Relations on computable structures

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0.1 Introduction

Gődel's incompleteness theorem from 1931 is an astonishing early result of computable mathematics. Gődel showed that "there are in fact relatively simple problems in the theory of ordinary whole numbers which cannot be *decided* from the axioms." The work of Gődel, Turing, Kleene, Church, Post and others in the mid-1930's established the rigorous mathematical foundations for the computability theory. However, even in 1930, van der Waerden in his *Moderne Algebra I* introduced an *explicitly* given field as one "whose elements are uniquely represented by distinguishable symbols with which addition, subtraction, multiplication and division can be performed in a finite number of operations." He showed that if a field is given explicitly, then every simple extension of that field is also given explicitly. In his pioneering paper on non-factorability of polynomials from 1930, van der Waerden essentially proved that an explicit field $(F, +, \cdot)$ does not necessarily have an algorithm for splitting polynomials in $F[x]$ into their irreducible factors. In the 1950's, Fröhlich and Shepherdson used the precise notion of a computable function to obtain a collection of results and examples about explicit rings and fields. Several years later, Rabin and Mal'tsev did an extensive investigation of computable groups and other *computable* (*recursive*, *constructive*) algebras. In the 1970's, Metakides and Nerode initiated a systematic study of computability in mathematical structures and constructions by using modern computability-theoretic tools, such as the *priority method*.

Computable mathematics explores the algorithmic content (effectiveness) of notions, constructions and theorems in classical mathematics. It starts by defining effective analogues of classical concepts in algebra and model theory. If we begin with structures, and effectivize the notion of a structure, we arrive at the notion of a computable structure. An algebraic structure for a finite language is *computable* if its domain is a computable set and its operations and relations are computable. For example, $(Q, +)$ is a computable group, $(Q, \leq)$ is a computable linear order, and $(Q, +, \cdot)$ is a computable field. On the other hand, if we begin with the notion of a theory and effectivize it, we arrive at the notion of a *decidable* (also called *computable*) theory. We then obtain the notion of a *decidable* structure, that is, a structure with a decidable complete diagram. Henkin's construction builds a decidable model for a consistent decidable theory. Notice that for a computable structure only its atomic diagram is a decidable theory. Thus, in the case of a computable structure, the starting point is semantic, while in the case of a decidable structure the starting point is syntactic. Two of the main problems in computable mathematics are: determining the decidability of certain theories, and determining whether any (or how many) computable or decidable structures with specific properties exist.

While some mathematical constructions can be replaced by effective ones, for others such replacement is impossible in principle. Thus, if a classical result does not effectivize, another important objective for computable mathematics is the discovery of its effective counterexamples or its effective substitutes. For example, Rabin proved that every computable field has a computable algebraic closure. On the other hand, Metakides and Nerode established that if $\mathcal{F} = (F, +, \cdot)$ is a computable field without a splitting algorithm, then for every computable algebraic closure $\mathcal{C}$ of $\mathcal{F}$ there is another computable algebraic closure which is not computably F-isomorphic to $\mathcal{C}$.

0.2 Definitions and notation

The computability-theoretic notation is standard and as in [23]. We review some of it. If φ is a partial function, then $dom(\varphi)$ is the domain of φ, $ran(\varphi)$ is the range of φ, and $\varphi(a)\downarrow$ denotes that $a \in dom(\varphi)$. We fix $\langle \cdot, \cdot \rangle$ to be a computable bijection from ω^2 to ω. If $\overrightarrow{s}$ is a finite sequence, then by $lh(\overrightarrow{s})$ we denote its length. The concatenation of sequences is denoted by $\hat{}$. We often identify a set X with its characteristic function χ_X. Let $X \subseteq \omega$. Then $\varphi_0^X, \varphi_1^X, \varphi_2^X, \ldots$ is a fixed effective enumeration of all unary X-computable functions. The function φ_e^X is also denoted by $\{e\}^X$. For $e \in \omega$, let $W_e^X =_{def} dom(\varphi_e^X)$. Hence $W_0, W_1, W_2, \ldots$ is an effective enumeration of all computably enumerable (c.e.) sets. We write $\varphi_{e,s}^X(n) = m$ if $e, n, m < s$, only numbers $z < s$ are used in the computation, and $\varphi_e^X(n) = m$ in fewer than s steps. Let $p \in 2^{<\omega}$. We write $\varphi_{e,s}^p(n) = m$ if $\varphi_{e,s}^Z(n) = m$ for some $Z \subseteq \omega$ such that $p \subset Z$ and only elements in $dom(p)$ are used in the computation. Let $Y \subseteq \omega$. The join $X \oplus Y$ is $\{2n : n \in X\} \cup \{2n+1 : n \in Y\}$. By $X \leq_T Y$ ($X \equiv_T Y$, respectively) we denote that X is Turing reducible to Y (X is Turing equivalent to Y, respectively). By

$X <_T Y$ we denote that $X \leq_T Y$ but $Y \not\leq_T X$. We have that $\mathbf{x} = \deg(X)$ is the Turing degree of X. Hence $\mathbf{0} = \deg(\emptyset)$ and $\mathbf{x}^{(n)} = \deg(X^{(n)})$, where $X^{(n)}$ is the n-th jump of X. Let $n \geq 1$ be a natural number. A set is Σ^0_n if it is c.e. relative to $\mathbf{0}^{(n-1)}$. A set is Π^0_n if its negation is Σ^0_n, and it is Δ^0_n if it is both Σ^0_n and Π^0_n. A set is *arithmetic* if it is Δ^0_n for some $n \geq 1$. The set of all Turing degrees is denoted by $\mathcal{D}$. Let $\mathcal{P}$ be an arbitrary computability-theoretic complexity class of sets. A Turing degree is $\mathcal{P}$ if it contains a set which is in $\mathcal{P}$.

We consider only computable first-order languages and only countable structures. A sequence of variables displayed after a formula contains all free variables occurring in the formula. Structures are denoted by script letters, and their domains by the corresponding capital Latin letters. The isomorphism of structures is denoted by $\cong$. Let $\mathcal{A}$ be a structure. The language of $\mathcal{A}$ is $L(\mathcal{A})$. The language of $L(\mathcal{A}) \cup \{\mathbf{a} : a \in A\}$ is denoted by $L(\mathcal{A})_A$. The expansion of the structure $\mathcal{A}$ to the language $L(\mathcal{A})_A$, where every constant $\mathbf{a}$ is interpreted by a, is $\mathcal{A}_A$. Let $\mathcal{P}$ be a class of formulae. Define the $\mathcal{P}$-*diagram* of a structure $\mathcal{A}$ to be the set of all $\mathcal{P}$-sentences in $L(\mathcal{A})_A$ which are true in $\mathcal{A}_A$. A basic sentence is an atomic sentence or the negation of an atomic sentence. Hence the atomic diagram of $\mathcal{A}$ is the set of all basic sentences of $L(\mathcal{A})_A$ which are true in $\mathcal{A}_A$. A structure is called 1-*computable* if its existential diagram is computable or, equivalently, its universal diagram is computable.

