

# RELATIVELY HYPERIMMUNE RELATIONS ON STRUCTURES

S. S. GONCHAROV, V. S. HARIZANOV, J. F. KNIGHT, AND C. F. D. MCCOY

ABSTRACT. Let  $R$  be a relation on the domain of a computable structure  $\mathcal{A}$ . We establish that the existence of an isomorphic copy  $\mathcal{B}$  of  $\mathcal{A}$  such that the image of  $R$  ( $\neg R$ , resp.) is  $h$ -simple ( $h$ -immune, resp.) relative to  $\mathcal{B}$  is equivalent to a syntactic condition, termed  $R$  is formally  $h$ -simple (formally  $h$ -immune, resp.) on  $\mathcal{A}$ .

## 1. INTRODUCTION

We consider only countable structures for computable relational languages, and investigate relatively hyperimmune and relatively hypersimple relations on these structures. We can identify the universe  $A$  of  $\mathcal{A}$  with a subset of  $\omega$ , and think of it as a set of constants. The language  $L_A$  is the extension of  $L$  by a constant symbol  $a$  for every  $a \in A$ , and the corresponding expanded structure is  $\mathcal{A}_A$ . The *atomic diagram* of  $\mathcal{A}$ ,  $D(\mathcal{A})$ , is the set of all atomic and negated atomic sentences of  $L_A$ , which are true in  $\mathcal{A}_A$ . A structure is *computable* if its atomic diagram is computable.

Here, we consider relative versions of classical computability-theoretic notions of hyperimmunity and hypersimplicity of relations on countable structures. A relation on  $\omega$  is called *hypersimple*, abbreviated by  *$h$ -simple*, if it is computably enumerable (c.e.) and its complement is hyperimmune. It is established in computability theory that a unary relation  $C$  on  $\omega$  is *hyperimmune*, abbreviated by  *$h$ -immune*, iff it is infinite and no computable function majorizes its principal function  $p_C$ , where  $p_C(n) =_{\text{def}} c_n$  provided that  $C = \{c_0 < c_1 < c_2 < \dots\}$ .

Let  $\mathcal{A}$  be an infinite computable structure for language  $L$ , and let  $R$  be an infinite and co-infinite relation on  $A$ . We consider the following problems.

**Problem 1.** *Under what syntactic conditions is there an isomorphism  $F$  from  $\mathcal{A}$  onto a copy  $\mathcal{B}$  such that  $\neg F(R)$  is  $h$ -immune relative to  $\mathcal{B}$ ?*

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**Problem 2.** *Under what syntactic conditions is there an isomorphism  $F$  from  $\mathcal{A}$  onto a copy  $\mathcal{B}$  such that  $F(R)$  is  $h$ -simple relative to  $\mathcal{B}$ ?*

In [8], Hird introduced the notions of *formal  $h$ -immunity* and *formal  $h$ -simplicity* for relations on computable structures. These are syntactic analogues of  $h$ -immunity and  $h$ -simplicity. It is not hard to see that if there is an isomorphic copy of  $\mathcal{A}$  on which the image of  $\neg R$  is relatively  $h$ -immune, then  $\neg R$  must be formally  $h$ -immune on  $\mathcal{A}$ . This necessary syntactic condition turns out to be sufficient. Thus, our main result on Problem 1 is the following equivalence of a semantic and a corresponding syntactic notion.

**Theorem 2.1.** *The following are equivalent for  $(\mathcal{A}, R)$ .*

- (i) *There is a copy  $\mathcal{B}$  of  $\mathcal{A}$  and an isomorphism  $F$  from  $\mathcal{A}$  onto  $\mathcal{B}$  such that  $\neg F(R)$  is  $h$ -immune relative to  $\mathcal{B}$ .*
- (ii) *The relation  $\neg R$  is formally  $h$ -immune on  $\mathcal{A}$ .*

Similarly, we obtain our main result on Problem 2.

**Theorem 3.1.** *The following are equivalent for  $(\mathcal{A}, R)$ .*

- (i) *There is a copy  $\mathcal{B}$  of  $\mathcal{A}$  and an isomorphism  $F$  from  $\mathcal{A}$  onto  $\mathcal{B}$  such that  $F(R)$  is  $h$ -simple relative to  $\mathcal{B}$ .*
- (ii) *The relation  $R$  is formally  $h$ -simple on  $\mathcal{A}$ .*

Unlike analogous results for computable copies  $\mathcal{B}$  of  $\mathcal{A}$ , these relative results do not involve extra decidability conditions on  $(\mathcal{A}, R)$ . Examples of such relative results are also presented in [1], [2], [3], and [5]. In particular, in [5] we investigated relative immunity and relative simplicity of relations on countable structures, and established equivalences of syntactic and corresponding semantic conditions.

