# Degrees of Structures

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- Consider *countable* structures *A* for *computable* languages.

  *Turing degree* of *A* is the Turing degree of the *atomic diagram of A*, $D$(*A*).
  $A$ is *computable* (*recursive*) if its Turing degree is $\mathbf{0}$.
  $D$(*A*) may be of much lower Turing degree than $Th$(*A*).

- (Tennenbaum) If *A* is a nonstandard model of $PA$, then
  *A* is not computable.

- (Harrington, Knight) There is a nonstandard model *A* of $PA$ such that
  *A* is *low* and $Th(A) \equiv_T \emptyset^{(\omega)}$.

- (Downey and Jockusch) Every Boolean algebra of *low* Turing degree has a computable copy.

- The *Turing degree spectrum* of $A$ is

$$DgSp(A) = \{\deg(B) : B \cong A\}.$$

- (Knight) A structure $A$ is *automorphically trivial* if there is a sequence $\overrightarrow{c} \in A^{<\omega}$ such that every permutation of $A$ that fixes $\overrightarrow{c}$ pointwise is an automorphism of $A$.

  $(i)$ If $A$ is automorphically trivial, then

$$|DgSp(A)| = 1.$$

  $(ii)$ If $A$ is automorphically nontrivial, then
  $DgSp(A)$ is closed upwards.

- (Harizanov, Knight and Morozov)

  $(i)$ If $A$ is automorphically trivial, then

$$(\forall B \simeq A)[D^e(B) \equiv_T D(B)].$$

  $(ii)$ If $A$ is automorphically nontrivial, and $X \geq_T D^e(A)$, there exists $B \cong A$ such that

$$D^e(B) \equiv_T D(B) \equiv_T X$$

- (Harizanov and R. Miller) If the language of $A$ is finite, then $A$ is trivial iff and $DgSp(A) = \{\mathbf{0}\}$.

- (Hirschfeldt, Khoussainov, Shore and Slinko) For every automorphically nontrivial structure $A$, there is a structure $B$, which can be:

  a symmetric irreflexive graph,

  a partial order,

  a lattice,

  a ring,

  an integral domain of arbitrary characteristic,

  a commutative semigroup,

  a 2-step nilpotent group, such that

$$DgSp(A) = DgSp(B).$$

- $\mathcal{D}$=the set of all Turing degrees

- (Wehner; Slaman)
  There is a structure $A$ such that
  $$DgSp(A) = \mathcal{D} - \{\mathbf{0}\}.$$

- (Hirschfeldt)
  There is a complete decidable theory, with all types computable, whose prime model $A$ has no computable copy, but has an $X$-decidable copy for every $X >_T \emptyset$.

- (Goncharov, Harizanov, Knight, McCoy, R. Miller, and Solomon) For each computable successor ordinal $\alpha$, there is a structure $A$ such that $DgSp(A)$ consists of the Turing degrees of sets $X$ such that $\Delta^0_\alpha(X)$ is not $\Delta^0_\alpha$.

- In particular, for every $n \in \omega$, there is a structure $A$ such that

$$DgSp(A) = \{\mathbf{c} \in \mathcal{D}\colon \mathbf{c}^{(\mathbf{n})} > \mathbf{0}^{(\mathbf{n})}\}.$$

A degree $\mathbf{c}$ is non-$low_n$ if $\mathbf{c}^{(\mathbf{n})} > \mathbf{0}^{(\mathbf{n})}$.

## Enumerations

- An *enumeration* of $S \subseteq P(\omega)$ is a binary relation $\nu$:

$$S = \{\nu(i) : i \in \omega\}, \text{ where } \nu(i) = \{x : (i, x) \in \nu\}.$$

  $\nu$ is *computable* (*c.e.*) if it is computable (c.e.) as a binary relation.

- (Wehner)
  There is a family $S$ such that for every $X >_T \emptyset$, $S$ has an enumeration computable in $X$, but $S$ has no computable enumeration.

- There is a family $S$ such that for every $X >_T \emptyset$, $S$ has an enumeration c.e. relative to $X$, but $S$ has no c.e. enumeration.

## Transforming $S$ into a graph

- Assign to $A \in S$, a *daisy graph* $G_A$ consisting of one *index* point $a$ at the center with $a \longrightarrow a$, and for each $n \in A$ a *petal* (disjoint from other petals)

$$a \longrightarrow a_0 \longrightarrow \cdots \longrightarrow a_n \longrightarrow a$$

- $G(S)$ is the union of a disjoint family of $G_A$ for each $A \in S$.
  $G(S)$ is a rigid graph.

- $G^\infty(S)$ consists of infinitely many copies of $G_A$ for each $A \in S$.
  $G^\infty(S)$ is not rigid.
  Copies of $G^\infty(S)$ correspond to enumerations of $S$.

Let $S \subseteq P(\omega)$, $X \subseteq \omega$.

- There is an enumeration of $S$ *c.e. in* $X$ iff there is a copy of $G^\infty(S)$ computable in $X$.

