

# COMPLEXITY OF DIAGRAMS OF COUNTABLE STRUCTURES

*Valentina S. Harizanov*

The George Washington University, Washington, DC  
harizanv@gwu.edu

*Julia F. Knight*

University of Notre Dame, Notre Dame, IN  
knight.1@nd.edu

*Andrei S. Morozov*

Sobolev Institute of Mathematics, Novosibirsk, Russia  
morozov@math.nsc.ru

## Fragments of Diagrams

Consider a *countable* structure  $\mathcal{A}$  for a *computable* language  $L$ .

- *Open diagram* of  $\mathcal{A}$ ,  $D(\mathcal{A}) = D_0(\mathcal{A})$ , is the set of all quantifier-free sentences of  $L_A$  true in  $\mathcal{A}_A$ .
- *$n$ -diagram* of  $\mathcal{A}$ ,  $D_n(\mathcal{A})$ , is the set of all  $\Sigma_n$  sentences of  $L_A$  that are true in  $\mathcal{A}_A$ .
- *Complete (elementary) diagram* of  $\mathcal{A}$ ,  $D^c(\mathcal{A})$ , is the set of all sentences of  $L_A$  that are true in  $\mathcal{A}_A$ .
- *Turing degree* of  $\mathcal{A}$  is the Turing degree of  $D(\mathcal{A})$ .
- $\mathcal{A}$  is *computable* if its Turing degree is  $\mathbf{0}$ .
- $\mathcal{A}$  is  *$n$ -decidable* if its  $n$ -diagram is computable.
- $\mathcal{A}$  is *decidable* if its complete diagram is computable.
- $B_n$  sentences are Boolean combinations of  $\Sigma_n$  (and  $\Pi_n$ ) sentences.
- $D_n(\mathcal{A}) = D^c(\mathcal{A}) \cap \Sigma_n \equiv_T D^c(\mathcal{A}) \cap B_n$
- $D_{n+1}(\mathcal{A})$  is c.e. in and above  $D_n(\mathcal{A})$ , uniformly in  $n$ .

## Examples

- $\mathcal{A}$  effectively eliminates quantifiers  
 $(\forall \mathcal{B} \cong \mathcal{A})[D^c(\mathcal{B}) \equiv_T D_0(\mathcal{B})]$   
(Intrinsic collapse of diagrams)
- (Moses)  
There is a linear order that is  $n$ -decidable, but has no  $(n + 1)$ -decidable copy.
- (Chisholm and Moses)  
There is a linear order that is  $n$ -decidable for every  $n$ , but has no decidable copy.
- $\mathcal{N} = (\omega, +, \cdot, S, 0)$   
 $D_n(\mathcal{N}) \equiv_T \emptyset^{(n)}$ , uniformly in  $n$
- (Knight)  
 $\mathcal{A}$  a nonstandard model of *Peano Arithmetic* ( $PA$ )  
 $(\exists \mathcal{B} \simeq \mathcal{A})[D_0(\mathcal{B}) <_T D_1(\mathcal{B}) <_T D_2(\mathcal{B}) <_T \dots]$
- (Knight)  
 $\mathcal{A}$  any model of  $PA$   
 $(\exists \mathcal{B} \simeq \mathcal{A})[D^c(\mathcal{B}) \equiv_T D_0(\mathcal{B})]$

- (Knight)

A structure  $\mathcal{A}$  is *trivial* if there is sequence  $\vec{c} \in A^{<\omega}$  such that every permutation of  $A$  that fixes  $\vec{c}$  pointwise is an automorphism of  $\mathcal{A}$ .

(i) If  $\mathcal{A}$  is trivial, then

$$(\forall \mathcal{B} \simeq \mathcal{A}) [D_0(\mathcal{B}) \equiv_T D_0(\mathcal{A})]$$

(ii) If  $\mathcal{A}$  is non-trivial, then

the set of Turing degrees of isomorphic copies of  $\mathcal{A}$  is closed upwards.

- (Harizanov, Knight, Morozov)

For any structure  $\mathcal{A}$ , there exists  $\mathcal{B} \simeq \mathcal{A}$  such that

$$D^c(\mathcal{B}) \equiv_T D_0(\mathcal{B})$$

(i)  $\mathcal{A}$  is trivial, via  $\vec{c}$ .

