Countable Models, Computability, and Enumerations, Part I

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Joint work with Sergei Goncharov, Julia Knight, Charlie McCoy, Russell Miller, and Reed Solomon.

Computable models

- Consider a *countable* structure *A* for a *computable* language L.

- *Turing degree* of *A* is the Turing degree of the atomic diagram of A, D(*A*).
  A is *computable* (*recursive*) if its Turing degree is $\mathbf{0}$.

- D(*A*) may be of much lower Turing degree than Th(*A*).
  N, the standard model of arithmetic, is computable.
  True Arithmetic, $TA = Th$(*N*), is of Turing degree $\mathbf{0}^{(\omega)}$.

- (Tennenbaum) There is no computable nonstandard model of PA.

Computable categoricity

Let A be a *computable* structure.

- A is *computably categorical* if for all computable $B \cong A$, there is a computable isomorphism f from A onto B.

- A is *relatively computably categorical* if for all $B \cong A$, there is an isomorphism f from A onto B, which is computable relative to $D(B)$.

- A is relatively computably categorical $\Rightarrow$ A is computably categorical.

Examples

- $(\omega, <)$ is not computably categorical.

- $(Q, <)$ is relatively computably categorical
(usual back-and-forth argument).
It is not computably stable.

- A computable structure A is *computably stable*
if for all computable $B \cong A$,
every isomorphism from A onto B is computable.

- (R. Miller) No computable tree $(T, \prec)$ of infinite height
is computably categorical.

(LaRoche, Goncharov-Dzgoev, Remmel)

- A computable linear order is computably categorical iff it has *finitely many successors*.

- A computable Boolean algebra is computably categorical iff it has *finitely many atoms*.

(Goncharov, Smith)

- Computably categorical abelian p-groups are those that can be written in one of the following forms:
  $(Z(p^\infty))^l \oplus G$ for $l \in \omega \cup \{\infty\}$ and G is finite, or
  $(Z(p^\infty))^n \oplus G \oplus (Z(p^k))^\infty$, where $n, k \in \omega$ and G is finite.

Computably categorical does not imply relatively computably categorical

- (Goncharov, 1980) There is a rigid computable graph that is computably categorical, but *not relatively* computably categorical.

- *Our goal:* Generalize Goncharov's result to higher levels of hyperarithmetical hierarchy.

- X is $\mathbf{\Sigma}^0_n \Leftrightarrow X$ is c.e. relative to $\emptyset^{(n-1)}$, for $1 \leq n < \omega$

- X is $\mathbf{\Sigma}^0_\alpha \Leftrightarrow X$ is c.e. relative to $H(a)$, for $|a| = \alpha \geq \omega$

Classification of computable formulas

- A computable Σ_0 (Π_0) formula is a finitary quantifier-free formula.
  A computable Σ_α formula, $\alpha > 0$, is a *c.e disjunction* of formulas

  $$\exists \overline{u}\, \psi(\overline{x}, \overline{u}),$$

  where ψ is computable Π_β for some $\beta < \alpha$.
  A computable Π_α formula, $\alpha > 0$, is a *c.e. conjunction* of formulas

  $$\forall \overline{u}\, \psi(\overline{x}, \overline{u}),$$

  where ψ is computable Σ_β for some $\beta < \alpha$.

Δ^0_α categoricity

Let A be a *computable* structure, α a computable ordinal.

- A is Δ^0_α *categorical* if for all computable $B \cong A$, there is a Δ^0_α isomorphism f from A onto B.

- A is *relatively* Δ^0_α *categorical* if for all $B \cong A$, there is an isomorphism f from A onto B, which is Δ^0_α relative to $D(B)$.

Scott families of formulas

Let *A* be a *countable* structure.

- A *Scott family* for *A* is a set Φ of formulas, with a fixed finite tuple of parameters $\overline{c}$ in A, such that
  each tuple in A satisfies some $\psi \in \Phi$,
  if $\overline{a}$, $\overline{b}$ are tuples in A satisfying the *same* formula $\psi \in \Phi$, then there is an automorphism of *A* taking $\overline{a}$ to $\overline{b}$.

- If *A* rigid, Scott family replaced by *defining family* of formulas with a single free variable:
  every $a \in A$ satisfies some $\psi(x) \in \Phi$,
  no ψ in Φ satisfied by more than one element in A.

