Miller, and R. Solomon, Enumerations in computable structure theory, *Annals of Pure and Applied Logic* 136 (2005), pp. 219–246.
- [17] S.S. Goncharov, V.S. Harizanov, J.F. Knight, and R.A. Shore,  $\Pi_1^1$  relations and paths through  $\mathcal{O}$ , *Journal of Symbolic Logic* 69 (2004), pp. 585–611.
- [18] S. Goncharov, S. Lempp, and R. Solomon, The computable dimension of ordered Abelian groups, *Advances in Mathematics* 175 (2003), pp. 102–143.
- [19] N.G. Khisamiev, Constructive Abelian  $p$ -groups, *Siberian Advances in Mathematics* 2 (1992), pp. 68–113 (English translation).
- [20] N.G. Khisamiev, Constructive Abelian groups, in: Yu.L. Ershov, S.S. Goncharov, A. Nerode, and J.B. Remmel, editors, *Handbook of Recursive Mathematics*, vol. 2 (North-Holland, Amsterdam, 1998), pp. 1177–1231.
- [21] B. Khossainov, A. Nies, and R.A. Shore, Computable models of theories with few models, *Notre Dame Journal of Formal Logic* 38 (1997), pp. 165–178.
- [22] O.V. Kudinov, An autostable 1-decidable model without a computable Scott family of  $\exists$ -formulas, *Algebra and Logic* 35 (1996), pp. 458–467 (Russian), pp. 255–260 (English translation).
- [23] O.V. Kudinov, A description of autostable models, *Algebra and Logic* 36 (1997), pp. 26–36 (Russian), pp. 16–22 (English translation).
- [24] P. LaRoche, Recursively presented Boolean algebras, *Notices AMS* 24 (1977), A552–A553.
- [25] S. Lempp, C. McCoy, R. Miller, and R. Solomon, Computable categoricity of trees of finite height, *Journal of Symbolic Logic* 70 (2005), pp. 151–215.
- [26] C.F.D. McCoy,  $\Delta_2^0$ -categoricity in Boolean algebras and linear orderings, *Annals of Pure and Applied Logic* 119 (2003), pp. 85–120.
- [27] G. Metakides and A. Nerode, Effective content of field theory, *Annals of Mathematical Logic* 17 (1979), pp. 289–320.
- [28] R. Miller, The computable dimension of trees of infinite height, *Journal of Symbolic Logic* 70 (2005), pp. 111–141.
- [29] A.T. Nurtazin, Strong and weak constructivizations and computable families, *Algebra and Logic* 13 (1974), pp. 311–323 (Russian), pp. 177–184 (English translation).
- [30] J.B. Remmel, Recursively categorical linear orderings, *Proceedings of the American Mathematical Society* 83 (1981), pp. 387–391.

- [31] J.B. Remmel, Recursive isomorphism types of recursive Boolean algebras, *Journal of Symbolic Logic* 46 (1981), pp. 572–594.
- [32] V.L. Selivanov, Enumerations of families of general recursive functions, *Algebra and Logic* 15 (1976), pp. 205–226 (Russian), pp. 128–141 (English translation).
- [33] R.L. Smith, Two theorems on autostability in  $p$ -groups, in *Logic Year 1979–80*, University of Connecticut, Storrs, Lecture Notes in Mathematics 859, Springer, Berlin (1981), pp. 302–311.
- [34] R.I. Soare, *Recursively Enumerable Sets and Degrees. A Study of Computable Functions and Computably Generated Sets* (Springer-Verlag, Berlin, 1987).