For every formula  $\psi(\vec{c}, \vec{x})$ , effectively find a quantifier-free formula  $\psi^*(\vec{c}, \vec{x})$ , in the language with equality,

$$\mathcal{A}_A \models [\psi(\vec{c}, \vec{x}) \Leftrightarrow \psi^*(\vec{c}, \vec{x})]$$

(ii)  $\mathcal{A}$  is non-trivial.

Let  $X$  be such that  $D^c(\mathcal{A}) \leq_T X$ .

There are structure  $\mathcal{B}$ , isomorphism  $f : \mathcal{A} \rightarrow \mathcal{B}$  with

$$f \leq_T X \leq_T D_0(\mathcal{B})$$

Thus,

$$X \leq_T D_0(\mathcal{B}) \leq_T D^c(\mathcal{B}) \leq_T f \oplus D^c(\mathcal{A}) \leq_T X$$

# Intrinsic Collapse of Fragments of Diagrams

## Questions

- (A) For fixed  $n$ , find syntactic conditions on  $\mathcal{A}$  such that  $(\forall \mathcal{B} \cong \mathcal{A})[D^c(\mathcal{B}) \equiv_T D_n(\mathcal{B})]$
- (B) For fixed  $n$ , find syntactic conditions on  $\mathcal{A}$  such that  $(\forall \mathcal{B} \cong \mathcal{A})[D_{n+1}(\mathcal{B}) \equiv_T D_n(\mathcal{B})]$ .

(Harizanov, Knight, Morozov)

- (A) The following are equivalent:
  - (i) For every structure  $\mathcal{B} \cong \mathcal{A}$  we have

$$D^c(\mathcal{B}) \leq_T D_n(\mathcal{B})$$

(ii) There is  $\vec{c} \in A^{<\omega}$  and a computable function  $d$  assigning to every finitary formula  $\gamma(\vec{x})$  a computable infinitary formula

$$d_\gamma(\vec{c}, \vec{x}) = \bigvee_{i \in W} \beta_i(\vec{c}, \vec{x}),$$

where  $W$  is c.e. and every  $\beta_i$  is a finitary  $\Sigma_{n+1}$  formula, such that

$$\mathcal{A}_A \models (\forall \vec{x})[\gamma(\vec{x}) \Leftrightarrow d_\gamma(\vec{c}, \vec{x})]$$

- (B) Similar characterization holds for

$$D_{n+1}(\mathcal{B}) \leq_T D_n(\mathcal{B})$$

In (ii), instead of an arbitrary finitary formula,  $\gamma(\vec{x})$  is an arbitrary finitary  $\Pi_{n+1}$  formula.

Proofs based of the following relativized Ash-Nerode theorem, similar to the one by Chisholm; Ash, Knight, Manasse and Slaman.

- (Harizanov, Knight, Morozov)

For a sequence  $(R_k)_{k \in \omega}$  of relations on a structure  $\mathcal{A}$ , the following are equivalent:

(i) For every isomorphism  $f$  from  $\mathcal{A}$  to some structure  $\mathcal{B}$ ,  $f(R_k)$  is c.e. relative to  $D_n(\mathcal{B})$ , uniformly in  $k$ .

(ii) For some  $\vec{c} \in A^{<\omega}$ , for each  $k$ , we can effectively find an index for a c.e. set of finitary  $\Sigma_{n+1}$  formulas with parameters  $\vec{c}$ , whose disjunction defines  $R_k$ .

- **Examples**

(i) For  $N \geq 1$ , if  $\mathcal{B}$  is a linear order of type  $\omega^N$ , then

$$D^c(\mathcal{B}) \equiv_T D_{2N-1}(\mathcal{B})$$

(ii) If  $\mathcal{B}$  is a linear order of type  $\omega^N \cdot \eta$ ,

where  $\eta$  is the order type of rationals, then

$$D^c(\mathcal{B}) \equiv_T D_{2N}(\mathcal{B})$$

## Turing Degrees of 1-Diagrams

- (Harizanov)

Let  $Y \subseteq \omega$  be c.e. There is a computable linear order  $\mathcal{B}$  of order type  $\omega$  whose successor relation  $S$  is Turing equivalent to  $Y$ . Hence

$$D_1(\mathcal{B}) \equiv_T Y,$$

since Moses showed that  $D_1(\mathcal{B}) \leq_T D_0(\mathcal{B}) \oplus S$ .

