Spectra of High$_n$ and Nonlow$_n$ Degrees

Andrey Frolov
N.G. Chebotarev Research Inst. of Mechanics and Mathematics
Kazan State University

Valentina Harizanov
George Washington University

Iskander Kalimullin
N.G. Chebotarev Research Inst. of Mechanics and Mathematics
Kazan State University

Oleg Kudinov
Sobolev Institute of Mathematics
Russian Academy of Sciences, Siberian Branch

Russell Miller
Queens College & C.U.N.Y. Graduate Center

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Correspondence:
Mathematics Department
Queens College – C.U.N.Y.
65-30 Kissena Blvd.
Flushing, NY 11367 U.S.A.
+001 (1) 718-997-5853
Russell.Miller@qc.cuny.edu

Abstract

We survey known results on spectra of structures and on spectra of relations on computable structures, asking when the set of all high_n degrees can be such a spectrum, and likewise for the set of nonlow_n degrees. We then repeat these questions specifically for linear orders and for relations on the computable dense linear order $\mathbb{Q}$. New results include realizations of the set of nonlow_n Turing degrees as the spectrum of a relation on $\mathbb{Q}$ for all $n \geq 1$, and a realization of the set of all nonlow_n Turing degrees as the spectrum of a linear order whenever $n \geq 2$. The state of current knowledge is summarized in a table in the concluding section.

Keywords: *computability, computable model theory, spectrum, relation, linear order.*

1 Introduction

Spectra of structures and spectra of relations are both natural and well-established topics of study in computable model theory. Logicians have come to a solid understanding of these notions over the last thirty years, yet many questions remain open, some of them unsolved more than a decade after being posed, despite heroic efforts.

The principal focus of this paper is on two specific kinds of possible spectra: spectra which consist of precisely the high_n degrees, and spectra which consist of precisely the nonlow_n degrees. These are both standard degree classes in computability theory, and we will survey these sets of degrees with respect to two familiar theories: the theory of graphs, and the theory of linear orders. We will recall known results and prove some new ones, to give as complete a picture as presently possible of the feasibility of realizing these degree classes as spectra of models of these theories.

At its base, the study of spectra examines the interaction between effectiveness properties of structures and the classical notion of isomorphism between two structures. For spectra of relations, we wish to measure the extent to which a classical isomorphism can map a simple subset of the structure to a complicated one, or vice versa. For spectra of structures, we ask a similar question about the image of the entire structure under the isomorphism. We now recall these concepts for the reader.

The Turing degree of a countable structure $\mathcal{M}$ with domain ω is the Turing degree of its atomic diagram. If the language is finite, this is the join of the degrees of the different functions $f^{\mathcal{M}}$ and relations $R^{\mathcal{M}}$, where f and R range over all function and relation symbols in the language of $\mathcal{M}$. (We will assume in this paper that the language is finite, unless otherwise stated.) By definition, the *spectrum* of (the isomorphism type of) $\mathcal{M}$ is the set of all Turing degrees of isomorphic copies of $\mathcal{M}$:

$$\text{Spec}(\mathcal{M}) = \{\deg_T(\mathcal{N}) : \mathcal{N} \cong \mathcal{M}\}.$$

(By convention, $\mathcal{N}$ must also have domain ω; it is not fair to make $\mathcal{N}$ complicated just by choosing its domain to be complicated. We wish to measure complexity of the functions and relations in $\mathcal{N}$, without interference from a complex way of naming the elements of $\mathcal{N}$.)

Intuitively, Spec($\mathcal{M}$) measures the intrinsic difficulty of computing a copy of $\mathcal{M}$: each degree $\boldsymbol{d}$ in Spec($\mathcal{M}$) is smart enough to build a structure isomorphic to $\mathcal{M}$. Conversely, for $\boldsymbol{d}$ to lie in Spec($\mathcal{M}$), $\mathcal{M}$ must be complicated enough to allow some way of coding $\boldsymbol{d}$ into a copy of $\mathcal{M}$. As seen in Theorem 1.1 below, the requirement of being "smart enough" is usually the difficult one when we ask whether $\boldsymbol{d}$ lies in Spec($\mathcal{M}$); coding is possible in all but certain trivial cases.

On the other hand, the *degree spectrum* of a relation R on a computable structure $\mathcal{A}$ is defined as:

$$\text{DgSp}_{\mathcal{A}}(R) = \{\deg_T(S) : (\exists \mathcal{B} \leq_T \emptyset)(\mathcal{B}, S) \cong (\mathcal{A}, R)\}.$$

The symbol R generally is not in the language of the structure $\mathcal{A}$; indeed, if it were, then $\text{DgSp}_{\mathcal{A}}(R)$ would contain only $\mathbf{0}$.

Again, the intuition we wish to capture by defining the degree spectrum of R is the question of how complicated we can make the relation R. Of course, if the definition allowed $\mathcal{B}$ to be *any* isomorphic copy of $\mathcal{A}$, then we would have much more freedom to increase the complexity of the image S of R (even if $\mathcal{B}$ has domain ω). Restricting the definition to computable structures $\mathcal{B}$ is our way of ruling out such tricks: for a degree $\boldsymbol{d}$ to lie in $\text{DgSp}_{\mathcal{A}}(R)$, we must be able to make the image of R have degree $\boldsymbol{d}$ while keeping the underlying structure computable.

A Turing degree $\boldsymbol{d}$ is *low* if its jump has the least possible complexity: $\boldsymbol{d}' = \mathbf{0}'$, the jump of the degree $\mathbf{0}$ of the computable sets. Likewise, it is *low*$_n$, for any $n \in \omega$, if its n-th jump $\boldsymbol{d}^{(n)}$ has the least possible complexity:

$\boldsymbol{d}^{(n)} = \mathbf{0}^{(n)}$. (We include here the degenerate case $n = 0$: the 0-th jump of a degree is just the degree itself, so $\mathbf{0}$ is the unique low_0 degree.) The concept was originally intended for the Δ^0_2 degrees, for which the jump of greatest possible complexity is the degree $\mathbf{0}''$. More generally, therefore, a degree $\boldsymbol{d}$ is defined to be *high* if $\mathbf{0}'' \leq \boldsymbol{d}'$, and *high*$_n$ if $\mathbf{0}^{(n+1)} \leq \boldsymbol{d}^{(n)}$. Again the case $n = 0$ will be considered: the high_0 degrees are just the degrees $\boldsymbol{d} \geq \mathbf{0}'$.

A valuable result for spectra of structures is the following theorem, proven in 1986 by Julia Knight.

Theorem 1.1 (Theorem 4.1 in [20]) *For all automorphically nontrivial stuctures $\mathcal{S}$, the spectrum of $\mathcal{S}$ is upwards closed under Turing reducibility: if $\boldsymbol{d} \in Spec(\mathcal{S})$ and $\boldsymbol{d} \leq \boldsymbol{c}$, then $\boldsymbol{c} \in Spec(\mathcal{S})$ as well.* ∎

$\mathcal{S}$ is *automorphically trivial* if there is a finite subset of $\mathcal{S}$ such that every permutation of ω which fixes that subset pointwise is an automorphism of $\mathcal{S}$. Finite structures satisfy this definition, of course, as does the complete graph on ω-many vertices, or an almost-complete such graph (missing only finitely many edges). For these structures, the spectrum is always a singleton, and if the language is finite, then the one degree in the spectrum must be $\mathbf{0}$. Automorphically trivial structures are of little interest to us, therefore; for all structures we consider, the spectrum will be upwards closed. For relations on computable structures, it is quite possible for the spectrum not to be upwards closed; indeed this holds of any definable relation (even of relations definable by infinitary formulas). However, it was shown by Harizanov and Miller in [12, Theorem 2.10 & Prop. 3.6] that for relations on the computable dense linear order, the spectrum either is upwards-closed or else is just the singleton $\{\mathbf{0}\}$, and likewise for relations on the computable random graph. Since our focus will be on relations on these two structures, we will again only be interested in upwards-closed sets of degrees as spectra of relations.

It follows that the class of low_n degrees is of no direct interest to us, for any n, since it is not closed upwards. Its complement, on the other hand, the set of *nonlow*$_n$ degrees, is upwards-closed and will be of intense interest. Our other focus will be the high_n degrees, for each n, as this class is also upwards-closed. (Some definitions of high_n require the degree in question to be Δ^0_2. For the sake of upwards-closure, we make no such restriction.)

Each of these classes is a natural candidate to be a spectrum. It is our intention to give a comprehensive picture of the current state of knowledge about the existence of spectra of high_n degrees and spectra of nonlow_n degrees: for structures in general, for linear orders in particular, for relations on computable structures in general, and for relations on the computable dense linear order in particular. We will therefore cite many known results, sketching proofs when possible. We also have new results to offer on this topic, however. Principal among these are the following.

Theorem 5.17 *For every $n \geq 1$, there exist a structure $\mathcal{S}$, and a relation R on the computable dense linear order $\mathbb{Q}$, such that Spec($\mathcal{S}$) and $DgSp_{\mathbb{Q}}(R)$ each contains precisely the high$_n$ degrees (i.e. those Turing degrees $\boldsymbol{d}$ with $\boldsymbol{d}^{(n)} \geq \mathbf{0}^{(n+1)}$).*

Theorem 4.6 *For every $n \geq 2$, there exists a linear order $\mathcal{L}_n$ such that Spec($\mathcal{L}_n$) contains exactly the nonlow$_n$ degrees, and there exists a set R_n which has this same spectrum when viewed as a relation on the computable dense linear order.*

Theorem 4.8 *There exists a relation R_1 on the computable dense linear order $\mathbb{Q}$, such that $DgSp_{\mathbb{Q}}(R_1)$ contains exactly the nonlow degrees.*

We will also generalize each of these results, replacing the target degrees $\mathbf{0}^{(n)}$ with arbitrary degrees $\boldsymbol{c}$. Additionally, we prove in Subsection 5.2 that there exists a set of Turing degrees which is not the spectrum of any linear order, but is the spectrum of a relation on the computable dense linear order $\mathbb{Q}$. This answers a question from [12].

Our computability-theoretic notation is standard; we recommend [28] as a source. We fix a single computable presentation $\mathcal{G}$ of the random graph, to be used throughout this paper, and likewise a single computable presentation $\mathbb{Q}$ of the countable dense linear order without end points. Since both of these structures are computably categorical, and since all properties of interest to us are preserved by computable isomorphism, it makes no difference which specific computable presentation of either we choose. For details about the random graph, we recommend [15]. Both of these structures are discussed in depth in [12] from the point of view of computable model theory.

2 Spectra of Structures

When we ask our questions about structures in general, the answers are mainly provided by the paper [11], by Goncharov, Harizanov, Knight, McCoy, Miller, and Solomon. Indeed, they stated the result for spectra of nonlow$_n$

degrees specifically, and extended it to nonlow$_\alpha$ degrees, for all successor ordinals $\alpha < \omega_1^{CK}$, using the iteration of the jump operator to define the transfinite jump $C^{(\alpha)}$ of a set $C \subseteq \omega$ for such α. Their result is as follows.