An ordinal is *computable* if it is finite or is the order type of a computable well ordering on ω. The computable ordinals form a countable initial segment of the ordinals. Kleene's $\mathcal{O}$ is the set of notations for computable ordinals, with the corresponding partial ordering $<_{\mathcal{O}}$ (see [22]). The least non-computable ordinal is denoted by ω_1^{CK}, where CK stands for Church-Kleene. To extend the arithmetic hierarchy, we define the representative sets in the hyperarithmetic hierarchy, H_a for $a \in \mathcal{O}$. The definition is recursive, and is based on iterating Turing jump: $H_1 = \emptyset$, $H_{2^a} = (H_a)'$, and $H_{3 \cdot 5^e} = \{2^x \cdot 3^n : x \in H_{\{e\}(n)}\}$. Let β be an infinite computable ordinal. Then a set is Σ^0_β if it is c.e. relative to some H_a such that β is represented by notation a. A set is Π^0_β if its negation is Σ^0_β, and it is Δ^0_β if it is both Σ^0_β and Π^0_β. A set is *hyperarithmetic* if it is Δ^0_α for some computable α. Hence, a set X is hyperarithmetic if $(\exists a \in \mathcal{O})[X \leq_T H_a]$. The hyperarithmetic sets coincide with the Δ^1_1 sets.

Let α be a computable ordinal. Ash [1] has defined computable Σ_α and Π_α formulae of $L_{\omega_1\omega}$, recursively and simultaneously, and together with their Gődel numbers. The computable Σ_0 and Π_0 formulae are the finitary quantifier-free formulae. The computable $\Sigma_{\alpha+1}$ formulae are of the form

$$\bigvee_{n \in W_e} \exists \overrightarrow{y_n} \psi_n(\overrightarrow{x}, \overrightarrow{y_n}),$$

where for $n \in W_e$, ψ_n is a Π_α formula indexed by its Gődel number, and

$\exists\overrightarrow{y_n}$ is a finite block of existential quantifiers. That is, $\Sigma_{\alpha+1}$ formulae are c.e. disjunctions of $\exists\Pi_\alpha$ formulae. Similarly, $\Pi_{\alpha+1}$ formulae are c.e. conjunctions of $\forall\Sigma_\alpha$ formulae. It can be shown that a computable Σ_1 formula is of the form

$$\bigvee_{n\in\omega} \exists\overrightarrow{y_n}\theta_n(\overrightarrow{x},\overrightarrow{y_n}),$$

where $(\theta_n(\overrightarrow{x},\overrightarrow{y_n}))_{n\in\omega}$ is a computable sequence of quantifier-free formulae. If α is a limit ordinal, then the Σ_α (Π_α, respectively) formulae are of the form $\bigvee\limits_{n\in W_e} \psi_n$ ($\bigwedge\limits_{n\in W_e} \psi_n$, respectively), such that there is a sequence $(\alpha_n)_{n\in W_e}$ of ordinals having limit α, given by the ordinal notation for α, and every ψ_n is a Σ_{α_n} (Π_{α_n}, respectively) formula. For a more precise definition of computable Σ_α and Π_α formulae see [1].

0.3 Ash-Nerode program

A structure $\mathcal{A}$ is computable if its domain A is a computable set and the relations and functions of $\mathcal{A}$ are uniformly computable. Equivalently, A is computable and the atomic diagram of $\mathcal{A}$ is computable. Thus, there is a computable enumeration $(a_i)_{i\in\omega}$ of A and an algorithm which determines for every quantifier-free formula $\theta(x_{i_0},\dots,x_{i_{n-1}})$ in $L(\mathcal{A})$ and for every sequence $(a_{i_0},\dots,a_{i_{n-1}})\in A^n$, whether $\mathcal{A}_A\vDash\theta(\mathbf{a}_{i_0},\dots,\mathbf{a}_{i_{n-1}})$.

One of the important and interesting questions in computable model theory is how a specific aspect of a computable structure may change if the structure is isomorphically transformed so that it remains computable. A structure $\mathcal{B}$ isomorphic to a computable structure $\mathcal{A}$ is not necessarily computable. However, even if $\mathcal{B}$ is computable, it can still lose many of the computable properties of $\mathcal{A}$.

A computable property of a computable structure $\mathcal{A}$ which Ash and Nerode [5] consider is an additional computable relation R on the domain of $\mathcal{A}$. That is, R is not named in $L(\mathcal{A})$. Ash and Nerode have investigated syntactic conditions on $\mathcal{A}$ and R under which for every isomorphism f from $\mathcal{A}$ onto a computable model $\mathcal{B}$, $f(R)$ is c.e. Such relations are called *intrinsically c.e.* on $\mathcal{A}$. In general, we have the following definition.

Definition 1 *Let $\mathcal{P}$ be a certain class of relations. An additional relation R on the domain of a computable structure $\mathcal{A}$ is called intrinsically $\mathcal{P}$ on $\mathcal{A}$ if the image of R under every isomorphism from $\mathcal{A}$ to a computable structure belongs to $\mathcal{P}$.*

Ash and Nerode showed that if $\mathcal{A}=(\omega,S)$, where S is the successor relation, then every computable relation on its domain is intrinsically computable. Such structures are called *computably stable*. On the other hand, neither the relation S, nor being an even number, are intrinsically computable on $(\omega, <)$. Ash and Nerode have introduced a computable syntactic condition, which under a certain extra decidability condition (D) on $\mathcal{A}$ involving R, characterizes those R which are intrinsically c.e. on $\mathcal{A}$. For an additional m-ary relation R on $\mathcal{A}$, the condition (D) is:

There is an algorithm which determines for every existential formula $\psi(x_0, \ldots, x_{m-1}, \overrightarrow{y})$ and every $\overrightarrow{c} \in A^{lh(\overrightarrow{y})}$, whether the following implication holds for every $a_0, \ldots, a_{m-1} \in A$:

$$[\mathcal{A}_A \models \psi(\mathbf{a}_0, \ldots, \mathbf{a}_{m-1}, \overrightarrow{\mathbf{c}})] \Rightarrow R(a_0, \ldots, a_{m-1}).$$

Condition (D) implies that R is a computable relation. It also implies that $\mathcal{A}$ is 1-computable. Condition (D) is satisfied in many natural examples of structures and relations.

Definition 2 (*Ash-Nerode [5]*) *Let R be an additional m-ary computable relation on the domain of a computable structure $\mathcal{A}$. Then the relation R is called formally c.e.* (*or formally Σ_1*) *on $\mathcal{A}$ if and only if there is a finite sequence $\overrightarrow{c}$ of elements in A and a computable Σ_1 formula $\psi(x_0, \ldots, x_{m-1}, \overrightarrow{\mathbf{c}})$ such that the following equivalence holds for every $a_0, \ldots, a_{m-1} \in A$:*

$$R(a_0, \ldots, a_{m-1}) \Leftrightarrow \mathcal{A}_A \models \psi(\mathbf{a}_0, \ldots, \mathbf{a}_{m-1}, \overrightarrow{\mathbf{c}}).$$

Furthermore, R is called formally computable on $\mathcal{A}$ if both R and its complement $\overline{R}$ are formally c.e. on $\mathcal{A}$.