- (Harizanov, Knight, Morozov)

Let  $Y \subseteq \omega$  be c.e. Let  $\mathcal{A}$  be 1-decidable. Assume that for every  $\vec{c} \in A^{<\omega}$ , we can *effectively* find a finitary  $\Pi_1$  formula  $\theta(\vec{c}, \vec{u})$  and  $\vec{a}$  such that

$$\mathcal{A}_A \models \theta(\vec{c}, \vec{a}),$$

and for every finitary  $\Sigma_1$  formula  $\sigma(\vec{c}, \vec{u})$ ,

$$[\mathcal{A}_A \models \sigma(\vec{c}, \vec{a})] \Rightarrow (\exists \vec{a}') [\mathcal{A}_A \models (\sigma(\vec{c}, \vec{a}') \wedge \neg \theta(\vec{c}, \vec{a}'))].$$

Then there is a *computable* structure  $\mathcal{B} \simeq \mathcal{A}$  such that

$$D_1(\mathcal{B}) \equiv_T Y.$$

## Turing Degrees of the First Two Diagrams

- (Harizanov, Knight, Morozov)

Let  $X, Y \subseteq \omega$  be such that  $Y$  is c.e. in and above  $X$  and  $D_0(\mathcal{A}) \leq_T X$ .

Assume that  $\mathcal{A}$  satisfies the following conditions.

(0) For every  $\vec{c} \in A^{<\omega}$ , we can *effectively* find a finitary  $\Delta_1$  formula (given by a pair of finitary  $\Sigma_1, \Pi_1$  formulas)  $\chi(\vec{c}, \vec{x})$  such that

$$\begin{aligned} \mathcal{A}_A \models (\exists \vec{x}) [ran(\vec{c}) \cap ran(\vec{x}) = \emptyset \wedge \chi(\vec{c}, \vec{x})] \\ \mathcal{A}_A \models (\exists \vec{x}) [ran(\vec{c}) \cap ran(\vec{x}) = \emptyset \wedge \neg \chi(\vec{c}, \vec{x})]. \end{aligned}$$

(1) For every  $\vec{c} \in A^{<\omega}$ , we can *effectively* find a finitary  $\Pi_1$  formula  $\theta(\vec{c}, \vec{x})$  and  $\vec{a} \in A^{<\omega}$  such that

$$\mathcal{A}_A \models \theta(\vec{c}, \vec{a}),$$

and for every finitary  $\Sigma_1$  formula  $\sigma(\vec{c}, \vec{x})$ ,

$$[\mathcal{A}_A \models \sigma(\vec{c}, \vec{a})] \Rightarrow (\exists \vec{a}') [\mathcal{A}_A \models (\sigma(\vec{c}, \vec{a}') \wedge \neg \theta(\vec{c}, \vec{a}'))]$$

Then there is  $\mathcal{B} \simeq \mathcal{A}$  such that

$$D_0(\mathcal{B}) \equiv_T X \wedge D_1(\mathcal{B}) \equiv_T Y$$

- **Example:**  $\mathcal{A} = (\omega, <)$

For a given  $\vec{c} = (c_0, \dots, c_{m-1})$ , a corresponding  $\Pi_1$  formula  $\theta(\vec{c}, x, y)$  is

$$[S(y, x) \wedge x > c_0 \wedge \dots \wedge x > c_{m-1}],$$

and a corresponding  $\Delta_1$  formula  $\chi(\vec{c}, x, y)$  is

$$[(x < y) \wedge x > c_0 \wedge \dots \wedge x > c_{m-1}].$$

Examples of  $\mathcal{A}$  such that there exists  $\mathcal{B} \cong \mathcal{A}$  with  $D_0(\mathcal{B}) \equiv_T X$  and  $D_1(\mathcal{B}) \equiv_T Y$ , where  $Y$  is c.e. in and above  $X$

- Let  $\mathcal{A}$  be the *Boolean algebra*  $I(\omega)$   
 A corresponding  $\Pi_1$  formula uses a unary *atom* relation, and a corresponding  $\Delta_1$  formula uses the binary relation of *being disjoint*.
- Let  $\mathcal{A}$  be the *Abelian group*  $Z_p^\omega \oplus Z_{p^2}^\omega$ .  
 A corresponding  $\Pi_1$  formula uses the unary relation of *not being divisible by  $p$* , and a corresponding  $\Delta_1$  formula uses the binary relation of one element *being a multiple by  $p$*  of the other element.

## Turing Degrees of Sequences of $n$ -Diagrams

- An  $(N + 1)$ -table is a sequence of sets  $(C_n)_{n \leq N}$  such that  $C_{n+1}$  is c.e. in and above  $C_n$  for  $n < N$ .
- An  $\omega$ -table is a sequence of sets  $(C_n)_{n \in \omega}$ , where  $C_{n+1}$  is c.e. in and above  $C_n$ , uniformly in  $n$ .
- We say these tables are *over*  $C_0$ .