- $S^+ =_{def} \{A \oplus \overline{A} : A \in S\}$.

- There is an enumeration of $S$ *computable in* $X$ iff there is a copy of $G^\infty(S^+)$ computable in $X$.

- (Goncharov, Harizanov, Knight, McCoy, R. Miller, and Solomon)

  Let $\alpha \geq 2$ be a computable successor ordinal.
  There is a structure with copies in exactly the Turing degrees of sets $X$ such that $\Delta^0_\alpha(X)$ is not $\Delta^0_\alpha$.

- *Proof sketch.* Relativize the proof for $\Delta^0_1$ to $\Delta^0_\alpha$.

  Get a graph $G$ such that the degrees of copies of $G$ are just the degrees of sets that are not $\Delta^0_\alpha$.

- Code a directed graph $G$ in a structure $G^*$ such that:

  $G$ has a $\Delta^0_\alpha$ copy iff $G^*$ has a computable copy.
  More generally, for any $X \subseteq \omega$,
  $G$ has a $\Delta^0_\alpha(X)$ copy iff $G^*$ has an $X$-computable copy.

- *Proof sketch.* Code a directed graph $G$ in a structure $G^*$, using a pair of structures $B_0$, $B_1$ such that
  $B_0$ codes $G \models a \to b$ and
  $B_1$ codes $G \models \neg(a \to b)$.

- $G^* = (G \cup U, G, U, Q, \ldots)$, where
  $G$ and $U$ are disjoint,
  $Q$ (a ternary relation) assigns to $a, b \in G$ an infinite set $U_{(a,b)}$:
  $(x \in U_{(a,b)} \Leftrightarrow Qabx)$,
  the sets $U_{(a,b)}$ form a partition of $U$,

$$(U_{(a,b)}, \ldots) \cong \begin{cases} B_0, & \text{if } G \models a \to b, \\ B_1, & \text{if } G \models \neg(a \to b). \end{cases}$$

Assume

- Pair $\{B_0, B_1\}$ is $\alpha$-friendly.
- $B_0$ and $B_1$ satisfy the same infinitary $\Pi_\beta$ sentences for $\beta < \alpha$.
- $B_0$ satisfies some computable $\Pi_\alpha$ sentence that is not true in $B_1$, and *vice versa.*

Then for any $\Delta^0_\alpha$ set $S$, there is a uniformly computable sequence $(C_n)_{n\in\omega}$ such that

$$C_n \cong \begin{cases} B_0, & \text{if } n \in S, \\ B_1, & \text{if } n \notin S. \end{cases}$$

- (R. Miller)
  There is a linear order $A$ such that

$$DgSp(A) \cap \mathbf{\Delta}_2^0 = \mathbf{\Delta}_2^0 - \{\mathbf{0}\}.$$

- (Harizanov and R. Miller)
  There exists a structure $A$ such that $DgSp(A)$ consists of the degrees that are high-or-above:

$$DgSp(A) = \{\mathbf{c} \in \mathcal{D}\colon\ \mathbf{c}' \geq \mathbf{0}''\}.$$

- A degree $\mathbf{c}$ is *high* if $\mathbf{c}' = \mathbf{0}''$.

- $(\omega, \prec)$ computable linear order
  Computable isomorphism $f : L = (\omega, \prec) \to (\mathbb{Q}, <)$.

- (Harizanov and R. Miller)
  For any relation $R$ on $L$, there exists a structure $A$ such that

$$DgSp(A) = DgSp_L(R).$$

- Define a relation $R$ on $L$ by:

$$\begin{aligned} f(R) \; &= \; \left(-1, -\frac{1}{2}\right) \bigcup \left( \bigcup_{n \in \emptyset'''} \left[ n, n + \frac{1}{2} \right) \right) \\ &\qquad \bigcup \left( \bigcup_{n \notin \emptyset'''} \left( n - \frac{1}{\pi}, n + \frac{1}{2} \right) \right) \end{aligned}$$

- $DgSp_{L}(R) = \{\mathbf{c} \in \mathcal{D}\colon \mathbf{c}' \geq \mathbf{0}''\}$.

- *Proof sketch.* Show
  $\mathbf{c} \in DgSp_{L}(R)$ iff $\emptyset''' \leq_1 Fin^C$
  for some set $C$ with $deg(C) = \mathbf{c}$

- $Fin^C = \{e : W_e^C \text{ is finite}\}$

- $\emptyset''' \leq_1 Fin^C \Leftrightarrow \emptyset'' \leq_T C'$

- (Jockusch) The (*Turing*) *degree* of the *isomorphism type* of *A*, if it exists, is the *least* Turing degree in $DgSp(A)$.

- (Richter) Assume that a structure *A* satisfies the effective extendability condition. If the degree of the isomorphism type of *A* exists, then it must be $\mathbf{0}$. ($DgSp(A)$ will contain a minimal pair of degrees.)