Effective Scott families

- A *formally* Σ^0_α *Scott family* is a Σ^0_α Scott family consisting of computable Σ_α formulas.

- A *formally c.e. Scott family* is a c.e. Scott family consisting of finitary existential formulas.

- If a computable structure A has a formally c.e. Scott family, then it is *relatively* computably categorical.

- *Proof sketch.*

  Let $(A, \overline{c}) \cong (B, \overline{d})$.

  Will construct an isomorphism f computable in $D(B)$.

  $f = \bigcup_s f_s, \ \ f_s \subset f_{s+1}$.

  Assume f_s maps $\overline{c}\hat{\ }\overline{a} \to \overline{d}\hat{\ }\overline{b}$,
  $a' \in A$, where $a' \notin \overline{c}\hat{\ }\overline{a}$.

  Find $\psi(\overline{c}, \overline{x}, y) \in \Phi$ and $b' \in B$ such that

  $$A \models \psi(\overline{c}, \overline{a}, a') \ \wedge \ B \models \psi(\overline{d}, \overline{b}, b').$$

  Let $f_{s+1}(a') = b'$.

Equivalence of semantic and syntactic conditions

- (Ash-Knight-Manasse-Slaman, Chisholm) Let A be a computable structure.
  A is *relatively* $\mathbf{\Delta}^0_\alpha$ categorical *iff*
  A has a formally $\mathbf{\Sigma}^0_\alpha$ Scott family *iff*
  A has a c.e. Scott family consisting of computable $\mathbf{\Sigma}_\alpha$ formulas.

  In particular, A is *relatively* computably categorical *iff*
  it has a formally c.e. Scott family.

- (Goncharov) Assume that the $\exists\forall$-diagram of A is computable.
  If A is computably categorical, then it has a formally c.e. Scott family.

- (Ash) Under some additional decidability on A,
  if A is $\mathbf{\Delta}^0_\alpha$ categorical, then it has a formally $\mathbf{\Sigma}^0_\alpha$ Scott family.

General result

(Goncharov-Harizanov-Knight-McCoy-Miller-Solomon, $APAL$ 2005)

- For every computable successor ordinal α,
  there is a computable structure A,
  which is $\mathbf{\Delta}^0_\alpha$ categorical,
  but A does not have a formally $\mathbf{\Sigma}^0_\alpha$ Scott family
  (not relatively $\mathbf{\Delta}^0_\alpha$ categorical).

Relations on structures

- Let R be an *additional* relation on a structure A.
  Let $\mathbb{P}$ be a computability-theoretic *complexity class*.

- (Ash-Nerode) R is *intrinsically* $\mathbb{P}$ on a computable A if in *all computable* isomorphic copies of A, the image of R is $\mathbb{P}$.

- R is *relatively intrinsically* $\mathbb{P}$ on a computable A if in *all* isomorphic copies B of A, the image of R is $\mathbb{P}$ *relative* to $D(B)$.

- Examples: (i) $Successor$ is intrinsically Π^0_1 on a computable linear order.
  (ii) $Dependence$ is intrinsically c.e. on a computable vector space.

Definability versus complexity of relations

- (Kueker) The following are equivalent for a relation R on a countable *A*:

  (i) R has fewer than $2^{\aleph_0}$ different images under automorphisms of *A*;

  (ii) R is definable in *A* by an $L_{\omega_1\omega}$ formula with finitely many parameters.

  Assume (i). There exists $\bar{c}$ such that for every $a \in R$
  there is a formula $\psi_a(x, \bar{c})$ satisfied by a but not by any $a' \notin R$.
  Hence, R is defined by $\bigvee_{a \in R} \psi_a(x, \bar{c})$.

- (Harizanov) *Turing degree spectrum* of a relation R on *A*,
  $DgSp(R)$, the set of Turing degrees of images of R
  in computable isomorphic copies of *A*.

Σ_α definability of relations

- (Ash) A relation defined in a countable structure A by a computable Σ_α (Π_α) formula is Σ^0_α (Π^0_α) relative to $D(A)$.

- The relation R is *formally* Σ^0_α on A if it is definable by a computable Σ_α formula with finitely many parameters.