- For any structure  $\mathcal{A}$ ,  $(D_n(\mathcal{A}))_{n \in \omega}$  is an  $\omega$ -table.

- (Knight)

For a sequence of Turing degrees  $(\mathbf{d}_n)_{n \in \omega}$ , the following are equivalent:

(i) There exists a *nonstandard* model  $\mathcal{A}$  of  $PA$  such that for all  $n$ ,

$$\text{deg}(D_n(\mathcal{A})) = \mathbf{d}_n.$$

(ii) There exists an  $\omega$ -table  $(C_n)_{n \in \omega}$ , over a completion of  $PA$ , such that for all  $n$ ,

$$\text{deg}(C_n) = \mathbf{d}_n.$$

## Questions

- For fixed  $N \in \omega$ , find conditions on  $\mathcal{A}$  guaranteeing that for every  $(N+1)$ -table  $(C_n)_{n \leq N}$  there exists  $\mathcal{B} \cong \mathcal{A}$  such that

$$(\forall n \leq N)[D_n(\mathcal{B}) \equiv_T C_n]$$

- Find conditions on  $\mathcal{A}$  guaranteeing that for every  $\omega$ -table  $(C_n)_{n \in \omega}$  there exists  $\mathcal{B} \cong \mathcal{A}$  such that

$$(\forall n \in \omega)[D_n(\mathcal{B}) \equiv_T C_n]$$

## Back and Forth Relations

For simplicity, assume  $\mathcal{A}$  is a structure for a finite relational language.

*Convention:* If  $\vec{a} \in A^{<\omega}$ , assume the elements in  $\vec{a}$  distinct. Assume concatenation  $\vec{a} \hat{\ } \vec{c}$  defined only if the elements in  $\vec{a}$  distinct from the elements in  $\vec{c}$ .

(Barwise; Ash and Knight)

- Let  $\vec{a}, \vec{b} \in A^{<\omega}$  be such that  $lh(\vec{a}) \leq lh(\vec{b})$ .
  - (i)  $\vec{a} \leq_0 \vec{b}$  iff  
the open formulas true of  $\vec{a}$  are all true of  $\vec{b}$ .
  - (ii)  $\vec{a} \leq_{n+1} \vec{b}$  iff  

$$(\forall \vec{d})(\exists \vec{c})[\vec{b} \hat{\ } \vec{d} \leq_n \vec{a} \hat{\ } \vec{c}]$$
- $\vec{a} \leq_n \vec{b}$  iff the infinitary  $\Pi_n$  formulas true of  $\vec{a}$  are also true of  $\vec{b}$ .

## Independence of Formulas

(Harizanov, Knight, Morozov)

- Formula  $\theta(\vec{u}, \vec{x})$  is *0-independent over  $\vec{u}$*  if it is open, and for every  $\vec{c} \in A^{lh(\vec{u})}$ , there exist  $\vec{a}$  and  $\vec{a}'$ :

$$\mathcal{A}_A \models \theta(\vec{c}, \vec{a}) \wedge \neg \theta(\vec{c}, \vec{a}')$$

- For  $n > 0$ ,  $\theta(\vec{u}, \vec{x})$  is *n-independent over  $\vec{u}$*  if it is  $\Pi_n$  and for every  $\vec{c}$ :

(i) There exists  $\vec{a}$  such that  $\mathcal{A}_A \models \theta(\vec{c}, \vec{a})$ ,

(ii) For every  $\vec{a}$  with  $\mathcal{A}_A \models \theta(\vec{c}, \vec{a})$ , and every  $\vec{a}_1$ ,

there exist  $\vec{a}'$  and  $\vec{a}'_1$  such that  $\mathcal{A}_A \models \neg \theta(\vec{c}, \vec{a}')$  and

$$\vec{c} \wedge \vec{a} \wedge \vec{a}_1 \leq_{n-1} \vec{c} \wedge \vec{a}' \wedge \vec{a}'_1$$

- **Examples:**  $\omega^\omega$ ,  $\omega^\omega \cdot \eta$

0-independent:  $x$  is greater than elements in  $\vec{u}$ ,

and  $x_0 < x_1$ ;

1-independent: ... and  $S(x_1, x_0)$ ;

2-independent: ... and  $x$  is a 1-limit

(first in its copy of  $\omega$ );

3-independent: ... and  $x_0$  and  $x_1$  are 1-limits,

where  $x_1$  is the next one after  $x_0$ ;

4-independent: ... and  $x$  is a 2-limit

(first in its copy of  $\omega^2$ ), etc.