- *Effective Extendability Condition* for *A*

  For every finite structure *C* isomorphic to a substructure of *A*, and every embedding $f$ of *C* into *A*, there is an algorithm that determines whether a given finite structure *D* extending *C* can be embedded into *A* by an embedding extending $f$.

- (Richter)

  (i) A *linear order* without a computable copy does not have the isomorphism type degree.

  (ii) A *tree* without a computable copy does not have the isomorphism type degree.

- Abelian $p$-group $G$

  $x \in (G - \{0\}) \Rightarrow (\exists n)[order(x) = p^n]$

- (A. Khisamiev)

  An *abelian $p$-group* without a computable copy does not have the isomorphism type degree.

- *Richter's Combination Method*

  Let $T$ be a theory in a finite language $L$ such that there is a computable sequence $A_0, A_1, A_2, \ldots$ of *finite* structures for $L$, which are *pairwise nonembeddable*. Assume that for every $X \subseteq \omega$, there is a model $A_X$ of $T$ such that

  $$A_X \leq_T X,$$

  and for every $i \in \omega$,

  $$A_i \text{ is embeddable in } A_X \Leftrightarrow i \in X.$$

  Then for every Turing degree $\mathbf{d}$, there is a model of $T$ whose isomorphism type has degree $\mathbf{d}$.

- For every Turing degree $\mathbf{d}$, there is an *abelian group* whose isomorphism type has degree $\mathbf{d}$.

- (Calvert, Harizanov, Shlapentokh)
  For every Turing degree $\mathbf{d}$, there are various *fields* whose isomorphism types have degree $\mathbf{d}$.

- *Proof sketch.* Let $M_0 = F$ be any computable finitely generated field.
  $\tilde{F}$ the algebraic closure of $F$.
  $\{f_i(t) \in F(t)\}_{i \geq 1}$ computable sequence of monic irreducible polynomials (over $F$).
  $\alpha_i$ a root of $f_i$, and $M_i = F(\alpha_i)$.
  Assume further that the sequence $\{M_i\}_i$ is *totally linearly disjoint* over $F$, and is *stable* with respect to $F$.

- Let $A_X = \prod_{i \in X} M_i$, where $X = D \oplus \overline{D}$.

$$DgSp(A_X) = \{\mathbf{c} \in \mathcal{D} : \mathbf{c} \geq \deg(D)\}.$$

- Let $F$ be a field, $\{L_i\}_{i\in\omega}$ a sequence of extensions of $F$.
  Let $L = \prod_{i\in\omega} L_i$.

- $\{L_i\}_{i\in\omega}$ is *totally linearly disjoint over* $F$ *if* the extensions are finite, and for all $i$, $L_i$ and $\prod_{j\in\omega\setminus\{i\}} L_j$ are linearly disjoint over $F$:

$$[L_i : F] = [L : \prod_{j\in\omega\setminus\{i\}} L_j] > 1.$$

- $\{L_i\}_{i\in\omega}$ is *stable with respect to* $F$ if for any embedding $\sigma : L \longrightarrow \tilde{F}$ ($\tilde{F}$ is the algebraic closure of $F$), such that $\sigma_{|F} = id$, then for all $i$,

$$\text{either } \sigma(L_i) = L_i \text{ or } \sigma(L_i) \not\subset L.$$

  $\{L_i\}_{i\in\omega}$ is *stable* if $F = \mathbb{Q}$, or $F$ is a finite field.

- Let $F = \mathbb{Q}$.
  $\{p_i\}_i$ listing of rational primes.
  $f_i(t) = t^2 - p_i$
  $M_i = \mathbb{Q}(\sqrt{p_i})$
  (Sequence $\{M_i\}_i$ is stable, and totally linearly disjoint over $\mathbb{Q}$.)

- Let $F = \mathbb{Q}(x)$, where $x$ is not algebraic over $\mathbb{Q}$.
  $M_i = \mathbb{Q}(x, \sqrt{p_i}, \sqrt[p_i]{x^2 + 1})$
  (Sequence $\{M_i\}_i$ is stable with respect to $\mathbb{Q}(x)$, and totally linearly disjoint over $\mathbb{Q}(x)$.)

- Let $F = \mathbb{F}_p$ be a field of $p$ elements for some rational prime $p$.
  Let $\alpha_i$ be of degree $p_i$ over $\mathbb{F}_p$.
  $M_i = \mathbb{F}_p(\alpha_i)$.
  (Sequence $\{M_i\}_i$ is stable, and totally linearly disjoint over $\mathbb{F}_p$.)

- Let $F = \mathbb{F}_p(x)$, where $x$ is not algebraic over $\mathbb{F}_p$.
  Let $M_i = \mathbb{F}_p(\sqrt{x^2 + i})$.
  (Sequence $\{M_i\}_i$ is stable with respect to $\mathbb{F}_p(x)$,
  and totally linearly disjoint over $\mathbb{F}_p(x)$.)