- (Ash-Nerode) Under some effectiveness condition (enough to have the existential diagram of (A, R) computable), R is *intrinsically c.e.* on a computable A iff R is *formally c.e.* on A.

- (Barker) Under some effectiveness conditions, R is *intrinsically* Σ^0_α on a computable A iff R is *formally* Σ_α on A.

- (Harizanov)

  Under some effectiveness condition
  (enough to have the existential diagram of (A, R) computable):

  (i) If R is *not intrinsically computable*,
  then $DgSp(R)$ includes all c.e. degrees.

  At least one of R, $\neg R$ is not definable in A by a computable Σ_1 formula.

  Example: $A = (\omega, <)$, $R = Succ$

  (ii) If $\neg R$ is not definable in (A, R) by a computable Σ_1 formula
  in which the symbol R occurs only positively,
  then $DgSp(R)$ includes all c.e. degrees realized via c.e. sets.

- Degrees coarser than Turing degrees:

$$X \leq_{\mathbf{\Delta}^0_\alpha} Y \Leftrightarrow X \leq_T Y \oplus \mathbf{\Delta}^0_\alpha$$

$$X \equiv_{\mathbf{\Delta}^0_\alpha} Y \Leftrightarrow (X \leq_{\mathbf{\Delta}^0_\alpha} Y \wedge Y \leq_{\mathbf{\Delta}^0_\alpha} X)$$

($\equiv_{\mathbf{\Delta}^0_1}$ is $\equiv_T$)

- (Ash-Knight) Under some effectiveness condition,
if R is *not intrinsically* $\mathbf{\Delta}^0_\alpha$ on a computable A,
then for every $\mathbf{\Sigma}^0_\alpha$ set C,
there is an isomorphism f from A onto a computable structure
such that $f(R) \equiv_{\mathbf{\Delta}^0_\alpha} C$.

Not possible to replace these by Turing degrees.

Equivalence of semantic and syntactic conditions

- (Ash-Knight-Manasse-Slaman, Chisholm)
R is *relatively intrinsically* $\mathbf{\Sigma}^0_\alpha$ on A *iff* R is *formally* $\mathbf{\Sigma}^0_\alpha$ on A.

- (Soskov) TFAE: (i) R is *relatively intrinsically* $\mathbf{\Delta}^1_1$ on A;
(ii) R is definable in A by a computable formula
with finitely many parameters;
(iii) R is *intrinsically* $\mathbf{\Delta}^1_1$ on A.

- (Soskov, Goncharov-Harizanov-Knight-Shore) TFAE:
(i) R is *relatively intrinsically* $\mathbf{\Pi}^1_1$ on A;
(ii) R is definable in A by a $\mathbf{\Pi}^1_1$ disjunction of computable formulas
with finitely many parameters;
(iii) R is *intrinsically* $\mathbf{\Pi}^1_1$ on A.

Intrinsically effective does not imply relatively intrinsically effective

- (Manasse)
  There is a computable structure with an intrinsically c.e.,
  but *not relatively* intrinsically c.e. relation.

- (Goncharov-Harizanov-Knight-McCoy-Miller-Solomon)
  For every computable successor ordinal α,
  there is a computable structure A with a relation R,
  such that R is intrinsically $\mathbf{\Sigma}^0_\alpha$ on A,
  but R is not definable in A by a $\mathbf{\Sigma}_\alpha$ formula with finitely many parameters
  (*not relatively* intrinsically $\mathbf{\Sigma}^0_\alpha$).

Turing degree spectrum of a structure

- $DgSp(A) = \{\mathrm{deg}(B) : B \cong A\}$.

- (Knight) A is *automorphically trivial* if it has a finite tuple $\bar{c}$ such that every permutation of A that fixes $\bar{c}$ pointwise is an automorphism of A.
  (i) A is automorphically trivial $\Rightarrow |DgSp(A)| = 1$.
  (ii) A is automorphically nontrivial $\Rightarrow DgSp(A)$ is closed upwards.

- (Harizanov-Knight-Morozov) (i) If A is automorphically trivial, then $(\forall B \simeq A)[D^e(B) \equiv_T D(B)]$.