That is, a relation is formally c.e. iff it is equivalent to an infinite disjunction of a computable sequence of existential formulae with finitely many parameters. It is easy to see that every formally c.e. relation on a computable structure is intrinsically c.e. Ash and Nerode have proven that, under condition (D), the converse holds; thus, establishing the equivalence of a syntactic and a semantic condition.

Theorem 1 (*Ash-Nerode [5]*) *Let R be an additional m-ary relation on the domain of a structure $\mathcal{A}$, satisfying the decidability condition* (*D*)*. Then*

$$(R \textit{ is intrinsically c.e. on } \mathcal{A}) \Rightarrow (R \textit{ is formally c.e. on } \mathcal{A}).$$

Proof sketch. For simplicity, assume that R is a unary relation. Assume that R is not formally c.e. on $\mathcal{A}$. We use a finite injury priority method to construct a computable structure $\mathcal{B}$ and an isomorphism f from $\mathcal{B}$ to $\mathcal{A}$ such that we satisfy the following requirement for every $e \in \omega$,

$$R_e : f^{-1}(R) \neq W_e.$$

The isomorphism f is obtained as $f = \lim f_s$, where f_s is a finite partial isomorphism constructed at stage s. The strategy for meeting a single requirement R_e is to wait for a stage s such that for some $b \in B$, $b \in W_{e,s}$, and to define f_s so that $f_s(b) \notin R$. It is possible to define such a partial isomorphism f_s because the relation R is not formally c.e. ■

Condition (D) is needed in the proof of Theorem 1 to effectively check for given $b \in W_{e,s}$, whether it is possible to define a partial finite isomorphism f_s so that $f_s(b) \notin R$. Goncharov and Manasse have independently shown that there is an intrinsically c.e. relation on a computable structure which is not formally c.e. Chisholm has further established that there is an intrinsically c.e. relation on a decidable structure which is not definable by any Σ_2 formula.

Barker [6] has extended Theorem 1 to all Σ^0_α relations, where α is a computable ordinal.

Definition 3 *An m-ary relation R is called formally Σ^0_α on $\mathcal{A}$ if there is a finite sequence $\overrightarrow{c}$ of elements in A and a computable Σ_α formula $\psi(x_0, \ldots, x_{m-1}, \overrightarrow{\mathbf{c}})$ such that the following equivalence holds for every $a_0, \ldots, a_{m-1} \in A$:*

$$R(a_0, \ldots, a_{m-1}) \Leftrightarrow \mathcal{A}_A \models \psi(\mathbf{a}_0, \ldots, \mathbf{a}_{m-1}, \overrightarrow{\mathbf{c}}).$$

We will prove that if R is formally Σ^0_α on $\mathcal{A}$, then R is intrinsically Σ^0_α on $\mathcal{A}$. Let f be an isomorphism from $\mathcal{A}$ to another computable structure $\mathcal{B}$. We have for every $b_0, \ldots, b_{m-1} \in B$:

$$[(b_0, \ldots, b_{m-1}) \in f(R)] \Leftrightarrow [\mathcal{B}_B \models \psi(\mathbf{b}_0, \ldots, \mathbf{b}_{m-1}, \mathbf{f}(\overrightarrow{\mathbf{c}}))].$$

Thus, $f(R)$ is defined by a computable Σ_α formula and, hence, it is Σ^0_α.

To establish that every intrinsically Σ^0_α relation is formally Σ^0_α, we will assume that $\mathcal{A}$ satisfies an extra decidability condition (D_α), expressed in terms of R, which has been introduced by Barker [6]. For example, let us consider the case when $\alpha = 2$ and, for simplicity, assume that R is a unary relation. Let $\overrightarrow{a} \in A^{<\omega}$. The *universal type* of $\overrightarrow{a}$ in $\mathcal{A}$, in symbols $Tp_\forall(\overrightarrow{a})$, is the set of all universal formulae satisfied in $\mathcal{A}$ by $\overrightarrow{a}$. Condition (D_2) consists of the following three subconditions:

(i) The structure $\mathcal{A}$ is 1-computable.

(ii) There is an algorithm which determines for every $\overrightarrow{b}, \overrightarrow{d} \in A^{<\omega}$ such that $lh(\overrightarrow{b}) = lh(\overrightarrow{d})$ whether

$$Tp_\forall(\overrightarrow{b}) \subseteq Tp_\forall(\overrightarrow{d}).$$

(iii) There is an algorithm which determines for every $\overrightarrow{c}, \overrightarrow{b} \in A^{<\omega}$ and every $r \in R$ whether there exist $\overrightarrow{d} \in A^{lh(\overrightarrow{b})}$ and $r' \in \overline{R}$ such that the sequence $\overrightarrow{c}\,\hat{}\,r'\,\hat{}\,\overrightarrow{d}$ satisfies every formula in $Tp_\forall(\overrightarrow{c}\,\hat{}\,r\,\hat{}\,\overrightarrow{b})$.

Theorem 2 (*Barker [6]*) *Let R be an additional computable relation on the domain A of a structure $\mathcal{A}$ such that the decidability condition (D_2) is satisfied. Then*

$$(R \textit{ is intrinsically } \Sigma^0_\alpha \textit{ on } \mathcal{A}) \Leftrightarrow (R \textit{ is formally } \Sigma^0_\alpha \textit{ on } \mathcal{A}).$$

In addition to considering the complexity of the relations on computable structures within arithmetic and hyperarithmetic hierarchy, we also consider their Turing degrees.

Definition 4 (*Harizanov [16]*) (*i*) *Turing degree spectrum of R on $\mathcal{A}$, in symbols $Dg_{\mathcal{A}}(R)$, is the set of all Turing degrees of the images of R under all isomorphisms from $\mathcal{A}$ onto computable structures. If for some such isomorphism f, $X = f(R)$ and $\mathbf{x} = deg(X)$, then we say that $\mathbf{x}$ is realized in $Dg_{\mathcal{A}}(R)$ via X.*

(*ii*) *For a computable structure $\mathcal{B}$ such that $\mathcal{B} \cong \mathcal{A}$, the* (*Turing*) *degree spectrum of R on $\mathcal{A}$ with respect to $\mathcal{B}$, in symbols $Dg_{\mathcal{A},\mathcal{B}}(R)$, is the set of all Turing degrees of the images $f(R) \subseteq B$ under all isomorphisms f from $\mathcal{A}$ to $\mathcal{B}$.*

Theorem 3 (*Harizanov [15]*) *If for a non-intrinsically c.e. relation R on a structure $\mathcal{A}$ condition* (*D*) *holds, then there are computable structures $\mathcal{B}$ and $\mathcal{C}$, and isomorphisms $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{A} \to \mathcal{C}$ such that $f(R)$ and $g(R)$ are Δ^0_2 relations which are Turing incomparable.*

0.4 Uncountable degree spectra

Since for every Turing degree $\mathbf{x}$, there are at most countably many Turing degrees which are $\leq \mathbf{x}$, and since every countable set of Turing degrees has an upper bound, a set of Turing degrees is uncountable if and only if it is unbounded.