Theorem 2.1 (Theorem 6.4 in [11]) *For each computable successor ordinal α, there is a structure with copies in just the Turing degrees of sets D such that $\Delta^0_\alpha(D)$ is not Δ^0_α. In particular, for each finite n, there is a structure with copies in just the nonlow$_n$ degrees.* ■

Lemma 5.5 of [11] describes a uniform procedure which takes any graph $\mathcal{G}$ and any computable successor ordinal α, and produces a corresponding structure $\mathcal{G}^*$ such that, for all $D \subseteq \omega$, $\mathcal{G}$ has a $\Delta^0_\alpha(D)$ copy iff $\mathcal{G}^*$ has a D-computable copy. (This $\mathcal{G}^*$ has other useful properties as well, relating to computability of isomorphisms, but we do not need those for our purposes in this paper.) Theorem 2.1 then follows promptly by relativizing a theorem proven independently by Slaman and Wehner.

Theorem 2.2 (See [27], [30]) *There exists a countable structure $\mathcal{S}$ whose spectrum contains every Turing degree except* $\mathbf{0}$*. More generally, for every set C, there exists a structure $\mathcal{S}_C$ with*

$$Spec(\mathcal{S}_C) = \{\boldsymbol{d} : \boldsymbol{d} > deg_T(C)\}.$$

■

By [14, Theorem 1.22] (discussed below), we may take $\mathcal{S}_{\emptyset^{(n)}}$ to be a graph, and then the procedure cited above, with $\alpha = n+1$, produces the structure $\mathcal{S}^*_{\emptyset^{(n)}}$ required by Theorem 2.1, whose spectrum is the set of nonlow$_n$ Turing degrees.

Despite having six authors, the paper [11] failed to note the simpler application of this procedure which proves the analogous result for high$_n$ degrees. We give it here.

Corollary 2.3 (of Lemma 5.5 in [11]) *For every $n \in \omega$, there exists a structure $\mathcal{H}_n$ with*

$$Spec(\mathcal{H}_n) = \{\boldsymbol{d} : \mathbf{0}^{(n+1)} \leq \boldsymbol{d}^{(n)}\}.$$

Proof. Richter proved in [24] that for every Turing degree $\boldsymbol{c}$, there is a structure (indeed a graph) whose spectrum contains exactly the degrees in the upper cone above $\boldsymbol{c}$, including $\boldsymbol{c}$ itself. So let $\mathcal{G}_n$ be a structure such that $\text{Spec}(\mathcal{G}_n) = \{\boldsymbol{d} : \mathbf{0}^{(n+1)} \leq \boldsymbol{d}\}$, the upper cone above $\mathbf{0}^{(n+1)}$. Then the procedure from Lemma 5.5 of [11], with $\alpha = n+1$, produces a structure $\mathcal{H}_n = \mathcal{G}_n^*$ which has copies in precisely those degrees $\boldsymbol{d}$ such that $\boldsymbol{d}^{(n)}$ computes a copy of $\mathcal{G}_n$, i.e. such that $\mathbf{0}^{(n+1)} \leq \boldsymbol{d}^{(n)}$. So $\mathcal{H}_n$ is the structure we need. ■

So, where Theorem 2.1 used the Slaman-Wehner structure relatived to $\emptyset^{(n)}$ and the procedure from [11], Corollary 2.3 applies this procedure to any structure with an upper cone of degrees as its spectrum. We note the following generalizations, which are immediate.

Proposition 2.4 *For every Turing degree* $\boldsymbol{c}$ *and every* $n \in \omega$*, there exist countable structures* $\mathcal{H}_{\boldsymbol{c},n}$ *and* $\mathcal{L}_{\boldsymbol{c},n}$ *with*

$$Spec(\mathcal{H}_{\boldsymbol{c},n}) = \{\boldsymbol{d} : \boldsymbol{c} \leq \boldsymbol{d}^{(n)}\}, \qquad Spec(\mathcal{L}_{\boldsymbol{c},n}) = \{\boldsymbol{d} : \boldsymbol{c} < \boldsymbol{d}^{(n)}\}.$$

■

3 Spectra of Relations

Turning to relations on computable structures, we now wish to prove analogous results about spectra of high$_n$ degrees and spectra of nonlow$_n$ degrees, The notion of a *spectrally universal* structure, as defined in [12], will make this a simple task.

Definition 3.1 A computable model $\mathcal{S}$ of a theory T is *spectrally universal* for T if for every countable nontrivial model $\mathcal{M}$ of T, there exists an embedding $g : \mathcal{M} \to \mathcal{S}$ such that

$$\text{DgSp}_{\mathcal{S}}(g(\mathcal{M})) = \text{Spec}(\mathcal{M}).$$

The authors of [12] consider there the computable random graph $\mathcal{G}$. Since this graph is computably categorical, it makes no difference which computable copy of it one chooses. We will write $\mathcal{G}$ for a fixed (but arbitrary) computable presentation of the random graph.

Theorem 3.2 (Theorem 3.2 of [12]) *The computable random graph* $\mathcal{G}$ *is spectrally universal for the theory of graphs.*

Since trivial graphs have spectrum $\{\mathbf{0}\}$, we immediately get:

Corollary 3.3 *Let $\mathcal{B}$ be any countable graph. Then there exists a unary relation R on an arbitrary computable presentation $\mathcal{G}$ of the random graph, such that*

$$DgSp_{\mathcal{G}}(R) = Spec(\mathcal{B}).$$

■

In concert with results by Hirschfeldt, Khoussainov, Shore, and Slinko in [14], this yields a far stronger theorem.

Theorem 3.4 ([14], Theorem 1.22) *For each nontrivial countable structure $\mathcal{S}$ (in any computable language, finite or infinite), there exists a symmetric irreflexive graph with the same spectrum as $\mathcal{S}$.*

If the language is finite, this holds for trivial structures as well. Moreover, in [14] the authors prove the same result for directed graphs, partial orders, lattices, rings, integral domains of arbitrary characteristic, commutative semigroups, and two-step nilpotent groups.

The results on graphs from [12] can be summarized as follows.

Theorem 3.5 (Theorem 3.10 of [12]) *Let $\mathcal{D}$ be any collection of Turing degrees. The following are equivalent:*

(1) $\mathcal{D}$ is the spectrum of some countable structure in some finite language.

(2) $\mathcal{D}$ is the spectrum of some countable graph.

(3) $\mathcal{D}$ is the degree spectrum of some unary relation R on the computable random graph $\mathcal{G}$.

(4) $(\forall n \geq 1)$ [$\mathcal{D}$ is the degree spectrum of some n-ary relation on $\mathcal{G}$].

(5) $(\exists n \geq 1)$ [$\mathcal{D}$ is the degree spectrum of some n-ary relation on $\mathcal{G}$]. ■

This allows us to transfer the results about spectra of structures from Proposition 2.4 to spectra of relations on $\mathcal{G}$.

Corollary 3.6 *For every Turing degree $\mathbf{c}$ and every $n \in \omega$, there exist unary relations $H_{\mathbf{c},n}$ and $L_{\mathbf{c},n}$ on the computable random graph $\mathcal{G}$ such that*

$$DgSp_{\mathcal{G}}(H_{\mathbf{c},n}) = \{\boldsymbol{d} : \boldsymbol{c} \leq \boldsymbol{d}^{(n)}\}, \qquad DgSp_{\mathcal{G}}(L_{\mathbf{c},n}) = \{\boldsymbol{d} : \boldsymbol{c} < \boldsymbol{d}^{(n)}\}.$$

■

With this result, we have answered all the questions originally posed for structures in general. The remainder of this paper is devoted to addressing the same questions for the specific case of linear orders. As we shall see below, this has proven to be a significantly more challenging question. However, we do have most of the same tools involving spectral universality. Just as the computable random graph is spectrally universal for graphs, Harizanov and Miller showed that the computable dense linear order $\mathbb{Q}$ is spectrally universal for linear orders. (This linear order is computably categorical. Therefore, as with the random graph $\mathcal{G}$ above, we fix an arbitrary computable presentation $\mathbb{Q}$ of the countable dense linear order without end points.)

Theorem 3.7 (Theorem 2.1 of [12]) *The structure $\mathbb{Q}$ is spectrally universal for the theory of linear orders.*

Indeed, here the restriction (in Definition 3.1) to trivial structures is unnecessary. The only trivial linear orders are the finite ones, and they all have spectrum $\{\mathbf{0}\}$, both as structures and as relations on $\mathbb{Q}$ under any embedding.

Corollary 3.8 *Let $\mathcal{A}$ be any countable linear order. Then there exists a unary relation R on $\mathbb{Q}$ such that*

$$DgSp_{\mathbb{Q}}(R) = Spec(\mathcal{A}).$$

■

Not all of Theorem 3.5 carries over readily to linear orders. The authors of [12] asked whether there is a converse to Theorem 3.7: is every spectrum of a unary relation on $\mathbb{Q}$ also the spectrum of a linear order? For $\mathcal{G}$ and the class of countable graphs, this holds, by Theorem 3.5, but in [12] some reasons are given why this should be less likely for linear orders. In Section 5 below, we provide a negative answer to this question.

4 Nonlow$_n$ Degrees and Spectra of Linear Orders

For consideration of nonlow$_n$ degrees for $n \geq 1$, the notion of the *shuffle sum* will be essential.

Definition 4.1 Let $\mathcal{L}_0, \mathcal{L}_1, \ldots$ be (finitely or countably many) linear order types. The *shuffle sum* of these orders is $\mathcal{L} = \sum_{q \in \mathbb{Q}} \mathcal{L}_{f(q)}$, where $f : \mathbb{Q} \to \omega$ is any function such that, for all $q_1, q_2 \in \mathbb{Q}$, and $k \in \omega$, $q_1 <_{\mathbb{Q}} q_2$ implies $k = f(q)$ for some $q \in \mathbb{Q}$ with $q_1 <_{\mathbb{Q}} q <_{\mathbb{Q}} q_2$.

The idea is that we take densely many pairwise-nonoverlapping copies of these orders $\mathcal{L}_i$: in between any copy of $\mathcal{L}_i$ and any copy of $\mathcal{L}_j$ in $\mathcal{L}$, there will be a copy of every $\mathcal{L}_k$. The same holds to the right and to the left of each copy of $\mathcal{L}_i$, so that no copy of any $\mathcal{L}_i$ can serve as an "end point." The order type of $\mathcal{L}$ is uniquely defined and independent of the choice of function f satisfying the condition in the definition. It also does not depend on the ordering of the $\mathcal{L}_i$'s in the given sequence, or on the number of repetitions of any $\mathcal{L}_i$ in this sequence.

4.1 Nonlow$_n$ Degrees, for $n \geq 2$

We begin by considering the classes of nonlow$_n$ degrees with $n \geq 2$. The following theorem of Downey and Knight from [7] will be key. Essentially it says that if you can build a linear order with spectrum $\mathfrak{S}$, then you can also build one whose spectrum contains all those degrees whose jumps land in $\mathfrak{S}$. Here η represents the isomorphism type of the usual order on the rational numbers.

Theorem 4.2 (Lemma 1.2 of [7]) *A linear order $\mathcal{L}$ has an X'-computable copy iff $(\eta+2+\eta)\cdot\mathcal{L}$ has an X-computable copy. Moreover, both constructions are uniform in X.*

Theorem 4.3 (Ash, Knight [3]) *For all sets S, the following are equivalent:*

1. $S \leq_T X'$;
2. *there exists a uniformly X-computable sequence of linear orders $\{\mathcal{D}_n\}_{n\in\omega}$ such that*
   (a) $\mathcal{D}_n \cong \omega$, *if* $n \in S$,
   (b) $\mathcal{D}_n \cong \omega^*$, *if* $n \notin S$.