In the remainder of this section, we review some computability-theoretic notation and concepts (see [13]). In Section 2, we review the notions of  $h$ -immunity and formal  $h$ -immunity, define the notion of relative  $h$ -immunity and prove our main theorem on relatively  $h$ -immune relations. In Section 3, we review the notions of  $h$ -simplicity and formal  $h$ -simplicity, define the notion of relative  $h$ -simplicity and prove our main theorem on relatively  $h$ -simple relations.

Let  $\varphi_0, \varphi_1, \varphi_2, \dots$  be a fixed effective enumeration of all unary partial computable functions. Let  $X \subseteq \omega$ . Then  $\varphi_0^X, \varphi_1^X, \varphi_2^X, \dots$  is a fixed effective enumeration of all unary  $X$ -partial computable functions. For a structure  $\mathcal{B}$ ,  $\varphi_e^{\mathcal{B}}$  stands for  $\varphi_e^{D(\mathcal{B})}$ . The following (canonical) indexing of finite sets is standard. Let  $D_0 =_{\text{def}} \emptyset$ . For  $m > 0$ , let  $D_m = \{d_0, \dots, d_{k-1}\}$ , where  $d_0 < \dots < d_{k-1}$  and  $m = 2^{d_0} + \dots + 2^{d_{k-1}}$ . A sequence  $(U_i)_{i \in \omega}$  of finite sets is a *strong array* if there is a unary

computable function  $f$  such that for every  $i \in \omega$ ,  $U_i = D_{f(i)}$ . A strong array is *disjoint* if its members are pairwise disjoint.

Throughout the paper, we will assume that  $\mathcal{A}$  is an infinite computable structure for language  $L$ , and that  $R$  is an infinite co-infinite relation on its domain. Without loss of generality, we assume that  $R$  is unary. By  $\vec{a}$  we denote a finite sequence (tuple) of elements. We often write  $a \in \vec{a}$  instead of  $a \in \text{ran}(\vec{a})$ , and  $\vec{a} \cap G = \emptyset$  instead of  $\text{ran}(\vec{a}) \cap G = \emptyset$ .

## 2. RELATIVELY $h$ -IMMUNE RELATIONS

The following definition of an  $h$ -immune relation on natural numbers is a usual one.

**Definition 2.1.** *Let  $S \subseteq \omega$ . The relation  $\neg S$  is  $h$ -immune (on  $\omega$ ) if it is infinite and there is no disjoint strong array  $(U_i)_{i \in \omega}$  such that for every  $n \in \omega$ , we have  $U_i \cap \overline{S} \neq \emptyset$ .*

We can similarly define  $h$ -immune relations on any computable set. Every  $h$ -immune set is *immune*, that is, infinite but without any infinite c.e. subset. Not every immune set is  $h$ -immune. However, Jockusch [9] proved that every immune initial segment of a computable linear order is  $h$ -immune. Hird [7] studied co-c.e. intervals of a computable linear order with  $h$ -immune (equivalently, immune) images on some computable copy of the linear order. Remmel [12] proved that if  $\mathcal{A}$  is a computable Boolean algebra, whose atoms form an infinite computable set, then there is a computable copy  $\mathcal{B}$  of  $\mathcal{A}$  such that the set of all atoms of  $\mathcal{B}$  is  $h$ -immune and of an arbitrary nonzero c.e. Turing degree.

The following definition of Hird introduces a syntactic property corresponding to  $h$ -immunity. We will term it “being formally  $h$ -immune on  $\mathcal{A}$ .”

**Definition 2.2.** (Hird [8]) (i) *A formal strong array on  $\mathcal{A}$  is a computable sequence of existential formulas in  $L$  with finitely many parameters  $\vec{c}$ ,  $(\psi_i(\vec{c}, \vec{x}_i))_{i \in \omega}$ , such that for every finite set  $G \subseteq A$  there is  $i \in \omega$  and a sequence  $\vec{a}_i \in A^{lh(\vec{x}_i)}$  with*

$$[\mathcal{A}_A \models \psi_i(\vec{c}, \vec{a}_i)] \wedge [\vec{a}_i \cap G = \emptyset].$$

(ii) *We say that the relation  $\neg R$  is formally  $h$ -immune on  $\mathcal{A}$  if there is no formal strong array  $(\psi_i(\vec{c}, \vec{x}_i))_{i \in \omega}$  on  $\mathcal{A}$  such that for every  $i \in \omega$ ,*

$$(\forall \vec{a}_i \in A^{lh(\vec{x}_i)}) [(\mathcal{A}_A \models \psi_i(\vec{c}, \vec{a}_i)) \Rightarrow (\vec{a}_i \cap \neg R \neq \emptyset)].$$