Let $\mathbf{b}$ be an arbitrary nonzero c.e degree. In [16], we give an example of a degree spectrum $Dg_{\mathcal{A},\mathcal{A}}(R)$ whose nonzero degrees form an upper cone with the bottom $\mathbf{b}$. That is,

$$Dg_{\mathcal{A},\mathcal{A}}(R) = \{\mathbf{0}\} \cup \{\mathbf{x} : \mathbf{x} \geq \mathbf{b}\}.$$

We also give an example of an uncountable degree spectrum $Dg_{\mathcal{A},\mathcal{B}}(R)$ which consists of pairwise incomparable Turing degrees.

Let $\mathcal{A}$, $\mathcal{B}$ be computable structures such that $\mathcal{A} \cong \mathcal{B}$. By $\mathcal{I}(\mathcal{A},\mathcal{B})$ we denote the set of all isomorphisms from $\mathcal{A}$ to $\mathcal{B}$. We say that a finite function p from A to B is a *partial isomorphism* from $\mathcal{A}$ to $\mathcal{B}$ if p is $1-1$, and for every atomic formula $\theta = \theta(x_0, \dots, x_{n-1})$ in $L(\mathcal{A})$, and every $a_0, \dots, a_{n-1} \in dom(p)$, we have

$$\mathcal{A}_A \models \theta(\mathbf{a}_0, \dots, \mathbf{a}_{n-1}) \Leftrightarrow \mathcal{B}_B \models \theta(\mathbf{b}_0, \dots, \mathbf{b}_{n-1}).$$

where $b_0 = (a_0), \dots, b_{n-1} = p(a_{n-1})$. By $\mathcal{I}_{fin}(\mathcal{A},\mathcal{B})$ we denote the set of all partial finite isomorphisms from $\mathcal{A}$ to $\mathcal{B}$. Let R be an additional, say unary, computable relation on A. We define an equivalence relation $\sim_R$ on $\mathcal{I}_{fin}(\mathcal{A},\mathcal{B})$ as follows:

$$q \sim_R r \Leftrightarrow (\forall b \in ran(q) \cap ran(r))[q^{-1}(b) \in R \Leftrightarrow r^{-1}(b) \in R].$$

Theorem 4 (*Harizanov [16]*) *The following are equivalent:*
(*i*) $Dg_{\mathcal{A}}(R)$ *is uncountable.*
(*ii*) $Dg_{\mathcal{A},\mathcal{B}}(R)$ *is uncountable.*
(*iii*) $Dg_{\mathcal{A},\mathcal{B}}(R)$ *has cardinality* 2^{ω}.
(*iv*) *There is a nonempty set* $\mathbb{P} \subseteq \mathcal{I}_{fin}(\mathcal{A}, \mathcal{B})$ *such that the following two conditions are satisfied:*

(*A*) $(\forall p \in \mathbb{P})(\forall a \in A)(\forall b \in B)(\exists q \in \mathbb{P})[q \supseteq p \wedge a \in dom(q) \wedge b \in ran(q)]$;
(*B*) $(\forall p \in \mathbb{P})(\exists q, r \in \mathbb{P})[q \supseteq p \wedge r \supseteq p \wedge \neg(q \sim_R r)]$.

Theorem 5 (*Harizanov [16]*) *Let* $\mathbb{P}$ *be as in* (*iv*) *of Theorem 4. Then for every set* $C \subseteq \omega$ *such that* $C \geq_T \mathbb{P}$, *there is an isomorphism* f *from* $\mathcal{A}$ *to* $\mathcal{B}$ *such that*

$$C \equiv_T f(R) \oplus \mathbb{P} \equiv_T f \oplus \mathbb{P}.$$

Proof sketch. Let $C \geq_T \mathbb{P}$. We use a finite extension argument, together with coding, to construct a sequence $(f_s)_{s\in\omega}$ of finite partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$, which is computable both in C and in $f(R) \oplus \mathbb{P}$, where $f =_{def} \bigcup_{s\in\omega} f_s$. Given f_{2e}, we find the first two finite functions r_0 and r_1 in $\mathbb{P}$ such that $r_0 \supseteq f_{2e}$, $r_1 \supseteq f_{2e}$; and $r_0^{-1}(n) \notin R$, $r_1^{-1}(n) \in R$, where n is the least number in $[ran(r_0) \cap ran(r_1)]$. This can be done computably in $\mathbb{P}$. Now we define

$$f_{2e+1} = \begin{cases} r_1 & \text{if } e \in C, \\ r_0 & \text{if } e \notin C. \end{cases}$$

■

In particular, if $\mathbb{P}$ in Theorem 5 is computable (or c.e.), then $Dg_{\mathcal{A},\mathcal{B}}(R)=\mathcal{D}$ and, moreover, for every set $C \subseteq \omega$, there is an isomorphism f from $\mathcal{A}$ to $\mathcal{B}$ such that

$$C \equiv_T f(R) \equiv_T f.$$

We now concentrate on degree spectra which coincide with all of $\mathcal{D}$. The following example further motivates the characterization of such degree spectra.

Let $\mathcal{A} = (Q, \leq)$, where Q is the set of all rational numbers. We will show that every Turing degree contains an initial segment of $\mathcal{A}$. Let $C \subseteq \omega$ be an infinite coinfinite set. We define a real number r_C by

$$r_C = \sum_{n\in C} \frac{1}{2^n}.$$

Let X be the initial segment of $\mathcal{A}$ determined by r_C. That is, $X = \{q \in Q : q < r_C\}$. We can easily establish that $C \equiv_T X$. That is, if $R =_{def} \{q \in Q : q < \sqrt{2}\}$, then $Dg_{\mathcal{A},\mathcal{A}}(R) = \mathcal{D}$.