Corollary 4.4 *For all families $\mathcal{F} = \{S_k\}_{k\in\omega}$, the following are equivalent:*

1. $\mathcal{F} \leq_T X''$;
2. there exists a uniformly X'-computable sequence of linear orders $\{\mathcal{D}_{n,k}\}_{n,k\in\omega}$ such that
   (a) $\mathcal{D}_{n,k} \cong \omega$, if $n \in S_k$, and
   (b) $\mathcal{D}_{n,k} \cong \omega^*$, if $n \notin S_k$;
3. there exists a uniformly X-computable sequence of linear orders $\{\mathcal{C}_{n,k}\}_{n,k\in\omega}$ such that
   (a) $\mathcal{C}_{n,k} \cong (\eta + 2 + \eta) \cdot \omega$, if $n \in S_k$, and
   (b) $\mathcal{C}_{n,k} \cong (\eta + 2 + \eta) \cdot \omega^*$, if $n \notin S_k$.

Now, let $\mathcal{F}$ be an X''-computable family. By Corollary 4.4, there is an X-computable sequence of linear orders $\{\mathcal{C}_{n,k}\}_{n,k\in\omega}$ such that the conditions (3a) and (3b) hold.

Define $\mathcal{L}_k(\mathcal{F}) = 4+\eta+3+\mathcal{C}_{0,k}+3+\mathcal{C}_{1,k}+3+\mathcal{C}_{2,k}+3+\cdots+4$. Let $\mathcal{L}(\mathcal{F})$ be the shuffle sum of $\mathcal{L}_k(\mathcal{F})$, as in Definition 4.1. So $\mathcal{L}(\mathcal{F})$ is an X-computable linear order, with $\mathcal{F}_1 = \mathcal{F}_2$ iff $\mathcal{L}(\mathcal{F}_1) \cong \mathcal{L}(\mathcal{F}_2)$.

Let $\mathcal{L}$ be an X-computable linear order such that $\mathcal{L} \cong \mathcal{L}(\mathcal{F})$. Define:

$$A_n = \{(x_1, x_2, \ldots, x_n) \mid (\forall 1 \leq i < n)\ \mathrm{Succ}(x_i, x_{i+1})\} \leq_T X'$$
$$B_n = \{(\bar{z}_1, \bar{z}_2) \in (A_n)^2 \mid \bar{z}_1 <_{\mathcal{L}} \bar{z}_2\ \&\ (\forall \bar{z} \in A_n)\neg(\bar{z}_1 <_{\mathcal{L}} \bar{z} <_{\mathcal{L}} \bar{z}_2)\} \leq_T X''.$$

Here $\bar{z}'_1 <_{\mathcal{L}} \bar{z}'_2$ means $x_n <_{\mathcal{L}} y_1$, where $\bar{z}'_1 = (x_1, \ldots, x_n)$ and $\bar{z}'_2 = (y_1, \ldots, y_n)$.

Each $(\bar{z}, \bar{z}') \in B_4$ allows us to build (X''-uniformly) some set S of $\mathcal{F}$. To do so, find $\bar{z} <_{\mathcal{L}} \bar{z}_0 <_{\mathcal{L}} \bar{z}_1 <_{\mathcal{L}} \cdots <_L \bar{z}_n <_{\mathcal{L}} \bar{z}_{n+1} <_{\mathcal{L}} \bar{z}'$ such that $(\bar{z}_i, \bar{z}_{i+1}) \in B_3$ for any i with $1 \leq i \leq n$, and the interval $[\bar{z}, \bar{z}_0] = \{t \mid x_n <_{\mathcal{L}} t <_{\mathcal{L}} y_1\}$ is dense, where $\bar{z}'_1 = (x_1, \ldots, x_n)$ and $\bar{z}'_2 = (y_1, \ldots, y_n)$. Now, if there is a pair $(x, y) \in A_2$ such that the interval $[\bar{z}_n, x]$ is dense, then set $n \in S$; while if there is a pair $(x, y) \in A_2$ such that the interval $[y, \bar{z}_{n+1}]$ is dense, then set $n \notin S$. (Exactly one of these two possibilities will hold.) Therefore, $\mathcal{F} \leq X''$.

So, $\mathcal{L}(\mathcal{F})$ has an X-computable copy iff $\mathcal{F}$ is X''-computable. For an arbitrary Turing degree $\boldsymbol{c}$, we now apply Wehner's result from [30, p. 2136], which was the key to his proof of Theorem 2.2.

Lemma 4.5 (Wehner) *For every Turing degree $\boldsymbol{c}$, there exists a family $\mathcal{F} = \{F_0, F_1, \ldots\}$ of finite sets such that for any Turing degree $\boldsymbol{d}$, $\mathcal{F}$ has an enumeration computable in $\boldsymbol{d}$ iff $\boldsymbol{d} > \boldsymbol{c}$.*

Now it is easy to see that $\mathrm{Spec}(\mathcal{L}(\mathcal{F})) = \{\boldsymbol{d} \mid \boldsymbol{d}'' > \boldsymbol{c}\}$. If $\boldsymbol{c} = \mathbf{0}''$, then $\mathrm{Spec}(\mathcal{L}(\mathcal{F}))$ contains exactly the nonlow$_2$ degrees. Furthermore, for $m \geq 0$, $\mathrm{Spec}((\eta + 2 + \eta)^m \cdot \mathcal{L}(\mathcal{F})) = \{\boldsymbol{d} \mid \boldsymbol{d}^{(m+2)} > \boldsymbol{c}\}$ (see for instance Frolov's work in [8]). In other words, the following theorem is true.

Theorem 4.6 *For every $n \geq 2$ and every Turing degree $\boldsymbol{c}$, there exists a linear order with spectrum $\{\boldsymbol{d} : \boldsymbol{d}^{(n)} > \boldsymbol{c}\}$. In particular, for each $n \geq 2$, there is a linear order whose spectrum contains exactly the nonlow$_n$ degrees.*

Applying Theorem 3.7 transfers this result to relations on $\mathbb{Q}$.

Corollary 4.7 *For every $n \geq 2$ and every Turing degree $\boldsymbol{c}$, there exists a unary relation on the computable dense linear order $\mathbb{Q}$ with spectrum $\{\boldsymbol{d} : \boldsymbol{d}^{(n)} > \boldsymbol{c}\}$.* ■

We remark that for any family $\mathcal{F}$, the linear order $\mathcal{L}(\mathcal{F})$ is strongly η-like. Note that there does not exist a strongly η-like linear order whose spectrum contain exactly all nonzero (i.e., all nonlow$_0$) degrees, because any low strongly η-like linear order has a computable copy, as shown by Frolov in [8].

4.2 Nonlow$_1$ Degrees

For nonlow degrees, we will be able to prove the analogue of Corollary 4.7, but not the analogue of Theorem 4.6. That is, our proof will work for relations on $\mathbb{Q}$, but not for arbitrary linear orders. It remains open whether the set of nonlow degrees can be the spectrum of a linear order.

Officially the domain of $\mathbb{Q}$ is ω. We let $<$ denote the usual less-than relation on ω, while $\prec$ will be the computable relation ordering ω densely without end points. Below, all open or closed intervals refer to the $\prec$ relation. References to the "least element" of an interval (a, b) denote the element x such that $x \leq y$ for all y with $a \prec y \prec b$. We will often trust the reader to interpret the explanation properly.

We fix a subset $P \subset \mathbb{Q}$ which is computable and dense in $\mathbb{Q}$ and also has complement dense in $\mathbb{Q}$. Intuitively, P might be the set of dyadic rationals, i.e. those with denominator a power of 2. However, no matter what computable dense order $\mathbb{Q}$ we chose, we can always build such a set P: just enumerate one element from each interval (a, b) into P and another into its complement.

Theorem 4.8 *For every Turing degree* $\boldsymbol{c}$*, there exists a unary relation* $\tilde{R}$ *on* $\mathbb{Q}$ *with:*

$$DgSp_{\mathbb{Q}}(\tilde{R}) = \{\boldsymbol{d} : \boldsymbol{d}' > \boldsymbol{c}\}.$$

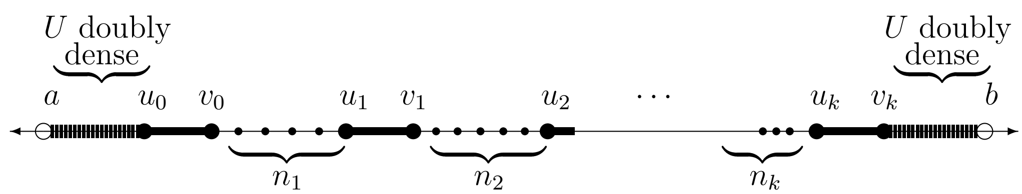
Corollary 4.9 *There exists a unary relation on* $\mathbb{Q}$ *whose degree spectrum contains precisely the nonlow Turing degrees.* ∎

Proof of Theorem 4.8. For every finite subset $F = \{n_1, \ldots, n_k\} \subset \omega$, with any order on its elements, and for any elements $a \prec b$ in $\mathbb{Q}$, we define the subset $U = U(a, b, n_1, \ldots, n_k) \subset (a, b)$ as follows.

Find the $(2k + 2)$ least elements (under $<$) of (a, b), and name them so that

$$a \prec u_0 \prec v_0 \prec u_1 \prec v_1 \prec \cdots \prec u_k \prec v_k \prec b.$$

Enumerate into U all of each closed interval $[u_i, v_i]$ with $i \leq k$. Next, for each $i < k$, enumerate into U the n_{i+1} least elements of the open interval (v_i, u_{i+1}). Finally, enumerate into U all of $P \cap (a, u_0)$ and $P \cap (v_k, b)$ (where P is the dense co-dense subset defined above). This defines $U(a, b, n_1, \ldots, n_k) = U$ as a subset of $\mathbb{Q}$, computable uniformly in a, b, k, and $n_1, \ldots, n_k$. It is useful to have a picture of this U within (a, b):



where the term "doubly dense" means that both U and its complement are dense in this interval of $\mathbb{Q}$. (In this picture $n_1 = 4$ and $n_2 = 5$.) Below we will often refer to the n_i*-subinterval*, meaning the interval (v_{i-1}, u_i) containing exactly n_i elements of R. Notice that this U depends not just on F, but even on the order $n_1, \ldots, n_k$ in which the finitely many elements of F are given. If F is empty, then $U(a, b, F)$ contains a single interval $[u_0, v_0]$, and is doubly dense in (a, u_0) and in (v_0, b).

Next we partition $\mathbb{Q}$ computably into infinitely many closed intervals $I_0, I_1, \ldots$ with rational end points. We ensure that for any $i \neq j$ and any k, there is an m such that the interval $I_{\langle k,m\rangle}$ lies between the intervals I_i and I_j. All this can easily be done in a computable fashion, with the end points $a_i \prec b_i$ of each I_i being computable uniformly in i.