  (ii) If A is automorphically nontrivial, and $X \geq_T D^e(A)$, there exists $B \cong A$ such that

$$D^e(B) \equiv_T D(B) \equiv_T X.$$

- $\mathcal{D}$=the set of all Turing degrees

- (Wehner, Slaman)
  There is a structure A such that
  $$DgSp(A) = \mathcal{D} - \{\mathbf{0}\}.$$

- (Hirschfeldt)
  There is a complete decidable theory, with all types computable,
  whose prime model A has no computable copy,
  but has an X-decidable copy for every noncomputable X.

- (Goncharov-Harizanov-Knight-McCoy-Miller-Solomon)
  For each computable successor ordinal α,
  there is a structure A whose
  $DgSp(A)$ consists of the Turing degrees of sets X
  such that $\Delta^0_\alpha(X)$ is not Δ^0_α.

- In particular, for every $n \in \omega$, there is a structure A such that

$$DgSp(A) = \{\mathbf{c} \in \mathcal{D}\colon \mathbf{c}^{(\mathbf{n})} > \mathbf{0}^{(\mathbf{n})}\}.$$

  A degree $\mathbf{c}$ is non-low_n if $\mathbf{c}^{(\mathbf{n})} > \mathbf{0}^{(\mathbf{n})}$.

Computable dimension of a structure

- *Computable dimension* of A is the number of computable isomorphic copies of A, up to computable isomorphism.

- (Metakides-Nerode, Nurtazin, Goncharov, Goncharov-Dzgoev, Remmel, LaRoche) The following classes have computable dimension 1 or ω:
  algebraically closed fields, and real closed fields,
  abelian groups,
  linear orders,
  Boolean algebras,
  Δ_2^0 categorical structures.

- (Goncharov) There are structures of computable dimension n
  for every finite $n \geq 1$.

Δ^0_α dimension of a structure

- Δ^0_α *dimension* of A is the number of computable copies of A, up to Δ^0_α isomorphism.

- (Goncharov-Harizanov-Knight-McCoy-Miller-Solomon)
  For each computable successor ordinal α and every finite $n \geq 1$,
  there is a computable structure A such that
  the Δ^0_α dimension of A is n.

Enumerations

- An *enumeration* of $S \subseteq P(\omega)$ is a binary relation ν:

$$S = \{\nu(i) : i \in \omega\}, \text{ where } \nu(i) = \{x : (i, x) \in \nu\}.$$

  ν is *computable* (*c.e.*) if it is computable (c.e.) as a binary relation.

- ν is *Friedberg* if it is 1-1: $i \neq j \Rightarrow \nu(i) \neq \nu(j)$.

- (Wehner)
  There is a family S such that for every noncomputable X,
  S has an enumeration computable in X (c.e. relative to X),
  but S has no computable (c.e.) enumeration.

Equivalent enumerations

- $\nu \leq \mu$ if there is a computable function f such that:

$$(\forall i)[\nu(i) = \mu(f(i))].$$

- ν and μ are *computably equivalent* if $\mu \leq \nu$ and $\nu \leq \mu$.

- (Goncharov)
  For every finite $n \geq 1$, there is a family of sets with
  exactly n c.e. Friedberg enumerations, up to computable equivalence.

- (Marchenkov)
  Not true for computable Friedberg enumerations if $n > 1$.

Discrete families

- $S \subseteq P(\omega)$ is *discrete* if for each $A \in S$,
  there exists $\sigma \in 2^{<\omega}$ such that for all $B \in S$,
  $\sigma \subseteq B \Leftrightarrow B = A$.

- S is *effectively discrete* if there is a *c.e. set* $E \subseteq 2^{<\omega}$ such that:
  $(\forall A \in S)(\exists \sigma \in E)[\sigma \subseteq A]$;
  $(\forall \sigma \in E)(\forall A, B \in S)[(\sigma \subseteq A \wedge \sigma \subseteq B) \Rightarrow A = B]$.

- (Selivanov)
  There exists S with unique computable Friedberg enumeration,
  (in fact, c.e. since it consists of the graphs of functions)
  up to computable equivalence, such that
  S is discrete but *not effectively discrete*.

Transforming S into a graph

- Assign to $A \in S$, a *daisy graph* G_A consisting of one *index* point a at the center with $a \to a$, and for each $n \in A$ a *petal* (disjoint from other petals)

$$a \to a_0 \to \cdots \to a_n \to a$$

- $G(S)$ is the union of a disjoint family of G_A for each $A \in S$. $G(S)$ is a rigid graph.