Being formally  $h$ -immune on  $\mathcal{A}$  turns out to be a necessary condition for the existence of a computable copy of  $\mathcal{A}$  such that the corresponding

image of  $R$  is  $h$ -immune (see [8]). Assume that  $\mathcal{B}$  is a computable copy of  $\mathcal{A}$  and that  $F$  is an isomorphism from  $\mathcal{A}$  onto  $\mathcal{B}$ . It is easy to show that if  $(\psi_i(\vec{c}, \vec{x}_i))_{i \in \omega}$  is a formal strong array on  $\mathcal{A}$ , then  $(\psi_i(F(\vec{c}), \vec{x}_i))_{i \in \omega}$  is a formal strong array on  $\mathcal{B}$  (see [6]). The following result establishes that, under some extra decidability conditions for  $(\mathcal{A}, R)$ , the existence of a computable copy  $\mathcal{B}$  of  $\mathcal{A}$  such that the image of  $\neg R$  is  $h$ -immune relative to  $\mathcal{B}$  is equivalent to  $\neg R$  being formally  $h$ -immune on  $\mathcal{A}$ .

**Theorem 2.1.** (Hird [8]) (i) *Assume that  $\mathcal{B}$  is a computable copy of  $\mathcal{A}$  and that  $F$  is an isomorphism from  $\mathcal{A}$  onto  $\mathcal{B}$ . If  $F(\neg R)$  is  $h$ -immune on  $\mathcal{B}$ , then  $\neg R$  is formally  $h$ -immune on  $\mathcal{A}$ .*

(ii) *Assume that there is an algorithm which decides for a given sequence  $\vec{c} \in A^{<\omega}$  and an existential formula  $\psi(\vec{u}, \vec{x})$  in  $L$ ,  $lh(\vec{u}) = lh(\vec{c})$ , whether*

$$(\forall \vec{a} \in A^{lh(\vec{x})})[(\mathcal{A}_A \models \psi(\vec{c}, \vec{a})) \Rightarrow (\vec{a} \cap \neg R \neq \emptyset)].$$

*If  $\neg R$  is formally  $h$ -immune on  $\mathcal{A}$ , then there is a computable structure  $\mathcal{B}$  and an isomorphism  $F$  from  $\mathcal{A}$  onto  $\mathcal{B}$  such that the relation  $F(\neg R)$  is  $h$ -immune on  $\mathcal{B}$ .*

An infinitary  $\Sigma_1$  formula is an  $L_{\omega\omega_1}$  formula of the form

$$\bigvee_{i \in I} \exists \vec{u}_i \psi_i(\vec{x}, \vec{u}_i),$$

where for every  $i \in I$ ,  $\psi_i(\vec{x}, \vec{u}_i)$  is a finitary quantifier-free formula. We assume that the finitary quantifier-free formulas are coded by some effective Gödel numbering, and  $\psi_i$  is the  $i^{\text{th}}$  formula under this numbering. If the index set  $I$  is c.e., then we have a *computable*  $\Sigma_1$  formula. We can continue recursively with the definition of computable infinitary formulas, by defining  $\Sigma_\alpha$  and  $\Pi_\alpha$  formulas for every computable ordinal  $\alpha$  (see [1]).

**Definition 2.3.** (i) *An infinitary formal strong array on  $\mathcal{A}$  is a computable sequence of computable  $\Sigma_1$  formulas,  $(\psi_i(\vec{c}, \vec{x}_i))_{i \in \omega}$ , such that for every finite set  $G \subseteq A$  there is  $i \in \omega$  and  $\vec{a}_i \in A^{lh(\vec{x}_i)}$  with*

$$[\mathcal{A}_A \models \psi_i(\vec{c}, \vec{a}_i)] \wedge [\vec{a}_i \cap G = \emptyset].$$

(ii) *The relation  $\neg R$  is infinitary formally  $h$ -immune on  $\mathcal{A}$  if there is no infinitary formal strong array  $(\psi_i(\vec{c}, \vec{x}_i))_{i \in \omega}$  on  $\mathcal{A}$  such that for every  $i \in \omega$ ,*

$$(\forall \vec{a}_i \in A^{lh(\vec{x}_i)})[(\mathcal{A}_A \models \psi_i(\vec{c}, \vec{a}_i)) \Rightarrow (\vec{a}_i \cap \overline{R} \neq \emptyset)].$$

The following proposition follows immediately.

**Proposition 2.2.** *The relation  $\neg R$  is infinitary formally  $h$ -immune on  $\mathcal{A}$  if and only if it is formally  $h$ -immune on  $\mathcal{A}$ .*

We now introduce a relative version of  $h$ -immunity.