We now use again Lemma 4.5 of Wehner, relativized to an arbitrary degree $\boldsymbol{c}$, to produce a family $\tilde{\mathcal{F}} = \{\tilde{F}_0, \tilde{F}_1, \ldots\}$ of finite sets such that for any Turing degree $\boldsymbol{d}$, $\tilde{\mathcal{F}}$ has an enumeration computable in $\boldsymbol{d}$ iff $\boldsymbol{d} > \boldsymbol{c}$. We may assign an ordering to the elements of each $\tilde{F}_i$, and assume without loss of generality that every possible ordering appears in $\tilde{\mathcal{F}}$. (Just replace each $\tilde{F}_i$ by $|\tilde{F}_i|!$-many copies of itself, one under each ordering.) We then define the relation $\tilde{R}$ on $\mathbb{Q}$ using this family $\tilde{\mathcal{F}}$: on each interval $I_{\langle i,j\rangle} = (a_{\langle i,j\rangle}, b_{\langle i,j\rangle})$ of $\mathbb{Q}$, we define $\tilde{R}$ to contain precisely the points of $U(a_{\langle i,j\rangle}, b_{\langle i,j\rangle}, \tilde{F}_i)$, using the given order on that $\tilde{F}_i$. The set of such intervals is dense, so $\tilde{R}$ may be viewed as the shuffle-sum of intervals of the form $U(a, b, \tilde{F}_i)$, for all $\tilde{F}_i \in \tilde{\mathcal{F}}$ and with all possible orderings of each $\tilde{F}_i$ included in the shuffle.

Now suppose that $(\mathbb{Q}, \tilde{R}) \cong (\mathbb{Q}, R)$. (Since $\mathbb{Q}$ is computably categorical, we need not consider other computable copies of the ground model $\mathbb{Q}$.) We claim that with an R' oracle, we can enumerate the family $\tilde{\mathcal{F}}$. To perform this enumeration, we start by considering all pairs $\langle x, y\rangle$ of points in $\mathbb{Q}$. If $x \prec y$ and our R'-oracle says that the interval $[x, y] \subset R$, then we begin enumerating a set $F = F_{\langle x,y\rangle}$ as follows.

Step 1. First we search for points $x' \prec v' \prec u$ in $\mathbb{Q}$ and an $n \in \omega$ such that $u \preceq x$ and $[u, x] \subset R$ and $[x', v'] \subset R$ and there are no more than n points of R in the interval (v', u). With our R'-oracle we will eventually find such points, if they exist. We then use the R'-oracle again to determine exactly how many points of R lie in (v', u), and enumerate this number into F. Then we repeat Step 1 with x' and v' in place of x and y.

Step 2. Simultaneously (for the same pair of points x and y), we search for points $v \prec u' \prec y'$ in $\mathbb{Q}$ and an $n \in \omega$ such that $y \preceq v$ and $[y, v] \subset R$ and $[u', y'] \subset R$ and there are no more than n points of R in the interval (v, u'). With our R'-oracle we will eventually find such points, if they exist. We then use the R'-oracle again to determine exactly how many points of R lie in (v, u'), and enumerate this number into F. Then we repeat Step 2 with u' and y' in place of x and y. (Notice that the repetition instruction applies each time we run Step 2, so Step 2 will never end; nor will Step 1.)

These two steps, running simultaneously, enumerate the set $F = F_{\langle x,y\rangle}$, and by doing so for each pair $\langle x, y\rangle$ as described, we enumerate a family $\mathcal{F}$ of sets, uniformly, from our R'-oracle. We claim that $\mathcal{F} = \tilde{\mathcal{F}}$. First, the only nontrivial closed intervals of $\mathbb{Q}$ contained in $\tilde{R}$ are those contained in an interval $[u_i, v_i]$ from the construction, corresponding to some $\tilde{F}_{i'} \in \tilde{\mathcal{F}}$, and so the same is true of R. Thus the interval $[x, y]$ must lie in one such interval, and our R'-computable algorithm then finds the end points corresponding to those u_i and v_i, locates the next interval $[u_{i-1}, v_{i-1}]$ or $[u_{i+1}, v_{i+1}]$ next to it, counts the number of points of R between those intervals, and enumerates that number into $F_{\langle x,y\rangle}$. Thus $\tilde{F}_{i'} \subseteq F_{\langle x,y\rangle}$. Moreover, when our process reaches the interval corresponding to $[u_0, v_0]$ or $[u_k, v_k]$, then the doubly-dense interval which follows ensures that we will never again find any points to make us enumerate any more numbers into $F_{\langle x,y\rangle}$. Hence $F_{\langle x,y\rangle} \subseteq \tilde{F}_{i'}$, and so the two sets are equal. Thus every set $F_{\langle x,y\rangle}$ which we enumerate lies in $\tilde{\mathcal{F}}$. Conversely, for every i', we will start enumerating $\tilde{F}_{i'}$ as soon as we find elements $x \prec y$ in $\mathbb{Q}$ from the interval corresponding to the interval $[u_0, v_0]$ of $I_{\langle i',0\rangle}$, so the family $\mathcal{F}$ we enumerate is precisely $\tilde{\mathcal{F}}$. Therefore, from an R'-oracle, we can enumerate $\tilde{\mathcal{F}}$, and Lemma 4.5 shows that $\deg_T(R') > \boldsymbol{c}$.

For the converse, fix some $C \in \boldsymbol{c}$ and let D be an arbitrary set such that $D' >_T C$. By Lemma 4.5, there is a Turing functional Φ such that $\Phi^{D'}(i)$ enumerates the i-th set G_i from some listing of the sets of $\tilde{\mathcal{F}}$. Formally, $G_i = \{j : \Phi^{D'}(\langle i,j\rangle)\downarrow = 1\}$, with $\Phi^{D'}$ total. We will use a D-oracle to build a relation R on $\mathbb{Q}$ such that $(\mathbb{Q}, \tilde{R}) \cong (\mathbb{Q}, R)$. Of course, the key is to use $\Phi^{D'}$, so we need to approximate D', via a D-computable enumeration $\langle D'_s\rangle_{s\in\omega}$. Write $G_{i,s}$ for the set $\{j : [\Phi_s^{D'_s}(\langle i,j\rangle)\downarrow = 1]\}$ of elements enumerated within s steps by $\Phi^{D'_s}$. Notice that $\langle G_{i,s}\rangle_{s\in\omega}$ is not an enumeration of G_i, since $G_{i,s} - G_{i,s+1}$ could be nonempty if D'_{s+1} changes below the use of $\Phi^{D'_s}(\langle i,j\rangle)$; it is only a D-computable approximation to G_i. All $G_{i,0}$ are empty, and we may assume that exactly one element enters or leaves exactly one set G_i at each stage $s+1$.

We build $R \leq_T D$ such that $(\mathbb{Q}, R) \cong (\mathbb{Q}, \tilde{R})$. By the upward closure of $\mathrm{DgSp}_{\mathbb{Q}}(\tilde{R})$ (see [12, Theorem 2.10]), this will complete the proof. We continue to use the partition of $\mathbb{Q}$ into open intervals $I_0, I_1, \ldots$ defined above, with each $I_q = (a_q, b_q)$.

At stage $t+1 = 2s+1$, we have already defined finitely many intervals I_r in $\mathbb{Q}$, which we call the *existing intervals*, each with its own *blueprint* $B(r,t)$. Say that $s = \langle i,j\rangle$, and consider each of the $|G_{i,s}|!$-many possible orderings of $G_{i,s}$. For each possible ordering, and in between each of the existing intervals, we choose a new interval I_q (with q as small as possible) and define a blueprint $B(q,t+1) = U(a_q, b_q, G_{i,s})$ for I_q under that ordering, which represents our current intention for the elements of I_q: those in $B(q,t+1)$ are intended to enter R eventually, while the rest of the elements of I_q are intended to enter $\overline{R}$. (Choosing all these new intervals ensures that every G_i under every ordering appears densely, of course, so that we build a shuffle sum.) We also put new intervals, one for each possible ordering, to the right of the rightmost existing interval and to the left of the leftmost one, with their own blueprints, in exactly the same way. All of the new intervals are designated as G_i-intervals, and will retain this designation throughout the construction. We also determine the $<$-least element z in the union of the existing intervals, say $z \in I_r$. We enumerate z into R (if $z \in B(r,t)$) or into $\overline{R}$ (if not).

At stage $t+1 = 2s+2$, let G_i be the unique set such that $G_{i,s+1} \neq G_{i,s}$, and let $G_{i,s} = \{n_1 < n_2 < \cdots < n_k\}$. We allow $k = 0$ if $G_{i,s} = \emptyset$. The *existing G_i-intervals* are all those G_i-intervals which were defined by stage $2s$, and which therefore need to be adjusted to reflect the change in G_i at stage $s+1$. (Any G_i-intervals defined at stage $2s+1$ have this change already built into them.) Each existing G_i-interval I_q has a *blueprint* $B(q,t)$, i.e. a subset of I_q containing those elements of $\mathbb{Q}$ which are currently planned to enter R. Only finitely many elements have already been enumerated into R or $\overline{R}$. The blueprint $B(q,t)$ is a subset of I_q isomorphic to $U(a_q, b_q, G_{i,s})$, with some order on the elements of $G_{i,s}$ (but not necessarily using the $<$-least elements of I_q as the end points of its n_j-subintervals, as $U(a_q, b_q, G_{i,s})$ does).

If a new element x entered $G_{i,s+1}$, then for each existing G_i-interval I_q, we change its blueprint so that $B(q,t+1)$ is isomorphic to $U(a_q, b_q, G_{i,s}, x)$, with x coming at the right end of the existing order of I_q on $G_{i,s}$. Specifically, we put a new x-subinterval in the new blueprint $B(q,t+1)$, to the right of the n_k-subinterval, making sure that any element of I_q already in R lies in $B(q,t+1)$, and that no element of I_q already in $\overline{R}$ lies in $B(q,t+1)$. Since only finitely many elements of I_q have already been enumerated into R or $\overline{R}$, this can easily be done.

Otherwise, some element $y \in G_{i,s}$ is not in $G_{i,s+1}$. Say that the blueprint $B(q,t)$ was of the form $U(a_q, b_q, n_1, \ldots, n_j, y, n_{j+2}, \ldots, n_k)$, and that v_j was the left end point of the y-subinterval, and that u is the $\prec$-least element to the right of v_j which is already enumerated into R or $\overline{R}$. (If there is no such u in (v_j, b_q), let $u = b_q$. It will be useful to refer to the diagram of $U(a, b, n_1, \ldots, n_k)$ on page 13.) We let $B(q,t+1) \cap (a_q, v_j] = B(q,t) \cap (a_q, v_j]$. In the interval (v_j, u), we define a new n_{j+2}-subinterval, then a new n_{j+3}-subinterval, and so on up to the new n_k-subinterval, followed by an interval entirely in $B(q,t+1)$ (choosing the right end point $\prec u$), and then an interval up to b_q in which $B(q,t+1)$ is doubly dense. Again we ensure that elements already in R lie in $B(q,t+1)$, and that those already in $\overline{R}$ do not. This fully defines the new blueprint $B(q,t+1)$, which now is isomorphic to $U(a_q, b_q, G_{i,s+1})$ under the restriction of the ordering from $G_{i,s}$ to $G_{i,t+1}$, completing stage $2s+2$.