Theorem 6 (*Harizanov [12]*) *The following are equivalent:*
(*i*) *The degree spectrum $Dg_{\mathcal{A},\mathcal{B}}(R) = \mathcal{D}$ and, moreover, for every set $C \subseteq \omega$, there is an isomorphism f from $\mathcal{A}$ to $\mathcal{B}$ such that $C \equiv_T f(R) \equiv_T f$.*
(*ii*) *There are $e \in \omega$ and $p \in 2^{<\omega}$ such that the set*

$$\mathbb{P}_{e,p} =_{def} \{\varphi_e^q : q \in 2^{<\omega} \wedge q \supseteq p\}$$

has the following properties:

$$\mathbb{P}_{e,p} \subseteq \mathcal{I}_{fin}(\mathcal{A}, \mathcal{B}),$$

$$(A) \text{ from Theorem 4 (iv) is satisfied for } \mathbb{P} = \mathbb{P}_{e,p}, \text{ and}$$

$$(\exists i \in \omega)(\forall q \supseteq p)(\forall a \in dom(q))[\varphi_i^{\varphi_e^q(R)}(a) \downarrow = q(a)].$$

(*iii*) *There is a nonempty computable (or c.e.) set $\mathbb{P} \subseteq \mathcal{I}_{fin}(\mathcal{A}, \mathcal{B})$ such that the conditions (A) and (B) from Theorem 4 (iv) are satisfied.*

Proof sketch. $\neg(ii) \Rightarrow \neg(i)$ Assume the negation of (ii). That is, for every $\langle e, i\rangle$ and every $p \in 2^{<\omega}$, there is $q \in 2^{<\omega}$ such that $q \supseteq p$ and

$$\begin{array}{rcl} \varphi_e^q & \notin & \mathcal{I}_{fin}(\mathcal{A}, \mathcal{B}), \text{ or} \\ (\exists a & \in & A)(\forall r \supseteq q)[a \notin dom(\varphi_e^r)], \text{ or} \\ (\exists b & \in & B)(\forall r \supseteq q)[b \notin ran(\varphi_e^r)], \text{ or} \\ (\exists a & \in & dom(q))[\varphi_i^{\varphi_e^q(R)}(a) \downarrow \neq q(a)]. \end{array}$$

We now use a finite extension argument to construct a set $C \subseteq \omega$ which satisfies the following requirement for every $\langle e, i\rangle$:

$$R_{\langle e,i\rangle} : \varphi_e^C \in \mathcal{I}(\mathcal{A}, \mathcal{B}) \Rightarrow \varphi_i^{\varphi_e^C(R)} \neq C.$$

Let $\mathbf{c} = \deg(C)$. The degree $\mathbf{c}$ cannot be realized in $Dg_{\mathcal{A},\mathcal{B}}(R)$ via an isomorphism of degree $\mathbf{c}$.

(*ii*) $\Rightarrow$ (*iii*) Fix the corresponding e and p. By assumption, $\mathbb{P}_{e,p} \subseteq \mathcal{I}_{fin}(\mathcal{A}, \mathcal{B})$ and (A) is satisfied for $\mathbb{P} = \mathbb{P}_{e,p}$. We can show that (B) is also satisfied for $\mathbb{P} = \mathbb{P}_{e,p}$.

(*iii*) $\Rightarrow$ (*i*) This is already proved in Theorem 5. ■

The equivalence of (i) and (iii) in Theorem 6 has also been established independently by Ash, Cholak and Knight in [2]. Their proof explicitly uses the forcing method. In the proof of $\neg(ii) \Rightarrow \neg(i)$ for Theorem 6, the construction of C can be done computably in $\emptyset''$. Hence $C \in \Delta_3^0$. Thus, if not every Turing degree is obtained in a degree spectrum $Dg_{\mathcal{A},\mathcal{B}}(R)$ via an isomorphism of the same Turing degree, then there is such a Δ_3^0 degree. This conclusion also follows from the proof in [2], since there is a generic Δ_3^0 set.

0.5 Realizing Δ^0_2 degrees in degree spectra

Let $\mathcal{A}_0 = (\omega, \prec)$ be the following computable linear order of order type $\omega + \omega^*$:

$$0 \prec 2 \prec 4 \prec \ldots \prec 5 \prec 3 \prec 1.$$

We define a computable relation R_0 to be the initial segment of type ω; that is, $R_0 = 2\omega$. An early result, obtained independently by Tennenbaum and Denisov, is that there is an isomorphic computable copy of $\mathcal{A}_0$ whose initial segment of type ω is not computable. It is easy to see that the relation R_0 is intrinsically Δ^0_2 on $\mathcal{A}_0$, because of the following definability of R_0 and $\neg R_0$:

$$x \in R_0 \Leftrightarrow \bigvee_{n\in\omega} \exists x_0 \ldots \exists x_n [x_0 \prec x_1 \prec \ldots \prec x_n \wedge x = x_n \wedge$$
$$\forall y[\neg(y \prec x_0) \wedge \neg(x_0 \prec y \prec x_1) \wedge \ldots \wedge \neg(x_{n-1} \prec y \prec x_n)],$$

and

$$x \notin R_0 \Leftrightarrow \bigvee_{n\in\omega} \exists x_0 \ldots \exists x_n [x_0 \succ x_1 \succ \ldots \succ x_n \wedge x = x_n \wedge$$
$$\forall y[\neg(y \succ x_0) \wedge \neg(x_0 \succ y \succ x_1) \wedge \ldots \wedge \neg(x_{n-1} \succ y \succ x_n)].$$

It can further be shown that the degree spectrum $Dg_{\mathcal{A}_0}(R_0)$ consists of all Δ^0_2 degrees (see [12]).

Now we will describe Ershov's classification of Δ^0_2 degrees. Let α be a computable ordinal. A set $C \subseteq \omega$ is α-c.e. if there are a computable function $f : \omega^2 \to \{0,1\}$ and a computable function $o : \omega \times \omega \to \alpha + 1$ with the following properties:

$$(\forall x)[\lim_{s\to\infty} f(x,s) = C(x) \wedge \ f(x,0) = 0],$$

$$(\forall x)(\forall s)[o(x, s+1) \leq o(x,s) \wedge o(x,0) = \alpha], \text{ and}$$

$$(\forall x)(\forall s)[f(x,s+1) \neq f(x,s) \Rightarrow o(x,s+1) < o(x,s)].$$

In particular, 1-c.e. sets are c.e. sets, and 2-c.e. sets are d-c.e. sets.

Let R be an additional unary computable relation on the domain of a computable structure $\mathcal{A}$. We are interested in syntactic conditions such that for an α-c.e. degree $\mathbf{c}$, there is an isomorphism f of degree $\mathbf{c}$ to a computable structure $\mathcal{B}$ for which $f(R)$ is also of degree $\mathbf{c}$. First we need the following definition.

The complement of R with respect to A is denoted $\overline{R}$. Let $\mathbf{R}$ be a symbol for R. If we are interested in the c.e. images of R, certain first-order formulae with positive occurrences of $\mathbf{R}$ in the expanded language $L(\mathcal{A})\cup\{\mathbf{R}\}$ play a special role. A Σ_1 formula in $L(\mathcal{A})\cup\{\mathbf{R}\}$ in which $\mathbf{R}$ occurs only positively is also called a Σ_1^Γ formula. This notation was introduced by Ash and Knight, who defined a hierarchy of infinitary formulae in a general setting in which Γ is a function assigning computable ordinals to relation symbols.