**Definition 2.4.** *Let  $S$  be an additional (unary) relation on the domain  $B$  of a countable structure  $\mathcal{B}$ .*

(i) *A sequence  $(U_i)_{i \in \omega}$  of finite sets is a strong array relative to  $\mathcal{B}$  if there is a unary  $\mathcal{B}$ -computable function  $f$  such that for every  $i \in \omega$ ,  $U_i = D_{f(i)}$ .*

(ii) *The relation  $\neg S$  is  $h$ -immune relative to  $\mathcal{B}$  if it is infinite and there is no disjoint strong array relative to  $\mathcal{B}$ ,  $(U_i)_{i \in \omega}$ , such that for every  $n \in \omega$ , we have  $U_i \cap \bar{S} \neq \emptyset$ .*

If there is an isomorphic copy of  $\mathcal{A}$  on which the image of  $\neg R$  is relatively  $h$ -immune, then  $\neg R$  must be formally  $h$ -immune on  $\mathcal{A}$ . This necessary syntactic condition turns out to be sufficient.

**Theorem 2.1.** *Let  $\mathcal{A}$  be a computable  $L$ -structure, and let  $R$  be a unary infinite and co-infinite relation on  $A$ . Then the following are equivalent:*

(i) *For all copies  $\mathcal{B}$  of  $\mathcal{A}$  and all isomorphisms  $F$  from  $\mathcal{A}$  onto  $\mathcal{B}$ ,  $\neg F(R)$  is not  $h$ -immune relative to  $\mathcal{B}$ .*

(ii) *The relation  $\neg R$  is not formally  $h$ -immune on  $\mathcal{A}$ .*

*Proof.* (ii)  $\Rightarrow$  (i) Assume that  $\neg R$  is not formally  $h$ -immune on  $\mathcal{A}$ . Let  $(\psi_i(\vec{c}, \vec{x}_i))_{i \in \omega}$  be a corresponding formal strong array on  $\mathcal{A}$ . Let  $\mathcal{B}$  be a copy of  $\mathcal{A}$  and  $F$  an isomorphism from  $\mathcal{A}$  onto  $\mathcal{B}$ . Hence,  $(\psi_i(F(\vec{c}), \vec{x}_i))_{i \in \omega}$  is a formal strong array on  $\mathcal{B}$ . Since for every  $\vec{a}_i \in A^{<\omega}$ , we have

$$(\vec{a}_i \cap \bar{R} \neq \emptyset) \Rightarrow (F(\vec{a}_i) \cap F(R) \neq \emptyset),$$

it follows that

$$(\forall \vec{b}_i \in B^{lh(\vec{x}_i)})[(\mathcal{B}_B \models \psi_i(F(\vec{c}), \vec{b}_i)) \Rightarrow (\vec{b}_i \cap F(R) \neq \emptyset)].$$

We now show that  $F(R)$  is not  $h$ -immune relative to  $\mathcal{B}$ , by enumerating a corresponding strong array relative to  $\mathcal{B}$ . We simultaneously enumerate finite sequences of elements in  $B$ , satisfying formulas in  $(\psi_i(F(\vec{c}), \vec{x}_i))_{i \in \omega}$ , such that none of these sequences intersects any of the previously enumerated ones. This is possible by the main property of a formal strong array. For every such sequence  $\vec{b}_i$  with  $\mathcal{B}_B \models \psi_i(F(\vec{c}), \vec{b}_i)$ , it follows that  $\vec{b}_i \cap F(R) \neq \emptyset$ , as required. The ranges of these sequences form a strong array relative to  $\mathcal{B}$ .

(i)  $\Rightarrow$  (ii) We build a generic copy  $(\mathcal{B}, S)$  of  $(\mathcal{A}, R)$ . Assuming that  $\neg S$  is not  $h$ -immune relative to  $\mathcal{B}$ , we will produce an infinitary formal

strong array on  $\mathcal{A}$ ,  $(\psi_i)_{i \in \omega}$ , which will witness that  $\neg R$  is not infinitary formally  $h$ -immune on  $\mathcal{A}$ . Let  $B = \{b_0, b_1, b_2, \dots\}$  be an infinite computable set, the universe of  $\mathcal{B}$ . Without loss of generality, we can assume that  $(\forall i)[b_i = i]$ . The set  $\mathcal{F}$  of *forcing conditions* consists of all finite 1-1 partial functions from  $B$  to  $A$ . We use letters  $p, q, r, \dots$  to denote elements of  $\mathcal{F}$ . Let  $\mathbf{R}$  be a unary relation symbol that is not in  $L$ . As a forcing language, we take a propositional language  $P^*$  in which the propositional variables are the atomic sentences in the expanded language  $(L \cup \{\mathbf{R}\})_B$ . Let  $P$  be the sublanguage of  $P^*$  consisting of atomic sentences in  $L_B$ . Let  $\mathcal{S}^*$  be the set of computable infinitary sentences in the language  $P^*$ , and let  $\mathcal{S}$  be the set of computable infinitary sentences in  $P$ . We consider only computable infinitary formulas in their *normal* form. Hence the negations can occur only in finitary open subformulas. For a sentence  $\psi$ , we write  $neg(\psi)$  for the computable infinitary sentence (in the normal form) that is dual to  $\psi$ , i.e. equivalent to the negation of  $\psi$ . A sentence  $\psi$  in forcing language  $P^*$  is propositional, but we may also think of it as a predicate sentence in the language  $(L \cup \{\mathbf{R}\})_B$ . The constants of  $\psi$  are the constants appearing in the propositional variables in  $\psi$ .