To see that this construction succeeds, consider the evolution of a G_i-interval I_q. Let $n_1, \ldots, n_k$ be the elements of G_i. There exists a stage s_0 with all $n_i \in G_{i,s}$ for every $s \geq s_0$, and then there exists another stage $s_1 > s_0$ such that for each $x \in G_{i,s_0} - G_i$, there exists a stage t with $s_0 < t < s_1$ for which $x \notin G_{i,t}$. So the leftmost k subintervals in $B(q, s_1)$ will be an n_1-subinterval, an n_2-subinterval, and so on, in some order. (Let us assume without loss of generality that the subintervals come in the order given here.) Let u_k be the right end point of the n_k-subinterval, and choose the rightmost v_k so that $[u_k, v_k] \subseteq B(q, s)$. (Thus v_k is the left end point of the n_{k+1}-subinterval, if there is an n_{k+1} in $G_{i,s}$, or else $B(q, s)$ is doubly dense in (v_k, b_q).) Then at all stages $s \geq s_1$, we will have $B(q, s) \cap (a_q, v_k] = B(q, s_1) \cap (a_q, v_k]$, since no n_i with $i \leq k$ ever again leaves G_i. Moreover, if there are cofinitely many s with $G_{i,s} = G_i$, then clearly R will be doubly dense in (v_k, b_q); while if there are not, then at infinitely many stages $s > s_1$, the element immediately to the right of n_k in the ordering on $G_{i,s}$ is removed from $G_{i,s+1}$, (since such an element cannot lie in G_i), and at each of these infinitely many stages, the new blueprint moves the left end point of the doubly dense interval to an element to the left of the leftmost element $\succ v_k$ which is already in R or $\overline{R}$. Clearly, then, R is doubly dense in (v_k, b_q), and so this interval I_q satisfies

$$(I_q, \prec, R) \cong (I_q, \prec, U(a_q, b_q, n_1, \ldots, n_k)).$$

Moreover, for every G_i and every order on its elements, after the stage s_0 described above, new G_i-intervals with that order on the elements of G_i are added to every gap among the existing intervals at every stage $2\langle i, j\rangle + 1$ of the construction. By the preceding argument, each such G_i-interval I_q turns out to be isomorphic to $(I_q, \prec, U(a_q, b_q, G_i))$ with the elements of G_i never changing their order. So $(\mathbb{Q}, \prec, R) \cong (\mathbb{Q}, \prec, \tilde{R})$, by a back-and-forth argument on intervals: every I_q on either side can be mapped onto an I_r in the appropriate gap on the other side, such that I_q and I_r are both G_i-intervals for the same i and with the same order on the elements of G_i. As argued above, this completes the proof of Theorem 4.8. ■

This result is not as strong as Theorem 4.6, because the latter established that actual linear orders could have spectra of nonlow$_n$ degrees, for $n > 1$, whereas here we have only shown that a relation on $\mathbb{Q}$ can have spectrum of nonlow$_1$ degrees. (Spectral universality of $\mathbb{Q}$ allows the spectrum of any linear order to be made into the spectrum of a relation on $\mathbb{Q}$, of course, as in Corollary 4.7.) It would be of interest to examine the spectrum of the linear order given by restricting $\prec$ from $\mathbb{Q}$ to $\tilde{R}$. The doubly dense intervals now become dense, of course, and it seems unlikely that any proof similar to this one could show that the nonlow degrees form the spectrum of this linear order, since we no longer have any notion of double density. When dealing with relations on $\mathbb{Q}$, the structure $\mathbb{Q}$ itself provides a context which is absent when one works with linear orders as structures, and in Theorem 4.8 we exploited this context, using the doubly dense intervals. The following question remains open.

Question 4.10 *Does there exist a linear order whose spectrum (as a structure) contains just the nonlow degrees?*

If we add the successor relation to the language, then we can prove a positive answer to this question. By analogy to Theorems 4.2 and 4.3, notice that a linear order $\mathcal{L}$ has an X-computable copy iff the order $(\eta+2+\eta)\cdot\mathcal{L}$ has an X-computable copy with X-computable successor relation S_L. Moreover, for all families $\mathcal{F}$ of subsets of ω, $\mathcal{F} \leq_T X'$ iff there exists a uniformly X-computable sequence of linear orders $\{\mathcal{C}_{n,k}\}$ with uniformly X-computable successor relations $S_{\mathcal{C}_{n,k}}$. Using Wehner's Lemma 4.5 and the techniques of Subsection 4.1, one can then build a linear order $\mathcal{L}$ such that $\mathrm{Spec}(\mathcal{L}, S_{\mathcal{L}})$ is exactly the class of all nonlow degrees. (Likewise for each $n > 1$, there exists a linear order $\mathcal{L}_n$ such that $\mathrm{Spec}(\mathcal{L}_n, S_{\mathcal{L}_n})$ contains exactly the nonlow$_n$ degrees.)

4.3 Nonlow$_0$ Degrees

Recall that a nonlow$_0$ degree is just a nonzero degree: its 0-th jump $\boldsymbol{d}^{(0)} = \boldsymbol{d}$ satisfies $\boldsymbol{d}^{(0)} > \mathbf{0}^{(0)} = \mathbf{0}$. Theorem 2.2 above showed the existence of a structure whose spectrum contains precisely the nonlow$_0$ degrees, but it remains unknown whether there exists a linear order with this spectrum. The closest approach is described in [23].

Theorem 4.11 (Thm. 4.1 of [23], and Chisholm, Downey) *There exists a linear order $\mathcal{L}$ whose spectrum contains every noncomputable Δ^0_2 degree, but not the degree $\mathbf{0}$. Indeed, Spec($\mathcal{L}$) contains all hyperimmune degrees.*

Proof. In [23], the statement about the Δ^0_2 degrees is proven. That proof is easily generalized to show that $\mathrm{Spec}(\mathcal{L})$ contains every degree $\boldsymbol{d}$ for which there exists a degree $\boldsymbol{b}$ with $\boldsymbol{b} < \boldsymbol{d} \leq \boldsymbol{b}'$: one just repeats the construction in [23], using Δ_2-permitting on a B-computable approximation to D (where $B \in \boldsymbol{b}$ and $D \in \boldsymbol{d}$), still diagonalizing against the computable linear orders (not against the B-computable ones). Chisholm and Downey have independently generalized this further, using hyperimmune permitting, to show that every hyperimmune degree lies in Spec($\mathcal{L}$). (These are the degrees of hyperimmune sets, as defined in [28, V.2.1].) Every degree $\boldsymbol{d}$ which is Δ_2 in and above some $\boldsymbol{b}$, as described above, is hyperimmune, so this result subsumes the earlier one. ∎

The order $\mathcal{L}$ built in [23], and its precursor in [16], each also has the properties of being the prime model of its theory, and of being elementarily equivalent to a computable linear order. Thus the theory of $\mathcal{L}$ provides the second known answer to a question from [26, p. 454], Rosenstein's book *Linear Orderings*, which closed by asking whether there exists a complete extension of the theory of linear orders which has a prime model and a computable model, but no computable prime model. The first solution to this question, which used a substantially different construction, was given by Hirschfeldt in [13].

Barmpalias has suggested that the basic module for the construction of $\mathcal{L}$ in [23] cannot be carried out below a hyperimmune-free degree, leading to the conjecture that Spec($\mathcal{L}$) may contain precisely the hyperimmune degrees. However, it is possible that the entire construction of $\mathcal{L}$ can be performed by other means, and so this remains a conjecture. Moreover, even if this $\mathcal{L}$ does not have all noncomputable degrees in its spectrum, there could still be another linear order, not computably presentable, which does. On the other hand, Corollary 5.5 below makes it clear that the construction of such a linear order could not be generalized to build spectra equal to strict upper cones above arbitrary Turing degrees $\boldsymbol{c}$; possibly it could be generalized to build a spectrum containing such a strict upper cone but not intersecting the lower cone of degrees $\leq \boldsymbol{c}$. The existence of a linear order with spectrum $\{\boldsymbol{d} : \boldsymbol{d} > \mathbf{0}\}$ is a significant open question in computable model theory.

5 High$_n$ Degrees and Spectra of Linear Orders

5.1 High$_0$ Degrees

Recall that a high$_0$ degree is just a degree $\boldsymbol{d} = \boldsymbol{d}^{(0)}$ which computes $\mathbf{0}^{(0+1)} = \mathbf{0}'$. Thus, a linear order whose spectrum contained just the high$_0$ degrees would have a least degree in its spectrum, namely $\mathbf{0}'$. Richter showed this to be impossible.

Theorem 5.1 (Theorem 3.3 of [24]) *If $\mathcal{L}$ is a linear order and there is a least degree in Spec($\mathcal{L}$), then that degree is $\mathbf{0}$.* ■

Specifically, Richter showed that there must exist a presentation $\mathcal{M} \cong \mathcal{L}$ such that $\deg_T(\mathcal{L}) \wedge \deg_T(\mathcal{M}) = \mathbf{0}$. The theorem follows immediately.

In their comparison of spectra for structures with spectra of relations in [12], Harizanov and Miller repeated Richter's proof to show the same for spectra of unary relations on $\mathbb{Q}$.

Proposition 5.2 (Proposition 2.16 in [12]) *If R is a unary relation on $\mathbb{Q}$ such that the degree $\mathbf{0}$ does not lie in $DgSp_{\mathbb{Q}}(R)$, then $DgSp_{\mathbb{Q}}(R)$ does not contain a least degree. Indeed $DgSp_{\mathbb{Q}}(R)$ contains a minimal pair of degrees.* ■

The proof in [12] is a straightforward adaptation of Richter's proof in [24] to the context of relations on $\mathbb{Q}$. So the situation for the high$_0$ case for linear orders is clear, both for structures and relations, and also for the general case of degrees whose 0-th jump computes some fixed degree $\boldsymbol{c}$.

5.2 High$_1$ Degrees

In [12], Harizanov and Miller proved the following.

Theorem 5.3 (Harizanov & Miller) *For every Turing degree $\boldsymbol{c}$, there exists a unary relation R on $\mathbb{Q}$ whose degree spectrum is $\{\boldsymbol{d} : \boldsymbol{c} \leq \boldsymbol{d}'\}$.*

The full result appears as Proposition 2.18 and Corollary 2.19, along with subsequent remarks, in [12].

On the other hand, Knight showed in [20] that a version of Richter's theorem also holds when we consider jumps of degrees of linear orders. The *jump degree* of a structure $\mathcal{A}$ is the least degree in the set $\{\boldsymbol{d}' : \boldsymbol{d} \in \text{Spec}(\mathcal{A})\}$, if such a degree exists.

Theorem 5.4 (Knight, Corollary 3.6 in [20]) *No degree except* $\mathbf{0}'$ *can be the jump degree of a linear order.* ■

Therefore, there is no linear order whose spectrum contains exactly the high degrees, and more generally, for any degree $\boldsymbol{c} > \mathbf{0}'$, there is no linear order $\mathcal{L}$ with $\text{Spec}(\mathcal{L}) = \{\boldsymbol{d} : \boldsymbol{c} \leq \boldsymbol{d}'\}$.

These two facts together yield an answer to a question of Harizanov and Miller from [12, p. 347]. They asked whether the spectrum of an arbitrary unary relation on $\mathbb{Q}$ can always be realized as the spectrum of a linear order. The relation R above provides a negative answer to the question, for exactly the reasons detailed here. This solution to the question was discovered independently by Csima and Knoll, as well as by the authors of this paper.