- $S^+ =_{def} \{A \oplus \overline{A} : A \in S\}$.

- If S has n c.e. (computable) Friedberg enumerations, up to computable equivalence, then $G(S)$ ($G(S^+)$) has computable dimension n.

Discrete families and defining families of graphs

- Assume S is discrete.
  Every element of $G(S^+)$ has a finitary existential definition without parameters.

- Assume S is discrete but not effectively discrete.
  Assume S has a computable Friedberg enumeration.
  $G(S^+)$ does not have a formally c.e. defining family.

- If S is a Selivanov's family, then $G(S^+)$ is computably categorical, but not relatively.

- The *cardinal sum* $B_0 \oplus B_1$ of disjoint structures B_0, B_1 in the same relational language: take the disjoint union of the structures and add predicates P_0 and P_1 which hold of the elements of B_0 and B_1, respectively.

- Let $A = G \oplus G$ for $G = G(S^+)$, where S a Selivanov family.
  Let R be the unique isomorphism.
  R is intrinsically c.e. (since G is computably categorical).

- R is not relatively intrinsically c.e.

  Assume otherwise. For any copy H of G, we take the disjoint union of the universes, and form a copy of A. There is an isomorphism from G onto H, computable in H. However, G is not relatively computably categorical.

- $S \subseteq P(\omega)$
  $G^{\infty}(S)$ consists of infinitely many copies of G_A for each $A \in S$.
  $G^{\infty}(S)$ is not rigid.
  Copies of $G^{\infty}(S)$ correspond to enumerations of S.

- $X \subseteq \omega$
  There is an enumeration of S *c.e. in* X (computable in X) *iff*
  there is an isomorphic copy of $G^{\infty}(S)$ ($G^{\infty}(S^{+})$) computable in X.

- (Goncharov-Harizanov-Knight-McCoy-Miller-Solomon)
  Let $\alpha \geq 2$ be a computable successor ordinal.
  There is a structure with copies in exactly the Turing degrees of sets X such that $\Delta^0_\alpha(X)$ is not Δ^0_α.

- *Proof sketch.* Relativize the proof for Δ^0_1 to Δ^0_α.

  Get a graph G such that the degrees of copies of G are just the degrees of sets that are not Δ^0_α.

- Code a directed graph G in a structure G^* such that:

  G has a Δ^0_α copy iff G^* has a computable copy.
  More generally, for any $X \subseteq \omega$,
  G has a $\Delta^0_\alpha(X)$ copy iff G^* has an X-computable copy.

- *Proof sketch continued.* Code (Δ^0_α) directed graph G in a (computable) structure G^*, using a pair of structures B_0, B_1 such that B_0 codes $G \models a \to b$ and B_1 codes $G \models \neg(a \to b)$.

- $G^* = (G \cup U, G, U, Q, \ldots)$, where G and U are disjoint, Q (a ternary relation) assigns to $a, b \in G$ an infinite set $U_{(a,b)}$: $(x \in U_{(a,b)} \Leftrightarrow Qabx)$, the sets $U_{(a,b)}$ form a partition of U,

$$(U_{(a,b)}, \ldots) \cong \begin{cases} B_0, & \text{if } G \models a \to b, \\ B_1, & \text{if } G \models \neg(a \to b). \end{cases}$$

- (Goncharov-Harizanov-Knight-McCoy-Miller-Solomon)
  Let $\alpha \geq 2$ be a computable successor ordinal.
  There is a computable structure that is $\mathbf{\Delta}^0_\alpha$ categorical but
  not relatively $\mathbf{\Delta}^0_\alpha$ categorical.

- *Proof sketch.* Relativize the proof for $\mathbf{\Delta}^0_1$ to $\mathbf{\Delta}^0_\alpha$.

  There is a rigid $\mathbf{\Delta}^0_\alpha$ directed graph G such that:

  (i) G has exactly one $\mathbf{\Delta}^0_\alpha$ isomorphic copy, up to $\mathbf{\Delta}^0_\alpha$ isomorphism,
  (ii) G does not have $\mathbf{\Sigma}^0_\alpha$ Scott family of finitary existential formulas.