We are especially interested in sentences in  $\mathcal{S}$  or  $\mathcal{S}^*$  expressing the following facts in  $(\mathcal{B}, S)$  for some fixed  $e \in \omega$ :

- (i)  $\varphi_e^{\mathcal{B}}$  is total (expressed in  $\mathcal{S}$ );
- (ii)  $D_{\varphi_e^{\mathcal{B}}(n)}$  and  $D_{\varphi_e^{\mathcal{B}}(n')}$  are disjoint for  $n \neq n'$  (expressed in  $\mathcal{S}$ );
- (iii)  $D_{\varphi_e^{\mathcal{B}}(n)} \cap \overline{S} \neq \emptyset$  for every  $n$  (expressed in  $\mathcal{S}^*$ ).

We express the sentence in (i) by

$$\bigwedge_{n \in \omega} \bigvee_{m \in \omega} \bigvee_{\{\sigma: \varphi_e^\sigma(n)=m\}} \theta_\sigma.$$

Here,  $\varphi_e^\sigma(n) = m$  is the halting computation of an oracle Turing machine with Gödel index  $e$  on input  $n$  with output  $m$  in  $\leq lh(\sigma)$  steps using the finite oracle  $\sigma \in 2^{lh(\sigma)}$ , where  $\theta_\sigma = \theta_{\sigma_{n,m}}(\vec{b}_{\sigma_{n,m}})$  is an open sentence in  $L_B$ , expressing information about  $D(\mathcal{B})$ , contained in  $\sigma$ .

We express the sentence in (ii) by

$$\bigwedge_{n \neq n'} \bigvee_{D_m \cap D_{m'} = \emptyset} \bigvee \{\theta_\sigma \wedge \theta_{\sigma'} : \varphi_e^\sigma(n) = m \wedge \varphi_e^{\sigma'}(n') = m'\}$$

where, again,  $\theta_\sigma$  and  $\theta_{\sigma'}$  (resp.) are open sentences in  $L_B$ , expressing suitable oracle information about  $D(\mathcal{B})$ , needed for the computations  $\varphi_e^\sigma(n) = m$  and  $\varphi_e^{\sigma'}(n') = m'$  (resp.).

We express the sentence in (iii) by

$$\bigwedge_{n,m} [\bigvee \{\theta_\sigma : \varphi_e^\sigma(n) = m\} \Rightarrow \bigvee_{\{j:j \in D_m\}} \neg \mathbf{R}(b_j)].$$

Let  $p \in \mathcal{F}$  and a sentence  $\psi$  in  $\mathcal{S}^*$ , we define a forcing relation,  $p \Vdash \psi$ , as follows:

- (1) If  $\psi$  is a finitary sentence, then  $p \Vdash \psi$  iff the constants of  $\psi$  are all in  $\text{dom}(p)$ , and  $p$  makes  $\psi$  true in  $(\mathcal{A}_A, R)$ ;
- (2)  $p \Vdash \bigvee_{i \in I} \psi_i$  iff  $(\exists i \in I)[p \Vdash \psi_i]$ ;
- (3)  $p \Vdash \bigwedge_{i \in I} \psi_i$  iff  $(\forall q \supseteq p)(\forall i \in I)(\exists r \supseteq q)[r \Vdash \psi_i]$ .

We have the usual forcing lemmas for every  $p, q \in \mathcal{F}$  and every sentence  $\psi$  in  $\mathcal{S}^*$ .

**Lemma 2.2.**  $[p \Vdash \psi \wedge q \supseteq p] \Rightarrow q \Vdash \psi$ .

**Lemma 2.3.**  $\neg[(p \Vdash \psi) \wedge (p \Vdash \text{neg}(\psi))]$ .

**Lemma 2.4.**  $(\exists r \supseteq p)[(r \Vdash \psi) \vee (r \Vdash \text{neg}(\psi))]$ .

We say that  $r$  *decides*  $\psi$  iff  $[(r \Vdash \psi) \vee (r \Vdash \text{neg}(\psi))]$ . A *complete forcing sequence* is a chain  $(p_n)_{n \in \omega}$  of forcing conditions such that for every sentence  $\psi$  in  $\mathcal{S}^*$ , there is  $n$  such that  $p_n$  decides  $\psi$ ; for every  $a \in A$ , there is  $n$  such that  $a \in \text{ran}(p_n)$ ; and for every  $b \in B$ , there is  $n$  such that  $b \in \text{dom}(p_n)$ . The existence of a complete forcing sequence  $(p_n)_{n \in \omega}$  follows from Lemma 2.4. It is easy to see that  $\bigcup_{n \in \omega} p_n$  a 1-1 function from  $B$  onto  $A$ . Let  $F =_{\text{def}} (\bigcup_{n \in \omega} p_n)^{-1}$ . Then  $F$  induces on  $B$  a copy  $(\mathcal{B}, S)$  of  $(\mathcal{A}, R)$ , where  $S = F(R)$ .