Definition 5 *Let $\overrightarrow{c} \in A^{<\omega}$ and $a \in A$.*

(i) (Harizanov) We say that a is free (also called 1-free) over $\overrightarrow{c}$ if $a \in \overline{R}$ and for every finitary Σ_1^Γ formula $\psi(\overrightarrow{z}, x)$, $lh(\overrightarrow{z}) = lh(\overrightarrow{c})$, if

$$(\mathcal{A}_A, R) \models \psi(\overrightarrow{\mathbf{c}}, \mathbf{a})$$

then

$$(\exists a' \in R)[(\mathcal{A}_A, R) \models \psi(\overrightarrow{\mathbf{c}}, \mathbf{a}')].$$

(ii) (Ash-Knight) Let β be a computable ordinal such that $\beta > 1$. The element a is β-free over $\overrightarrow{c}$ if $a \in \overline{R}$ and for every ordinal γ such that $1 \leq \gamma < \beta$,

$$(\forall \overrightarrow{u})(\exists a' \in R)(\exists \overrightarrow{v})[\overrightarrow{c}\,\hat{}\,a\,\hat{}\,\overrightarrow{u} \leq_\gamma \overrightarrow{c}\,\hat{}\,a'\,\hat{}\,\overrightarrow{v}].$$

Here, $\leq_\gamma$ is a binary relation defined on finite sequences of equal length from $A^{<\omega}$ by: $\overrightarrow{s} \leq_\gamma \overrightarrow{s'}$ iff every Π_γ formula true of $\overrightarrow{s}$ is also true of $\overrightarrow{s'}$ (equivalently, every Σ_γ formula true of $\overrightarrow{s'}$ is also true of $\overrightarrow{s}$).

Let the set of all free elements over $\overrightarrow{c}$ be denoted by $fr(\overrightarrow{c})$. Note that $fr(\overrightarrow{c}) \subseteq \overline{R}$. Clearly, if $a \in fr(\overrightarrow{c})$ and $\overrightarrow{d}$ is a subsequence of $\overrightarrow{c}$, then $a \in fr(\overrightarrow{d})$. Let

$$bd(\overrightarrow{c}) =_{def} \{a \in \overline{R}\text{: } a \text{ is not free over } \overrightarrow{c}\}.$$

Thus, if $a \in bd(\overrightarrow{c})$ and $\overrightarrow{c}$ is a subsequence of $\overrightarrow{d}$, then $a \in bd(\overrightarrow{d})$. A maximal relation on $\overline{R}$ (with respect to the set-theoretic inclusion) which is definable by a computable Σ_1^Γ formula with parameters $\overrightarrow{c}$ is of the form $bd(\overrightarrow{c})$. Conversely, if $bd(\overrightarrow{c})$ is definable by a computable Σ_1^Γ formula with parameters $\overrightarrow{c}$, then $bd(\overrightarrow{c})$ is a maximal relation on $\overline{R}$ definable by such a formula. For example, if $(\mathcal{A}, R) = (\mathcal{A}_0, R_0)$ then

$$a \in fr(\overrightarrow{c}) \Leftrightarrow a \prec c_{i_0},$$

where c_{i_0} is the $\prec$-least element in $\overline{R} \cap ran(\overrightarrow{c})$. If $\mathcal{A}$ is $(\omega, =)$ and R is a computable infinite coinfinite subset of ω, then

$$a \in fr(\overrightarrow{c}) \Leftrightarrow a \notin ran(\overrightarrow{c}).$$

Theorem 7 *(Harizanov [15]) Assume that there is an algorithm which for every $\overrightarrow{c} \in A^{<\omega}$ outputs an element $a \in A$ such that a is free over $\overrightarrow{c}$. Let*

$C \subseteq \omega$ be a c.e. set. Then there exists a computable structure $\mathcal{B}$ and an isomorphism $f : \mathcal{A} \to \mathcal{B}$ such that

$$f(R) \equiv_T f \equiv_T C \wedge (f(R) \textit{ is c.e.}).$$

Proof sketch. Let $\{C_s\}_{s\in\omega}$ be a computable enumeration of C such that at every stage s, C receives at most one new element, and that element is $\leq s$. Let $B = \omega$ and let $(\theta_e)_{e\in\omega}$ be an effective list of all atomic sentences in $L(\mathcal{A})_B$. For every e, either θ_e or $\neg\theta_e$ will be enumerated in the diagram of $\mathcal{B}$. At every stage s of the construction, we define a finite isomorphism f_s from $\mathcal{B}$ to $\mathcal{A}$. Let $X_s = f_s^{-1}(R)$ and

$$\overline{X_s} = \{d_0^s < d_1^s < d_2^s < \dots\}.$$

We set $X =_{def} \bigcup_{s\in\omega} X_s$. During the construction, we define a partially computable function $h(n, s)$ such that for every s,

$$h(n, s) \downarrow \Leftrightarrow n \in \{0, \dots, s+1\}.$$

For every n, there exists $\lim_{s\to\infty} h(n, s)$, and for every c,

$$c \in C_s - C_{s-1} \Rightarrow d^{s-1}_{\gamma(c,s-1)} \in X_s.$$

■

Let α be a nonzero ordinal. By Cantor's normal form theorem, there is a unique representation

$$\alpha = \omega^{\alpha_1} \cdot n_1 + \omega^{\alpha_2} \cdot n_2 + \dots + \omega^{\alpha_k} \cdot n_k,$$

where $\alpha_1 > \alpha_2 > \dots > \alpha_k$ and $0 < n_1, n_2, \dots, n_k < \omega$. Let

$$c_n(\alpha) =_{def} \omega^{\alpha_1} \cdot (nn_1) + \omega^{\alpha_2} \cdot (nn_2) + \dots + \omega^{\alpha_k} \cdot (nn_k),$$

and

$$c^*(\alpha) =_{def} \sup\{c_n(\alpha) : n \in \omega\}.$$

Hence, if β is the greatest ordinal such that $\omega^\beta \leq \alpha$, then $c^*(\alpha) = \omega^{\beta+1} \leq \alpha \cdot \omega$.

Theorem 8 *(Ash-Cholak-Knight [2]) Let $C \subseteq \omega$ be an α-c.e. set, where $\alpha \geq 2$. Assume that the relations $(\leq_\gamma)_{1\leq\gamma<c^*(\alpha)}$ are uniformly c.e. Assume that for every $n \in \omega$, for every sequence $\overrightarrow{c} \in A^{<\omega}$, there is $a \in \overline{R}$ such that a is $c_n(\alpha)$-free over $\overrightarrow{c}$. Then there is a computable model $\mathcal{B}$ and an isomorphism f from $\mathcal{A}$ to $\mathcal{B}$ such that*

$$f(R) \equiv_T C \equiv_T f.$$

0.6 Finite degree spectra

The following result shows how the Ash-Nerode decidability condition (D) affects the cardinality of the degree spectrum.