Also, Knight's theorem on jump degrees eliminates most strict upper cones from being the spectrum of any linear order.

Corollary 5.5 *Suppose* $\boldsymbol{c}$ *is a nonlow Turing degree. Then* $\{\boldsymbol{d} : \boldsymbol{d} > \boldsymbol{c}\}$ *cannot be the spectrum of a linear order.*

Proof. If $\text{Spec}(\mathcal{L}) = \{\boldsymbol{d} : \boldsymbol{d} > \boldsymbol{c}\}$, then $\mathcal{L}$ has jump degree $\boldsymbol{c}'$, since some $\boldsymbol{d} > \boldsymbol{c}$ must have $\boldsymbol{d}' = \boldsymbol{c}'$. Theorem 5.4 then shows that $\boldsymbol{c}$ is low. ■

5.3 High$_n$ Degrees

For linear orders and the class of high$_n$ degrees with $n \geq 2$, we have results from the work in [1] by Ash, Jockusch, and Knight.

Theorem 5.6 (Lemma 2.3 in [1]) *For any* $C \subseteq \omega$, *let* $\mathcal{L}$ *be the shuffle sum of* ω *and all* $(c+1)$*-point finite linear orders with* $c \in (C \oplus \overline{C})$. *Then* $Spec(\mathcal{L}) = \{\boldsymbol{d} : deg_T(C) \leq \boldsymbol{d}''\}$. ■

The authors of [1] remarked that one can extend this result to get spectra of the form $\{\boldsymbol{d} : \boldsymbol{c} \leq \boldsymbol{d}^{(2n+2)}\}$, for all $n \in \omega$. The more recent technique described in Theorem 4.2 allows us to extend it also to odd jumps > 2.

Theorem 5.7 *For all degrees* $\boldsymbol{c}$ *and all* $n \geq 2$, *there exists a linear order with spectrum* $\{\boldsymbol{d} : \boldsymbol{c} \leq \boldsymbol{d}^{(n)}\}$.

Proof. Use Theorem 5.6 to get an order $\mathcal{L}$ with spectrum $\{\boldsymbol{d} : \boldsymbol{c} \leq \boldsymbol{d}''\}$. Then $(n-2)$ applications of Theorem 4.2 yield $\text{Spec}((\eta + 2 + \eta)^{(n-2)} \cdot \mathcal{L}) = \{\boldsymbol{d} : \boldsymbol{c} \leq (\boldsymbol{d}'')^{(n-2)}\}$, as desired. ∎

Corollary 5.8 *For all $n \geq 2$, there exists a linear order whose spectrum contains exactly the high$_n$ degrees.* ∎

Spectral universality of $\mathbb{Q}$ enables one to transfer this result immediately to relations on $\mathbb{Q}$. However, we give a separate construction, in which the coding of the set $C \in \boldsymbol{c}$ into the relation is far more obvious. First we have three preliminary definitions.

Definition 5.9 For any ordinal of the form ω^m with $m \in \omega$, the *standard presentation* of ω^m is given by the reverse lexicographic ordering on m-tuples of natural numbers. The *standard presentation* of the ordinal ω^ω has domain $\omega^{<\omega}$, i.e. all finite strings of natural numbers, ordered first by length and then with each subset ω^k in the reverse lexicographic order.

For instance, the reverse lexicographic order on ω^2 is given by:

$$\langle 0,0\rangle \prec \langle 1,0\rangle \prec \langle 2,0\rangle \prec \cdots \prec \langle 0,1\rangle \prec \langle 1,1\rangle \prec \cdots .$$

Using this order allows us to embed each ω^m as the initial segment of ω^{m+1} by extending each string $\sigma \in \omega^m$ to $\sigma\hat{\ }\langle 0\rangle \in \omega^{m+1}$.

Definition 5.10 In a linear order $(\mathcal{L}, \prec)$, every point is a limit point at level 0. A point $x \in \mathcal{L}$ is a limit point at level $k+1$ iff $(\forall y \prec x)$ [there is a limit point z at level k with $y \prec z \prec x$].

Notice that in the standard presentation of any ordinal ω^m or ω^ω, the property of being a limit point at level k is computable uniformly in k and m. In general, this property is uniformly Π^0_{2k}-definable. Also, by our definition, every limit point at any level $k+1$ must also be a limit point at level k.

Definition 5.11 For $m \in \omega$, and all computable formulas $\theta(x, z_1, \ldots, z_m)$, the formula

$$(\exists^\infty z_m) \cdots (\exists^\infty z_1)\ \theta(x, z_1, \ldots, z_m)$$

is *in Inf$_m$ form*, where $\exists^\infty z_i$ abbreviates $(\forall y_i \exists z_i > y_i)$. (The point is mainly that the variables y_i may *not* appear in θ.) A set is *Inf$_m$-definable* if it is definable by a formula in this form.

Every Inf_m formula is Π^0_{2m}, of course, but we need to know which Turing degrees contain Inf-definable sets. The answer is provided in [22, Lemma 2] by Kreisel, Shoenfield, and Wang.

Lemma 5.12 (Kreisel, Shoenfield, & Wang) *For all $m \in \omega$, every Π^0_{2m}-set R is Inf_m-definable.*

Proof. Proofs appear in [22] and [25], but we give one briefly here as well, proceeding by induction on m. The $m = 0$ case is trivial, but to illustrate our approach, we demonstrate the $m = 1$ case before beginning our induction. If $R = \{x : \forall y \exists z \varphi(x, y, z)\}$, then let $\psi(x, y, z)$ be the formula

$$\varphi(x, y, z) \ \& \ (\forall z' < z) \ \neg\varphi(x, y, z').$$

Then $\forall y \exists z \psi$ also defines R, and since to each x and y there corresponds at most one z such that $\psi(x, y, z)$, we see that

$$R = \{x : (\exists^\infty \sigma)(\forall t < \mathrm{lh}(\sigma))\psi(x, t, \sigma(t))\},$$

which is in Inf_1 form as required. (Here σ varies over $\omega^{<\omega}$, of course.)

For the inductive step, assume that $R = \{x : \forall y \exists z \varphi(x, y, z)\}$ where φ is any Π^0_{2m} formula. The procedure above may now fail, since ψ includes $\neg\varphi(x, y, z')$, which may not be Π^0_{2m} when $m > 0$. We prove first, therefore, that there exists a Σ^0_{2m+1} formula analogous to $\exists z \psi(x, y, z)$ above, with at most one witness z for each x and y. We may assume that φ itself is of the form $\forall w \alpha(x, y, z, w)$ with α in Σ^0_{2m-1}, since $m > 0$. So, for fixed x and y, we have:

$$\begin{aligned}
\exists z \varphi(x, y, z) &\iff \exists z[\varphi(x, y, z) \ \& \ (\forall z' < z) \ \neg\varphi(x, y, z')] \\
&\iff \exists z[\varphi(x, y, z) \ \& \ (\forall z' < z) \exists w \ \neg\alpha(x, y, z', w)] \\
&\iff \exists \tau \in \omega^{<\omega}[\varphi(x, y, \mathrm{lh}(\tau)) \ \& \ (\forall t < \mathrm{lh}(\tau)) \ \neg\alpha(x, y, t, \tau(t))] \\
&\qquad \text{(think of } z \text{ as } \mathrm{lh}(\tau)) \\
&\iff \exists \sigma \in \omega^{<\omega}[\varphi(x, y, \mathrm{lh}(\sigma)) \ \& \ (\forall t < \mathrm{lh}(\sigma))[\neg\alpha(x, y, t, \sigma(t)) \ \& \\
&\qquad (\forall u < \sigma(t))\alpha(x, y, t, u)]]
\end{aligned}$$

Define β so that the final formula on this list is $\exists \sigma \in \omega^{<\omega} \ \beta(x, y, \sigma)$. The point is that τ picks out witnesses w (for each $t < \mathrm{lh}(\tau)$) such that $\neg\alpha(x, y, t, w)$, thereby ensuring that $z = \mathrm{lh}(\tau)$ really is the least witness to $\exists z \varphi(x, y, z)$. However, for a given $\langle x, y, z \rangle$, there may be more than one such

τ, so we refine the formula to search for a σ (in ω^z) which picks out the least such witness w for each $t < z$. For each x and y, this σ must be unique (if it exists at all).

Now let

$$S = \{x : (\exists^\infty \rho)(\forall y < \mathrm{lh}(\rho))\beta(x, y, \rho(y))\}.$$

We claim that $R = S$. (Notice that here $\rho(y)$ is itself in $\omega^{<\omega}$, so in fact ρ lies in $\omega^{<(\omega^{<\omega})}$. Nevertheless, $\exists^\infty \rho$ is still a first-order quantifier.) If $x \in R$, then $\forall y \exists z \varphi(x, y, z)$, and so for every n we let $\rho \in (\omega^{<\omega})^n$ be defined (for $y < n$) by taking $\rho(y)$ to be the σ such that $\beta(x, y, \sigma)$ holds. Since we have such a ρ for every n, we have infinitely many such ρ, and so $x \in S$. Conversely, we remarked above that for any fixed x and y, there is at most one $\sigma \in \omega^{<\omega}$ such that $\beta(x, y, \sigma)$ holds, and that this σ has length equal to the least z such that $\varphi(x, y, z)$ holds. Therefore, for a fixed $x \in S$, any ρ satisfying the requirements of S is determined by $\mathrm{lh}(\rho)$, and indeed all such ρ are compatible as strings. So, for $x \in S$, the union of all corresponding ρ gives a total function $f : \omega \to \omega$ such that $\varphi(x, y, \mathrm{lh}(f(y)))$ holds for all y. Therefore $x \in R$.

The final step is to notice that the formula $(\forall y < \mathrm{lh}(\rho))\beta(x, y, \rho(y))$ is Π^0_{2m}, since $\beta(x, y, \sigma)$ is in the form $[\Pi^0_{2m}\ \&\ \neg\Sigma^0_{2m-1}\ \&\ \Sigma^0_{2m-1}]$, as written out above. (A bounded quantifier may always be pulled across an unbounded one, although this requires yet more uses of $\omega^{<\omega}$.) So, by inductive hypothesis, the formula $(\forall y < \mathrm{lh}(\rho))\beta(x, y, \rho(y))$ may be rewritten in Inf_m form. But $R = S = \{x : (\exists^\infty \rho)(\forall y < \mathrm{lh}(\rho))\beta(x, y, \rho(y))\}$, so R is definable by a formula in Inf_{m+1} form. ■

The preceding proof revealed a result about Σ^0_{2m+1} formulas as well. We first define the concept.

Definition 5.13 For any $m \in \omega$ and any formula $\theta(x)$ in Inf_m-form, the formula

$$\exists y \theta(\langle x, y \rangle)$$

is said to be *in* Single_m *form* if, for every x, there exists at most one y such that $\theta(\langle x, y \rangle)$ holds.

So every formula in Single_m form is Σ^0_{2m+1}, and the result from the proof of Lemma 5.12 establishes the converse.

Corollary 5.14 *For every $m \in \omega$, every Σ^0_{2m+1}-formula is equivalent to a formula in Single_m-form.* ■

Next we need a specific inductive process for building linear orders.