We have the following *Truth-and-Forcing Lemma*.

**Lemma 2.5.** *For any  $\psi \in \mathcal{S}^*$ , we have  $(\mathcal{B}_B, F(R)) \models \psi$  iff there is  $n \in \omega$  such that  $p_n \Vdash \psi$ .*

To complete the proof of Theorem 2.1, we need the following lemma.

**Lemma 2.6.** *The relation  $F(\neg R)$  is  $h$ -immune relative to  $\mathcal{B}$ .*

*Proof of Lemma 2.6.* Suppose otherwise. Then there is  $e \in \omega$  such that  $(D_{\varphi_e^{\mathcal{B}(n)}})_{n \in \omega}$  is a strong array on  $B$ , such that for every  $n \in \omega$ ,  $D_{\varphi_e^{\mathcal{B}(n)}} \cap \vec{S} = \emptyset$ . By the *Truth-and-Forcing Lemma*, there is  $p \in \mathcal{F}$  ( $p = p_n$  for some  $n$ ) such that  $p$  forces the statements expressing this fact. Let  $p$  map  $\vec{d}$  onto  $\vec{c}$ .

We will define a computable sequence of computable  $\Sigma_1$  formulas in  $L$ , with parameters  $\vec{c}$ . Assume that  $\vec{x}_0, \vec{x}_1, \vec{x}_2, \dots$  is an effective

enumeration of all finite sequences of variables of  $L$ . For  $i \in \omega$ , let  $\psi_i(\vec{c}, \vec{x}_i)$  be the following formula

$$\bigvee_{q \supseteq p} \bigvee \{q \models \text{“}\varphi_e^{\mathcal{B}}(n) = m\text{”} : n, m \in \omega \wedge D_m = \vec{b}_i \wedge q(\vec{b}_i) = \vec{x}_i\}$$

Let us prove that  $(\psi_i(\vec{c}, \vec{x}_i))_{i \in \omega}$  is an infinitary formal strong array on  $\mathcal{A}$ . Let  $G \subseteq A$  be a finite set. Let

$$M =_{\text{def}} \{n \in \omega : D_{\varphi_e^{\mathcal{B}}(n)} \cap F(G) \neq \emptyset\}.$$

Clearly,  $M$  is a finite set. If there is  $p' \supseteq p$ , such that for every  $a \in G$ , there is  $n \in \omega$  such that  $p' \Vdash \text{“}F(a) \in D_{\varphi_e^{\mathcal{B}}(n)}\text{”}$ , then choose such  $p'$ . Otherwise, let  $p' \supseteq p$  be such that no  $p'' \supseteq p$  is such that  $p''$  forces any new elements of  $F(G)$  (that is, the ones that  $p'$  does not force) into  $\bigcup_{n \in \omega} D_{\varphi_e^{\mathcal{B}}(n)}$ . Choose  $n \notin M$  such that for some  $q \supseteq p'$ , for some  $m$ , we have  $q \Vdash \text{“}\varphi_e^{\mathcal{B}}(n) = m\text{”}$ , where  $D_m \subseteq \text{dom}(q)$ . Let  $\vec{a}_i =_{\text{def}} q(D_m)$ . Then  $\mathcal{A}_A \models \psi_i(\vec{c}, \vec{a}_i)$ .

Now assume that for some  $\vec{a}_i \in A^{lh(\vec{x}_i)}$ , we have  $\mathcal{A}_A \models \psi_i(\vec{c}, \vec{a}_i)$ . Then for some  $q \supseteq p$ , there are  $\vec{b}_i, n, m$  such that  $q(\vec{b}_i) = \vec{a}_i$ ,  $D_m = \vec{b}_i$ , and  $q \Vdash \text{“}\varphi_e^{\mathcal{B}}(n) = m\text{”}$ . Then there is  $q' \supseteq q$  such that for some  $j \in D_m$ , we have  $q' \Vdash \neg \mathbf{R}(b_j)$ . Hence  $a_j \in \overline{R}$ .  $\square$

### 3. RELATIVELY $h$ -SIMPLE RELATIONS

The following definition of an  $h$ -simple relation on natural numbers is standard.