Theorem 9 (*Harizanov [15]*) *(i) If for a non-intrinsically c.e. relation R on a structure $\mathcal{A}$, condition (D) holds, then the degree spectrum $Dg_{\mathcal{A}}(R)$ is infinite.*

(ii) There is a computable non-intrinsically c.e. relation R on a computable structure $\mathcal{A}$ such that the degree spectrum $Dg_{\mathcal{A}}(R) = \{\mathbf{0}, \mathbf{x}\}$, where $\mathbf{x} \leq \mathbf{0}''$ but $\mathbf{x} \not\leq \mathbf{0}'$.

Theorem 9 (ii) uses Goncharov's results from the theory of enumerations.

Theorem 10 (*Harizanov [14]*) *There is a computable structure $\mathcal{A}$ and a computable unary relation R on its domain such that $Dg_{\mathcal{A}}(R) = \{\mathbf{0}, \mathbf{x}\}$, where $\mathbf{0} < \mathbf{x} \leq \mathbf{0}'$ but $\mathbf{x}$ cannot be realized via a c.e. set.*

Proof sketch. We use Goncharov's infinite injury method. First, we construct a family $\mathcal{S}$ of c.e. sets and a subfamily $\mathcal{P}$, $\mathcal{P} \subseteq \mathcal{S}$, with specific properties. We call a function $\nu : \omega \to \mathcal{S}$ a computable enumeration of $\mathcal{S}$ if there is a uniformly computable sequence $\{\nu_t\}_{t\in\omega}$ of functions from ω to the set of finite subsets of ω such that for every $n \in \omega$, $\nu(n) = \bigcup\limits_{t\in\omega} \nu_t(n)$. The family $\mathcal{S}$ constructed has two injective computable enumerations, ν and μ, such that every other injective computable enumeration λ of $\mathcal{S}$ is computably equivalent to ν or μ. Here, λ is computably equivalent to ν if the function $f : \omega \to \omega$ such that $\nu = \lambda f$ is computable. While the set $\nu^{-1}(\mathcal{P})$ is a computable subset of ω, the set $\mu^{-1}(\mathcal{P})$ is a non-c.e. Δ^0_2 set.

We then encode the enumeration ν into a rigid computable structure $\mathcal{A}$. The category of injective computable enumerations of $\mathcal{S}$, whose morphisms are equivalences (computable equivalencies, respectively) of the enumerations, is equivalent to the category of computable structures isomorphic to $\mathcal{A}$ whose morphisms are isomorphisms (computable isomorphisms, respectively) of the structures. The set R which encodes $\nu^{-1}(\mathcal{P})$ in $\mathcal{A}$ is computable and its degree spectrum on $\mathcal{A}$ is as required. ■

Theorem 11 (*Khoussainov and Shore [19], Goncharov and Khoussainov [9]*) *There is a computable structure $\mathcal{A}$ and a computable unary relation R on its domain such that $Dg_{\mathcal{A}}(R) = \{\mathbf{0}, \mathbf{x}\}$, where $\mathbf{x}$ is a nonzero degree realized via a c.e. set.*

Proof sketch. (1) Khoussainov and Shore directly construct a rigid directed graph $\mathcal{G}$ with a unary relation U on its domain such that $\mathcal{G}$ has, up to isomorphism, exactly two computable presentations, $\mathcal{A}$ and $\mathcal{B}$. The image R of U on A is computable, while the image X of U on B is c.e. and noncomputable. Moreover, the relation $\{(b, a) : b \in X \wedge a \in R \wedge$ (there is an isomorphism from $\mathcal{B}$ to $\mathcal{A}$ which extends the map $b \to a)\}$ is computable.

(2) Goncharov and Khoussainov first construct a family $\mathcal{S}$ of c.e. sets and a subfamily $\mathcal{P}$, with specific properties. The family $\mathcal{S}$ has, up to computable equivalence, exactly two injective computable enumerations, ν and μ. The set $\nu^{-1}(\mathcal{P})$ is a computable subset of ω, and the set $\mu^{-1}(\mathcal{P})$ is a noncomputable c.e. set. Let $\mathcal{A}$ be a rigid computable model which encodes the enumeration ν. A computable relation R is the subset which encodes $\nu^{-1}(\mathcal{P})$ in $\mathcal{A}$. ■

The following result broadly generalizes Theorem 11.

Theorem 12 (*Khoussainov and Shore [19]*) *Let $(\mathcal{P}, \preceq)$ be a computable partially ordered set. Then there are a computable structure $\mathcal{A}$ and a computable unary relation R on its domain such that $(Dg_{\mathcal{A}}(R), \leq) \cong (\mathcal{P}, \preceq)$ and every degree in $Dg_{\mathcal{A}}(R)$ is realized via a c.e. set.*

0.7 Relations with Post-type properties on computable structures

In 1944, Post introduced the notion of simplicity in computability theory. Let $X \subseteq \omega$. We say that X is *simple* if X is c.e., its complement $\overline{X}$ is infinite, and there is no infinite c.e. set W such that $W \subseteq \overline{X}$. Post established the existence of a simple set. Dekker showed that there is a simple set in every nonzero c.e. degree.

Let $\mathcal{B}$ be a computable structure for L, and let X be a new relation on its domain B. For simplicity, assume that X is unary. Let $\mathbf{R}$ be a new unary relation symbol. A unary relation F on B is definable by a computable Σ_1^Γ formula $\psi(\overrightarrow{\mathbf{d}}, x)$ (with parameters $\overrightarrow{\mathbf{d}}$, where $\overrightarrow{d} \in B^{<\omega}$) if for every $b \in B$:

$$F(b) \Leftrightarrow (\mathcal{B}_B, X) \models \psi(\overrightarrow{\mathbf{d}}, \mathbf{b}).$$

The subsets of $\overline{X}$ which are definable by computable Σ_1^Γ formulae, and their subsets, play the role of finite sets to some extent.

Definition 6 (*Hird [17]*) *The relation X is called quasi-simple on $\mathcal{B}$ if X is c.e., $\overline{X}$ is not definable by a computable Σ_1^Γ formula with finitely many parameters, and for every c.e. $W \subseteq \overline{X}$, there is a unary relation F definable by a computable Σ_1^Γ formula with finitely many parameters such that*

$$W \subseteq F \subseteq \overline{X}.$$

Hence quasi-simple sets on $(\omega, =)$ are classical simple sets. Let R be an additional unary relation on the domain of a computable structure $\mathcal{A}$. Hird established a certain natural decidability condition on $(\mathcal{A}, R)$ which is sufficient for obtaining a quasi-simple relation X as the image of R on the domain of a computable model $\mathcal{B}$ isomorphic to $\mathcal{A}$. Ash, Knight and Remmel gave conditions on $(\mathcal{A}, R)$ which are sufficient for obtaining a quasi-simple relation of an arbitrary c.e. degree.