Lemma 5.15 *Fix $m > 0$, and let $R(n, x_1, \dots x_m)$ be any computable predicate. Let $\varphi(n)$ be the Inf$_m$ formula*

$$\exists^\infty z_1 \cdots \exists^\infty z_m \; R(n, \vec{z})$$

and let $A = \{n : \varphi(n)\}$ be the set defined by φ. Assume that there exists an algorithm, uniform in n and in R (that is, in a Δ_1-index for R, as in [28, Definition II.2.1]), which builds for each n a nonempty computable linear order $\mathcal{L}^n_m$ such that $\mathcal{L}^n_m \cong \omega^m$ if $n \in A$ and $\mathcal{L}^n_m \cong \omega^{m-1} + \cdots + \omega^{m-1}$ if $n \notin A$. (In the second case, $\mathcal{L}^n_m$ may contain any finite number of copies of ω^{m-1}, and the number of copies may be different for different elements $n \notin A$.) Finally, assume also that we have an algorithm for computing the left end point of each $\mathcal{L}^n_m$ uniformly in n and R.

Then there exists an algorithm, uniform in n and R, which builds for each n an order $\mathcal{L}^n_{m+1}$ such that

$$\mathcal{L}^n_{m+1} \cong \begin{cases} \omega^{m+1}, & \text{if } \exists^\infty z_{m+1} \varphi(\langle n, z_{m+1}\rangle); \\ \omega^m + \cdots + \omega^m, & \text{if not.} \end{cases}$$

Again, in $\mathcal{L}^n_{m+1}$, any finite number of copies of ω^m is allowed, for any n not satisfying the Inf$_{m+1}$ predicate. Moreover, we may compute the left end point of $\mathcal{L}^n_{m+1}$ uniformly in n and R.

Notice that the proof of this lemma relativizes easily to any oracle C, using a Δ^C_1-index for R and building a C-computable order $\mathcal{L}^n_{m+1}$.

Proof. To build $\mathcal{L}^n_{m+1}$, we define the set

$$S = \{n \in \omega : \exists^\infty z_{m+1} \varphi(\langle n, z_{m+1}\rangle)\}$$

and the sets $A_x = \{n \in \omega : \varphi(\langle n, x\rangle)\}$. The lemma gives us algorithms uniform in n which build computable orders $\mathcal{L}^{n,x}_m$ of order type ω^m or $\omega^{m-1} + \cdots + \omega^{m-1}$, depending on whether $n \in A_x$ or not. We adjoin these to get the computable order $\mathcal{L}^n_{m+1} = \mathcal{L}^{n,0}_m + \mathcal{L}^{n,1}_m + \cdots$, whose left end point is computable uniformly in n. Then we have

$$n \in S \;\text{ iff }\; \exists^\infty x (n \in A_x) \;\text{ iff }\; \exists^\infty x (\mathcal{L}^{n,x}_m \cong \omega^m) \;\text{ iff }\; \mathcal{L}^n_{m+1} \cong \omega^{m+1}$$

$$n \notin S \;\text{ iff }\; \exists y \forall x > y (n \notin A_x) \;\text{ iff }\; \mathcal{L}^n_{m+1} \cong \omega^m + \cdots + \omega^m$$

since in the latter case cofinitely many x have $\mathcal{L}^{n,x}_m \cong \omega^{m-1} + \cdots + \omega^{m-1}$. ∎

Lemma 5.16 *Let $\exists y\theta(\langle x,y\rangle)$ be a formula in Single$_m$-form defining a set B. Then there exists an algorithm building computable linear orders $\mathcal{K}^x_{m+1}$ uniformly in x such that*

$$\mathcal{K}^x_{m+1} \cong \begin{cases} \omega^m + \omega^m, & \text{if } \exists y\theta(\langle x,y\rangle); \\ \omega^m, & \text{if not.} \end{cases}$$

Moreover, the left end point of $\mathcal{K}^x_{m+1}$ is computable uniformly in x.

Proof. For each $y \in \omega$ we let

$$A_y = \{x \in \omega : \theta(\langle x,y\rangle)\}.$$

Thus A_y is defined in Inf$_m$ form, uniformly in y, and we can compute a Δ_1-index for the corresponding computable predicate, so by Lemma 5.15 we have algorithms, uniform in y, building orders $\mathcal{L}^{x,y}_m$ such that:

$$\mathcal{L}^{x,y}_m \cong \omega^m \text{ iff } x \in A_y \text{ iff } \theta(\langle x,y\rangle)$$

$$\mathcal{L}^{x,y}_m \cong \omega^{m-1} + \cdots + \omega^{m-1} \text{ iff } x \notin A_y \text{ iff } \neg\theta(\langle x,y\rangle).$$

Define $\mathcal{K}^x_{m+1}$ to be the computable order $L^{x,0}_m + \mathcal{L}^{x,1}_m + \cdots$. (Its left end point is the left end point of $\mathcal{L}^{x,0}_m$, which is computable in x by Lemma 5.15.)

Now for each $x \in B$, we have a unique corresponding witness y satisfying $\theta(\langle x,y\rangle)$, so $\mathcal{L}^{x,y}_m \cong \omega^m$ and for all $z \neq y$, $\mathcal{L}^{x,z}_m \cong \omega^{m-1} + \cdots + \omega^{m-1}$. Hence $\mathcal{K}^x_{m+1} \cong \omega^m + \omega^m$. On the other hand, for each $x \notin B$, $\mathcal{K}^x_{m+1} \cong \omega^m$. ■

Now we may extend Theorem 5.3 from the class of the high$_1$ degrees to the class of high$_n$ degrees relative to all possible degrees $\boldsymbol{c}$.

Theorem 5.17 *For every $n > 0$ and every Turing degree $\boldsymbol{c}$, there exists a relation R on the structure $\mathbb{Q}$ such that $DgSp_{\mathbb{Q}}(R)$ contains precisely those Turing degrees $\boldsymbol{d}$ such that $\boldsymbol{d}^{(n)} \geq \boldsymbol{c}$.*

Proof. We first consider the case $n = 2m$ (with $m > 0$). Given a set C in $\boldsymbol{c}$ with jump C', we define a unary relation R on $\mathbb{Q}$ as follows. Fix a computable $\prec$-increasing sequence $a_0 \prec b_0 \prec a_1 \prec b_1 \prec \cdots$ in $\mathbb{Q}$ such that $\{a_j : j \in \omega\}$ is unbounded in $\mathbb{Q}$. (The prototype for this sequence is the set of natural numbers, $a_i = 2i$ and $b_i = 2i+1$, as a subset of the rationals.) Our relation R will contain every closed interval $[b_i, a_{i+1}]_\prec$ in $\mathbb{Q}$.

Next, for each j, we let $\mathcal{K}^j$ be the linear order ω^m if $j \notin C'$, and $\omega^m + \omega^m$ if $j \in C'$. Let g_j embed $\mathcal{K}^j$ into the interval $(a_j, b_j)_\prec$, so that Image(g_j) is unbounded above in $[a_j, b_j)_\prec$, and so that for every limit point $y \in \mathcal{K}^j$ at level 1, $g_j(y)$ is the least upper bound in $\mathbb{Q}$ of the image of the set of its predecessors in $\mathcal{K}^j$. Let $R = \cup_j([a_j, b_j] \cup \text{Image}(g_j))$.

Now let $(\mathbb{Q}, \tilde{R}) \cong (\mathbb{Q}, R)$ via some automorphism f of $\mathbb{Q}$. (Since $\mathbb{Q}$ is computably categorical, we need not worry about other computable copies of $\mathbb{Q}$; see [12, Lemma 1.6].) Now $\tilde{R}''$ can compute the functions $j \mapsto \tilde{a}_j = f(a_j)$ and $j \mapsto \tilde{b}_j = f(b_j)$. Given b_j, search first for some $\tilde{a} \succ \tilde{b}_j$ and $\tilde{c} \succ \tilde{a}$ with $[\tilde{b}_j, \tilde{a}] \subset \tilde{R}$ and $(\tilde{a}, \tilde{c}) \cap \tilde{R} = \emptyset$. This $\tilde{a}$ must be $f(a_j)$. Then $\tilde{b}_{j+1}$ will be the unique $\tilde{b} \succ \tilde{a}$ such that the complement of $\tilde{R}$ is dense in $[\tilde{a}, \tilde{b}]_\prec$ but there exists $\tilde{d} \succ \tilde{b}$ with $[\tilde{b}, \tilde{d}] \subset \tilde{R}$. All this is definable with two quantifiers over $\tilde{R}$, so an $\tilde{R}''$-oracle can compute $\tilde{a}_0$ and $\tilde{b}_0$ (as finitely much information), then $\tilde{a}_1$, and so on.

We claim that the set C' is 1-reducible to $\tilde{R}^{(2m+1)}$ (and hence that $C \leq_T R^{(2m)}$, by [28, Theorem III.2.3(v)]). Indeed, $j \in C'$ iff $\mathcal{K}^j \cong \omega^m + \omega^m$, so by our construction above

$$C' = \{j : \exists x (x \text{ is a level-}m \text{ limit point of } (\tilde{R}, \prec) \ \& \ \tilde{a}_j \prec x \prec \tilde{b}_j\}.$$

Thus C' is a $\Sigma^{\tilde{R}}_{2m+1}$ set, hence $\leq_1 \tilde{R}^{(2m+1)}$, forcing $\tilde{R}^{(2m)} \geq_T C$. (Notice that we do require $m > 0$ in order for this to work, since computing $\tilde{a}_j$ and $\tilde{b}_j$ requires an $\tilde{R}''$-oracle.)

Conversely, suppose that $D^{(2m)} \geq_T C$. Then $C' \leq_1 D^{(2m+1)}$, via some computable function h. Applying Lemma 5.16 relativized to D, we see that D computes linear orders $\mathcal{K}^k$, uniformly in k, of type $\omega^m + \omega^m$ if $k \in D^{(2m+1)}$ and of type ω^m if $k \notin D^{(2m+1)}$. (Corollary 5.14, relativized to D, allows us to express $D^{(2m+1)}$ in Single$_m$-form relative to D, so that Lemma 5.16 may be applied.) Using the structure $\mathbb{Q}$ and the sequence $a_0 \prec b_0 \prec a_1 \prec \cdots$ already built, we may embed each order $\mathcal{K}^{h(j)}$ into the interval $[a_j, b_j)_\prec$ uniformly in j, with left end point a_j and with image having no upper bound in $[a_j, b_j)_\prec$, and moreover with the image being computable in D. (The image is clearly c.e. in D, and we may assume that at stage s in the construction of the embedding, no element $\leq s$ from $\mathbb{Q}$ is chosen to enter the image.) Moreover, we may ensure that if $x \in \mathcal{K}^j$ is a limit point at level 1, then its image is the least upper bound in $\mathbb{Q}$ of the image of the set of its predecessors. (This is essential for the isomorphism from $(\mathbb{Q}, R)$ to $(\mathbb{Q}, \tilde{R})$, since we built R using this same rule.) Let $\tilde{R}$ be the union of the images of all these embeddings with $\cup_j[b_j, a_{j+1}]_\prec$, so $\tilde{R} \leq_T D$.