**Definition 3.1.** *Let  $S \subseteq \omega$ . The relation  $S$  is  $h$ -simple (on  $\omega$ ) if  $S$  is c.e. and  $\neg S$  is  $h$ -immune.*

We can similarly define  $h$ -simple relations on any computable set. The set of all  $h$ -simple sets is a proper subset of the set of all *simple* sets, that is, the c.e. sets whose complements are *immune*. Dekker [4] showed that the deficiency set of a computable 1-1 enumeration of a noncomputable c.e. set is  $h$ -simple. Hence every nonzero c.e. Turing degree contains an  $h$ -simple set.

In [8], Hird introduced a syntactic analogue of  $h$ -simplicity for relations on computable structures, which we will term formal  $h$ -simplicity. Since we are considering c.e. relations, computable  $\Sigma_1$  formulae with positive occurrences of  $\mathbf{R}$  in the expanded language  $L \cup \{\mathbf{R}\}$  play an important role.

**Definition 3.2.** (i) *An infinitary  $\mathbf{R}^+$ -formal strong array on  $(\mathcal{A}, R)$  is a computable sequence of computable  $\Sigma_1$  formulae in  $L \cup \{\mathbf{R}\}$  with only positive occurrences of  $\mathbf{R}$  and with finitely many parameters  $\vec{c}$ ,*

$(\psi_i(\vec{c}, \vec{x}_i))_{i \in \omega}$ , such that for every finite set  $G \subseteq A$  there is  $i \in \omega$  and a sequence  $\vec{a}_i \in A^{lh(\vec{x}_i)}$  such that

$$[(\mathcal{A}_A, R) \models \psi_i(\vec{c}, \vec{a}_i)] \wedge [\vec{a}_i \cap G = \emptyset].$$

If computable  $\Sigma_1$  formulas are just finitary existential ones, then we have an  $\mathbf{R}^+$ -formal strong array.

(ii) We say that the relation  $R$  is formally  $h$ -simple on  $\mathcal{A}$  if  $R$  is c.e. and there is no  $\mathbf{R}^+$ -formal strong array  $(\psi_i(\vec{c}, \vec{x}_i))_{i \in \omega}$  on  $(\mathcal{A}, R)$  such that for every  $i \in \omega$ ,

$$(\forall \vec{a}_i \in A^{lh(\vec{x}_i)}) [((\mathcal{A}_A, R) \models \psi_i(\vec{c}, \vec{a}_i)) \Rightarrow (\vec{a}_i \cap \overline{R} = \emptyset)].$$

Hird [8] established that, under a suitable decidability condition,  $R$  is formally  $h$ -simple on  $\mathcal{A}$  iff there is a computable copy  $\mathcal{B}$  of  $\mathcal{A}$  such that the image of  $R$  under an isomorphism from  $\mathcal{A}$  onto  $\mathcal{B}$  is  $h$ -simple on  $\mathcal{B}$ . Harizanov [6] gave sufficient general conditions for the existence of an  $h$ -simple relation on a computable copy of  $\mathcal{A}$ , in arbitrary nonzero c.e. Turing degree.

Next, we introduce a relative version of  $h$ -simplicity.

**Definition 3.3.** Let  $S$  be an additional (unary) relation on the domain  $B$  of a countable structure  $\mathcal{B}$ . The relation  $S$  is  $h$ -simple relative to  $\mathcal{B}$  if  $S$  is c.e. relative to  $\mathcal{B}$  and  $\neg S$  is  $h$ -immune relative to  $\mathcal{B}$ .

In the next theorem we give a relative analogue of Hird's result for  $h$ -simple relations on computable copies.

**Theorem 3.1.** Let  $\mathcal{A}$  be an infinite computable structure in a relational language  $L$ , and let  $R$  be a computable unary infinite and co-infinite relation on  $A$ . Then the following are equivalent:

- (i) For all copies  $\mathcal{B}$  of  $\mathcal{A}$  and all isomorphisms  $F$  from  $\mathcal{A}$  onto  $\mathcal{B}$ ,  $F(R)$  is not  $h$ -simple relative to  $\mathcal{B}$ .
- (ii) The relation  $R$  is not formally  $h$ -simple on  $\mathcal{A}$ .

*Proof.* We can prove that (ii)  $\Rightarrow$  (i) as in Theorem 2.1. We now prove by contrapositive that (i)  $\Rightarrow$  (ii). Assume that  $R$  is formally  $h$ -simple on  $\mathcal{A}$ . If  $R$  is definable in  $\mathcal{A}$  by a computable  $\Sigma_1$  formula with finitely many parameters, then in any copy  $\mathcal{B}$  of  $\mathcal{A}$ , the image of  $R$  is c.e. relative to  $\mathcal{B}$ . Hence, if  $\mathcal{B}$  is a copy in which the image of  $\neg R$  is relatively  $h$ -immune, then the image of  $R$  is relatively  $h$ -simple.