Theorem 13 (*Ash, Knight and Remmel [3]*) *Assume that* $(\mathcal{A}, R)$ *satisfy the following conditions.*

(*i*) *For every* $\overrightarrow{c} \in A^{<\omega}$, *there is an element* a *such that* $a \in fr(\overrightarrow{c})$.

(*ii*) *For every* $\overrightarrow{c} \in A^{<\omega}$, $bd(\overrightarrow{c})$ *is definable by a computable* Σ_1^Γ *formula with parameters* $\overrightarrow{\mathbf{c}}$.

(*iii*) *There is an algorithm which for a given* $\overrightarrow{c} \in A^{<\omega}$ *and* $a \in A$, *decides whether* $a \in bd(\overrightarrow{c})$.

(*iv*) *Let* $\overrightarrow{c} \in A^{<\omega}$, $a, b, b_1, \ldots, b_k, d_1, \ldots, d_m \in \overline{R}$, *and let* ψ *be a finitary* Σ_1^Γ *formula with* $k + m + 2$ *free variables such that*

$$(\mathcal{A}_A, R) \models \psi(\overrightarrow{\mathbf{c}}, \mathbf{a}, \mathbf{b}, \mathbf{b}_1, \ldots, \mathbf{b}_k, \mathbf{d}_1, \ldots, \mathbf{d}_m).$$

Assume that

$$a, b \in fr(\overrightarrow{c});$$

for every $i \in \{1, \ldots, k\}$, *there is a sequence* $\overrightarrow{u_i}$ *such that*

$$(\overrightarrow{u_i} \supseteq \overrightarrow{c}\hat{\ }b\hat{\ }b_1\hat{\ }\ldots\hat{\ }b_{i-1}) \wedge b_i \in fr(\overrightarrow{u_i}) \wedge a \in bd(\overrightarrow{u_i}); \text{ and}$$

for every $j \in \{1, \ldots, m\}$, *there is a sequence* $\overrightarrow{v_j}$ *such that*

$$(\overrightarrow{c} \supseteq \overrightarrow{v_j}) \wedge d_j \in fr(\overrightarrow{v_j}) \wedge d_j \in bd(\overrightarrow{c}).$$

Then there are $a', b_1', \ldots, b_k' \in R$, $b \in A$ *and* $d_1', \ldots, d_m' \in \overline{R}$ *such that*

$$(\mathcal{A}_A, R) \models \psi(\overrightarrow{\mathbf{c}}, \mathbf{a}', \mathbf{b}', \mathbf{b}_1', \ldots, \mathbf{b}_k', \mathbf{d}_1', \ldots, \mathbf{d}_m') \text{ and}$$

for every $j \in \{1, \ldots, m\}$,

$$d_j' \in fr(v_j).$$

There is a computable structure $\mathcal{B}$ *isomorphic to* $\mathcal{A}$ *such that the image of* R *on* B *is quasi-simple on* $\mathcal{B}$ *and of an arbitrary nonzero c.e. Turing degree.*

Shore [21] introduced the concepts of nowhere simple and effectively nowhere simple sets in computability theory. Let $X \subseteq \omega$. We say that X is *nowhere simple* if X is c.e., and for every c.e. set W such that $W - X$ infinite, there is an infinite c.e. set W' such that $W' \subseteq W - X$. Thus, nowhere simplicity is definable in the lattice of all c.e. sets. Similarly, X is *effectively nowhere simple* if X is c.e., and there is a unary computable function g such that for every $e \in \omega$, $W_{g(e)} \subseteq W_e - X$ and

$$(W_e - X \text{ is infinite}) \Rightarrow (W_{g(e)} \text{ is infinite}).$$

The function g is called a *witness function* for X. Clearly, every computable set is effectively nowhere simple. Miller and Remmel [20] established that X is effectively nowhere simple if there is a c.e. set W such that $W \cap X = \emptyset$ and for every $e \in \omega$,

$$(W_e - X \text{ is infinite}) \Rightarrow (W_e \cap W \text{ is infinite}).$$

The set W is called a *witness set* for X. Hence, effective nowhere simplicity is definable in the lattice of all c.e. sets. Shore [21], and Miller and Remmel [20] proved that every c.e. Turing degree contains an effectively nowhere simple set.

We consider the following general definition of nowhere simplicity and effective nowhere simplicity of relations on computable structures.

Definition 7 (*Harizanov [10]*) *Let $\mathcal{B}$ be a computable structure and let X be a new unary relation on B.*

(i) *The relation X is nowhere simple on $\mathcal{B}$ if X is c.e., and for every c.e. set W, there is a c.e. set $W' \subseteq W - X$ such that if $W - X$ is not contained in any subset F of $\overline{X}$ definable by a computable Σ_1^Γ formula with finitely many parameters, then the same is true of W'.*

(ii) *The relation X is effectively nowhere simple on $\mathcal{B}$ if there is a computable unary function g such that for every $e \in \omega$, $W_{g(e)} \subseteq W_e - X$ and if $W_e - X$ is not contained in any subset F of $\overline{X}$ definable by a computable Σ_1^Γ formula with finitely many parameters, then the same is true of $W_{g(e)}$.*

The following theorem generalizes Shore's and Miller-Remmel's result.

Theorem 14 (*Harizanov [10]*) *Let the following conditions hold for a computable structure $\mathcal{A}$ and a new computable unary relation R on A:*

(i) *for every $\overrightarrow{c} \in A^{<\omega}$, there is an element a such that $a \in fr(\overrightarrow{c})$,*

(ii) *for every $\overrightarrow{c} \in A^{<\omega}$, $bd(\overrightarrow{c})$ is definable by a computable Σ_1^Γ formula with parameters $\overrightarrow{\mathbf{c}}$,*

(iii) *there is an algorithm which for given $\overrightarrow{c} \in A^{<\omega}$ and $a \in A$, decides whether $a \in bd(\overrightarrow{c})$,*

(iv) *for every $\overrightarrow{c} \in A^{<\omega}$ and $a, v \in \overline{R}$, if $a \in fr(\overrightarrow{c})$, and $v \in bd(\overrightarrow{c})$, then $a \in fr(\overrightarrow{c}\,\hat{}\,v)$,*

(v) *for every $\overrightarrow{c} \in A^{<\omega}$, $\overrightarrow{u} \in \overline{R}^{<\omega}$ and $a, v \in \overline{R}$, if $a \in fr(\overrightarrow{c}\,\hat{}\,\overrightarrow{u})$, $v \in fr(\overrightarrow{c}\,\hat{}\,a)$, and $v \in fr(\overrightarrow{u})$, then $a \in fr(\overrightarrow{c}\,\hat{}\,\overrightarrow{u}\,\hat{}\,v)$.*

Then there is a computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ such that the image of R on B is of an arbitrary c.e. degree and effectively nowhere simple on $\mathcal{B}$.

For example, as a consequence of Theorem 7, we obtain that for every c.e. degree $\mathbf{c}$, there is a computable linear order of type η with a dense co-dense effectively nowhere simple subset of degree $\mathbf{c}$.

0.8 References

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