These successor relations and initial element relations were used by Moses.

## Turing Degrees of Arbitrary Sequences of Diagrams

- (Harizanov, Knight, Morozov)

Suppose  $D_N(\mathcal{A})$  is computable, and the relations  $\leq_n$  are c.e., for  $n < N$ . Suppose also that for each tuple  $\vec{u}$  of distinct variables, and each  $n \leq N$ , we can find a formula that is  $n$ -independent over  $\vec{u}$ . Then for any  $(N+1)$ -table  $(C_n)_{n \leq N}$ , there exists  $\mathcal{B} \cong \mathcal{A}$  such that

$$(\forall n \leq N)[D_n(\mathcal{B}) \equiv_T C_n]$$

- (Harizanov, Knight, Morozov)

Suppose  $D^c(\mathcal{A})$  is computable, and the relations  $\leq_n$  are c.e., uniformly in  $n$ , for  $n \in \omega$ . Suppose also that for each tuple  $\vec{u}$  of distinct variables, and each  $n$ , we can find a  $\Pi_n$  formula that is  $n$ -independent over  $\vec{u}$ . Then for any  $\omega$ -table  $(C_n)_{n < \omega}$ , there exists  $\mathcal{B} \cong \mathcal{A}$  such that

$$D_n(\mathcal{B}) \equiv_T C_n, \text{ uniformly in } n.$$

The uniformity implies that

$$D^c(\mathcal{B}) \equiv_T \bigoplus_n C_n$$

## Examples

- For any  $2N$ -table  $(C_n)_{n \leq 2N-1}$ , there is a linear order  $\mathcal{B}$  of type  $\omega^N$  such that for all  $n \leq 2N - 1$ ,  $D_n(\mathcal{B}) \equiv_T C_n$ .
- For any  $(2N + 1)$ -table  $(C_n)_{n \leq 2N}$ , there is a linear order  $\mathcal{B}$  of type  $\omega^N \cdot \eta$  such that for all  $n \leq 2N$ ,  $D_n(\mathcal{B}) \equiv_T C_n$ .
- For any  $\omega$ -table  $(C_n)_{n \in \omega}$ , there is a linear order  $\mathcal{B}$  of type  $\omega^\omega$  or of type  $\omega^\omega \cdot \eta$  such that for all  $n$ ,  $D_n(\mathcal{B}) \equiv_T C_n$ , uniformly in  $n$ .

## Open Problems

- Weaken the definition of  $n$ -independent formulas so that the general results still hold, but can be applied to Boolean algebras and other structures.
- What are the possible sequences  $(deg(D_n(\mathcal{B})))_{n \in \omega}$ , for  $\mathcal{B} \cong \mathcal{N}$ ?
- Characterize the sequences of Turing degrees of  $n$ -diagrams for models of a given completion of  $PA$ .

## Knight's Conjecture

Let  $S$  be a completion of  $PA$  and let  $(C_n)_{n \in \omega}$  be an  $\omega$ -table. Suppose that there exist an enumeration  $R$  of a Scott set and a family of functions  $(t_n)_{n \geq 1}$  such that:

- $R \leq_T C_0$ ,
- $t_n \leq_T C_{n-1}$  uniformly in  $n$ ,
- $\lim_s t_n(s)$  is an  $R$ -index for  $S \cap \Sigma_n$ ,
- for all  $s$ ,  $t_n(s)$  is an  $R$ -index for a subset of  $S \cap \Sigma_n$ .

Then there is a (nonstandard) model  $\mathcal{A}$  of  $S$  such that

$$D_n(\mathcal{A}) \equiv_T C_n, \text{ uniformly in } n.$$

## Open Problem

- If  $\mathcal{A}$  is a non-standard model of  $PA$ , must there be  $\mathcal{B} \cong \mathcal{A}$  such that  $D_1(\mathcal{A}) \leq_T D_1(\mathcal{B})$  and  $D_0(\mathcal{B}) <_T D_0(\mathcal{A})$ ?

Knight showed that for any nonstandard model of  $PA$ , there is an isomorphic copy whose atomic diagram has strictly lower Turing degree. The problem is to produce such a copy without lowering the degree of the 1-diagram.