- Code G in a *computable* structure $G^* = (G \cup U, G, U, Q, \ldots)$,
  using a pair of structures B_0, B_1 such that
  B_0 codes $G \models a \rightarrow b$ and
  B_1 codes $G \models \neg(a \rightarrow b)$.

- B_0 and B_1 are *computable* structures, for which the standard *back-and-forth relations* $\leq_\beta$ for $\beta < \alpha$ are uniformly c.e. (Pair $\{B_0, B_1\}$ is α-*friendly*.)

- B_0 and B_1 satisfy the *same* infinitary Π_β sentences for $\beta < \alpha$. (If α were a limit ordinal, then B_0 and B_1 would also satisfy the same Π_α sentences.)

- B_0 satisfies *some* computable Π_α sentence that is not true in B_1, and *vice versa.*

- Then for any Δ^0_α set S, there is a uniformly computable sequence $(C_n)_{n\in\omega}$ such that

$$C_n \cong \begin{cases} B_0, & \text{if } n \in S, \\ B_1, & \text{if } n \notin S. \end{cases}$$

Back-and-forth relations

- $\leq_\beta$ on the set of pairs $\{(i,\overline{b}) : \overline{b} \in B_i\}$, are defined inductively as follows:

  (i) $(i,\overline{b}) \leq_1 (j,\overline{c})$ *iff*
  the existential formulas true of $\overline{c}$ in B_j are true of $\overline{b}$ in B_i;

  (ii) if $\beta > 1$, $(i,\overline{b}) \leq_\beta (j,\overline{c})$ *iff* for all $\overline{c}'$ in B_j, and all γ with $1 \leq \gamma < \beta$, there exists $\overline{b}'$ in B_i such that

  $$(j,\overline{c},\overline{c}') \leq_\gamma (i,\overline{b},\overline{b}').$$

- $(i,\overline{b}) \leq_\beta (j,\overline{c})$ *iff* all Π_β formulas of $L_{\omega_1\omega}$ true of $\overline{b}$ in B_i are true of $\overline{c}$ in B_j.

Existence of structures B_0 and B_1

- Case $\alpha = 2$: orders ω and ω^*.
- Can be distinguished by finitary $\mathbf{\Pi}_2$ sentences saying that there is no first, or last, element.
- $\omega \leq_1 \omega^*$ and $\omega^* \leq_1 \omega$ (since both orders are infinite).
- Each order is rigid, with a c.e. defining family consisting of finitary $\mathbf{\Sigma}_2$ formulas $\psi_n(x)$ saying that there are exactly n elements to the left, or right, of x. Any tuple of elements $\overline{x}$ in ω or ω^* can be defined by a conjunction of such formulas. Formally $\mathbf{\Sigma}^0_2$ Scott family without parameters.

$\{\omega, \omega^*\}$ is 2-friendly

Facts about linear orders L_0, L_1.

- $L_0 \leq_1 L_1$ *iff* either both orders are infinite or L_0 is at least as big as L_1.
- $(L_0, \overline{a}) \leq_\gamma (L_1, \overline{b})$ *iff* for $L_0 = A_0 + a_1 + A_1 + \ldots + a_n + A_n$ and $L_1 = B_0 + b_1 + B_1 + \ldots + b_n + B_n$, we have $A_i \leq_\gamma B_i$ for $i = 0, \ldots, n$.
- Can enumerate the $\leq_1$ relation between tuples in our orders.

Uniformly relatively Δ^0_α categorical structures

- B is *uniformly relatively* Δ^0_α *categorical* if
  given an X-computable index for C with $C \cong B$,
  we can find a $\Delta^0_\alpha(X)$ index for an isomorphism from B onto C.

- B has a formally Σ^0_α Scott family with no parameters
  $\Longrightarrow$ B is uniformly relatively Δ^0_α categorical.

- Assume, in addition, B_0 and B_1 are uniformly relatively Δ^0_α categorical.
  Can show that:
  (i) G^* is Δ^0_α categorical
  (G had exactly one Δ^0_α isomorphic copy, up to Δ^0_α isomorphism);
  (ii) G^* does not have formally Σ^0_α Scott family
  (G did not have Σ^0_α Scott family of finitary existential formulas).