Assume that  $R$  is not definable by a computable  $\Sigma_1$  formula with finitely many parameters. Then in a generic copy  $\mathcal{B}$  of  $\mathcal{A}$ , the image of  $R$  is not c.e. relative to  $\mathcal{B}$  (as established in [2] and [3]). Therefore, as in [5], we introduce an expanded language,  $L^* = L \cup \{\mathbf{R}'\} \cup \{\mathbf{Q}\}$ , where  $\mathbf{R}'$  is a new unary relation symbol and  $\mathbf{Q}$  is a new binary relation

symbol. We extend the  $L$ -structure  $\mathcal{A}$  into an  $L^*$ -structure  $\mathcal{A}^*$ . The structure  $\mathcal{A}^*$  is obtained by extending the universe  $A$  by another infinite computable set  $R'$ , and expanding  $\mathcal{A}$  to include the unary relation  $R'$  and a binary relation  $Q$ , which is a 1-1 mapping from  $R'$  onto  $R$ . In  $\mathcal{A}^*$ , the open  $L^*$ -formula  $\neg \mathbf{R}'(x)$  defines  $A$ , and the existential  $L^*$ -formula  $\exists y \mathbf{Q}(y, x)$  defines  $R$ . The following lemma follows from a proof in [5].

**Lemma 3.2.** *Let  $D \subseteq A$ . If the set  $D$  is definable in  $\mathcal{A}^*$  by a computable  $\Sigma_1$  formula  $\psi^*(\vec{c}^*, x)$  of  $L^*$ , then it is definable in  $(\mathcal{A}, R)$  by some computable  $\Sigma_1$  formula  $\psi(\vec{c}, x)$  in  $L \cup \{\mathbf{R}\}$ , with finitely many parameters  $\vec{c}$  and only positive occurrences of  $\mathbf{R}$ . Moreover, we can obtain  $\psi(\vec{c}, x)$  effectively from  $\psi^*(\vec{c}^*, x)$ , uniformly in the Gödel code of  $\psi^*$ .*

Hence, it follows that the relation  $\neg(R \cup R')$  is not formally  $h$ -immune on  $\mathcal{A}^*$ . If we apply Theorem 2.1 to the relation  $R \cup R'$  on structure  $\mathcal{A}^*$ , we get a copy  $\mathcal{B}^*$  of  $\mathcal{A}^*$  and an isomorphism  $F$  from  $\mathcal{A}^*$  onto  $\mathcal{B}^*$ , with  $\mathcal{B}$  corresponding to  $\mathcal{A}$  under  $F$ , such that the following holds:

- (a) the structure  $\mathcal{B}$  is computable in structure  $\mathcal{B}^*$ ;
- (b) the relation  $F(R)$  is c.e. relative to  $\mathcal{B}^*$ ;
- (c) the relation  $(B^* - F(R \cup R'))$  is  $h$ -immune relative to  $\mathcal{B}^*$ .

Thus, by (a), any function computable relative to  $\mathcal{B}$  is computable relative to  $\mathcal{B}^*$ . Clearly,  $B - F(R) = B^* - F(R \cup R')$ . Consequently,  $B - F(R)$  is  $h$ -immune relative to  $\mathcal{B}$ . However, we still cannot conclude that  $F(R)$  is  $h$ -simple relative to  $\mathcal{B}$ , because  $F(R)$  is not necessarily c.e. relative to  $\mathcal{B}$ .

Now, using Knight's construction in [10], we can prove, as in [5], that either there is an automorphism  $H$  of  $\mathcal{A}$  for which  $H(R)$  is  $h$ -simple, or there is an isomorphism  $H$  from  $\mathcal{B}$  onto a structure  $\mathcal{C}$  such that  $H(F(R))$  is  $h$ -simple relative to  $\mathcal{C}$ . Hence, the structure  $\mathcal{C}$  is an isomorphic copy of  $\mathcal{A}$ , on which the corresponding image of  $R$  is relatively  $h$ -simple.  $\square$

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SIBERIAN BRANCH OF THE RUSSIAN ACADEMY OF SCIENCES, SOBOLEV INSTITUTE OF MATHEMATICS, 630090 NOVOSIBIRSK, RUSSIA, GONCHAR@MATH.NSC.RU

DEPARTMENT OF MATHEMATICS, THE GEORGE WASHINGTON UNIVERSITY, WASHINGTON, D.C. 20052, USA, HARIZANV@GWU.EDU

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, NOTRE DAME, IN 46556, USA, KNIGHT.1@ND.EDU

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, MADISON, WI 53706, USA, MCCOY@MATH.WISC.EDU