Then $(\mathbb{Q}, R) \cong (\mathbb{Q}, \tilde{R})$ as follows. Every $[b_j, a_{j+1}]_\prec$ maps to itself via the identity map. Within $(a_j, b_j)_\prec$ every element of R maps to the corresponding element of $\tilde{R}$, since the restriction of $\prec$ to each is the same well-order. Each element $r \in R$ has an immediate successor $r' \in R$, and similarly in $\tilde{R}$, so the countable dense linear order $(r, r')_\prec$ maps onto the interval between the corresponding elements of $\tilde{R}$. We claim that this completes the construction of the isomorphism from $(\mathbb{Q}, R)$ onto $(\mathbb{Q}, \tilde{R})$. Every element $t \notin R$ has an "immediate R-successor" in R, since $(R \cap (a_j, b_j)_\prec, \prec)$ is a well-order, and t must also have an immediate R-predecessor, because our construction of R ensured that every increasing sequence of elements of R has a least upper bound in $\mathbb{Q}$, and that this least upper bound must be either b_j or an element of R. This guarantees that we have defined our map on all elements of $(\mathbb{Q}, R)$. Moreover, our construction of $\tilde{R}$ ensured the same, so every element of $(\mathbb{Q}, \tilde{R})$ lies in the image of our map, making it an isomorphism.

Since $\tilde{R} \leq_T D$, the upwards closure of $\mathrm{DgSp}_{\mathbb{Q}}(R)$ (see [12, Theorem 2.10]) shows that $\deg_T(D) \in \mathrm{DgSp}_{\mathbb{Q}}(R)$. This proves the result for the high$_{2m}$ case.

For the $n = 2m + 1$ case, we need a different strategy on the same structure $\mathbb{Q}$. We may assume that $m > 0$, referring to Theorem 5.3 for the case $n = 1$. Again, let C be a set in the degree $\boldsymbol{c}$, with jump C'. We again define a computable sequence $a_0 \prec b_0 \prec a_1 \prec b_1 \prec \cdots$ with no upper bound in $\mathbb{Q}$. Each interval $[b_j, a_{j+1}]_\prec$ is defined to lie in R. In each interval $(a_j, b_j)_\prec$, we define R to contain a subset R_j with greatest lower bound a_j and least upper bound b_j such that if $j \in C'$, then $(R_j, \prec) \cong \omega^m \cdot \omega^*$, and if $j \notin C'$, then $(R_j, \prec) \cong \omega^m \cdot \zeta$. (Recall that ζ is the order type of the integers $\mathbb{Z}$, often written as $\omega^* + \omega$, with ω^* denoting the order type of the negative integers.) In doing so, we ensure that for every limit point y of $(R_j, \prec)$ at level 1, the image of y is the least upper bound in $\mathbb{Q}$ of the set of its predecessors in R_j. We claim that $\mathrm{DgSp}_{\mathbb{Q}}(R) = \{\boldsymbol{d} : \boldsymbol{c} \leq \boldsymbol{d}^{(2m+1)}\}$.

Suppose first that $(\mathbb{Q}, R) \cong (\mathbb{Q}, \tilde{R})$, via an isomorphism f. Again, by [12, Lemma 1.6], we need not consider other copies of $\mathbb{Q}$. An $\tilde{R}''$-oracle allows us to compute the images $\tilde{a}_0 \prec \tilde{b}_0 \prec \tilde{a}_1 \prec \cdots$ of the a_j and b_j under f. Given $\tilde{a}_j$, search for $\tilde{d} \succ \tilde{b} \succ \tilde{a}_j$ such that $[\tilde{b}, \tilde{d}]_\prec \subset \tilde{R}$ and the complement of $\tilde{R}$ is dense in $(\tilde{a}_j, \tilde{b})_\prec$, and then $\tilde{b}_j$ must be this $\tilde{b}$. Then, given $\tilde{b}_j$, search for $\tilde{d} \succ \tilde{a} \succ \tilde{b}_j$ with $[\tilde{b}_j, \tilde{a}]_\prec \subset \tilde{R}$ and with the complement of $\tilde{R}$ dense in $(\tilde{a}, \tilde{d})_\prec$, and $\tilde{a}_{j+1}$ must be this $\tilde{a}$.

We need an arbitrary point in the interval $(\tilde{a}_j, \tilde{b}_j)_\prec$, so fix $\tilde{c}_j$ to be the $<$-least point in this interval. Then for any j, we know that $j \in C'$ iff

$(\tilde{R}_j, \prec) \cong \omega^m \cdot \omega^*$ iff the set of limit points of $\tilde{R}$ at level m in the interval $(\tilde{c}_j, \tilde{b}_j)_\prec$ is a finite set. Let g be a computable function defined so that $\Phi_{g(j)}^{\tilde{R}^{(2m)}}$ is the oracle Turing program which converges on exactly the level-m limit points y of the order $(\tilde{R}, \prec)$ such that $\tilde{c}_j \prec y \prec \tilde{b}_j$. This can be done uniformly in an $\tilde{R}^{(2m)}$-oracle (which can determine each $\tilde{a}_j$ and $\tilde{b}_j$, since $m > 0$, and then determine each $\tilde{c}_j$). Then g gives a computable 1-reduction from C' to $\text{Fin}^{\tilde{R}^{(2m)}}$, so $C' \leq_1 \tilde{R}^{(2m+2)}$, and by [28, Theorem III.2.3(v)] again, we have $C \leq_T \tilde{R}^{(2m+1)}$, as desired.

For the converse, suppose that $D^{(2m+1)} \geq_T C$, so that $C' \leq_1 D^{(2m+2)}$ via some computable function h. We use the original sequence $a_0 \prec b_0 \prec a_1 \prec \cdots$, and build a unary relation $\tilde{R} \leq_T D$ on $\mathbb{Q}$ as follows. Start by putting all intervals $[b_j, a_{j+1}]_\prec$ into $\tilde{R}$. Next, for each j, find the $<$-least element c_j in the interval $(a_j, b_j)_\prec$ and put a computable copy of $\omega^m \cdot \omega^*$ into the interval $(a_j, c_j)_\prec$, with a_j as its greatest lower bound and c_j as its least upper bound, and with each limit point at level 1 being the least upper bound of its predecessors in $\tilde{R}$. Then let $\mathcal{L}^j$ be the D-computable linear order $\mathcal{L}_{m+1}^{h(j)}$ given by Lemma 5.15, such that $\mathcal{L}^j \cong \omega^{m+1}$ if $h(j) \notin D^{(2m+2)}$ and $\mathcal{L}^j \cong \omega^m + \cdots + \omega^m$ if $h(j) \in D^{(2m+2)}$. (Here we use Lemma 5.12, relativized to D, to express the complement of $D^{(2m+2)}$ in Inf_{m+1}-form relative to D, so that Lemma 5.15 may be applied.) We can embed each $\mathcal{L}^j$ into the interval $[c_j, b_j)_\prec$, uniformly in j, with least upper bound b_j, and (using the same trick as in the $2m$-case) so that the image is also D-computable. Moreover, we may ensure again that if $x \in \mathcal{L}^j$ is a level-1 limit point, then its image is the least upper bound in $\mathbb{Q}$ of the image of the set of its predecessors in $\mathcal{L}^j$. Let the image of each $\mathcal{L}^j$ under this embedding also be enumerated into $\tilde{R}$. Notice that this still leaves $\tilde{R} \leq_T D$, so we need only show that $(\mathbb{Q}, R) \cong (\mathbb{Q}, \tilde{R})$. But for $j \in C'$ we have $\mathcal{L}^j \cong \omega^m + \cdots + \omega^m$, so

$$(\tilde{R} \cap (a_j, b_j), \prec) \cong \omega^m \cdot \omega^* + \mathcal{L}^j \cong \omega^m \cdot \omega^* \cong (R \cap (a_j, b_j), \prec)$$

and for $j \notin C'$

$$(\tilde{R} \cap (a_j, b_j), \prec) \cong \omega^m \cdot \omega^* + \omega^{m+1} \cong \omega^m \cdot (\omega^* + \omega) \cong (R \cap (a_j, b_j), \prec).$$

Moreover, $\tilde{R}$ has no upper or lower $\prec$-bound in any $(a_j, b_j)_\prec$, so indeed $(\mathbb{Q}, R) \cong (\mathbb{Q}, \tilde{R})$, by an argument much the same as in the $(2m)$-case. Thus $\deg_T(\tilde{R}) \in \text{DgSp}_{\mathbb{Q}}(R)$, and by Theorem 2.10 from [12], $\boldsymbol{d} \in \text{DgSp}_{\mathbb{Q}}(R)$ as well. ■

Corollary 5.18 *For every $n > 0$, there exists a relation R on the structure $\mathbb{Q}$, such that $DgSp_{\mathbb{Q}}(R)$ contains precisely those Turing degrees $\boldsymbol{d}$ which are high$_n$-or-above, namely those for which $\boldsymbol{d}^{(n)} \geq \boldsymbol{0}^{(n+1)}$.* ■

6 Conclusions and Questions

It seems appropriate to summarize the (new and old) results given in this article. Here we list possible spectra of graphs and linear orders, and of unary relations on the computable random graph $\mathcal{G}$ and the computable dense linear order $\mathbb{Q}$. The variable n ranges over all integers ≥ 2, and $\boldsymbol{c}$ is allowed to be an arbitrary Turing degree. "Y" indicates that all such sets can be realized as spectra; "N" indicates that no such spectrum can be realized; and "??" indicates an open question. Results in the first two columns come from Sections 2 and 3; for the last two columns, we refer to results in this article. For instance, the"Y(4.6)" on the seventh line in the column "LO's" means that for every $n \geq 2$ and every degree $\boldsymbol{c}$ with $\boldsymbol{c} \geq \boldsymbol{0}^{(n)}$, there exists a countable linear order with spectrum $\{\boldsymbol{d} : \boldsymbol{d}^{(n)} > \boldsymbol{c}\}$, and that this result appears as Theorem 4.6 above. (In this case, the result also holds trivially for degrees $\boldsymbol{c} \not\geq \boldsymbol{0}^{(n)}$. We have avoided trivial cases in the table.)

Spectrum	Restrictions	Graphs	Relations on $\mathcal{G}$	LO's	Relations on $\mathbb{Q}$
$\{\boldsymbol{d} > \boldsymbol{0}\}$		Y	Y	??	??
$\{\boldsymbol{d}' > \boldsymbol{0}'\}$		Y	Y	??	Y(4.9)
$\{\boldsymbol{d}^{(n)} > \boldsymbol{0}^{(n)}\}$		Y	Y	Y(4.6)	Y(4.7)
$\{\boldsymbol{d} > \boldsymbol{c}\}$	$\boldsymbol{c}$ nonlow	Y	Y	N(5.5)	??
$\{\boldsymbol{d} > \boldsymbol{c}\}$	$\boldsymbol{c}$ low	Y	Y	??	??
$\{\boldsymbol{d}' > \boldsymbol{c}\}$	$\boldsymbol{c} \geq \boldsymbol{0}'$	Y	Y	??	Y(4.8)
$\{\boldsymbol{d}^{(n)} > \boldsymbol{c}\}$	$\boldsymbol{c} \geq \boldsymbol{0}^{(n)}$	Y	Y	Y(4.6)	Y(4.7)
$\{\boldsymbol{d} \geq \boldsymbol{0}'\}$		Y	Y	N(5.1)	N(5.2)
$\{\boldsymbol{d}' \geq \boldsymbol{0}''\}$		Y	Y	N(5.4)	Y(5.3)
$\{\boldsymbol{d}^{(n)} \geq \boldsymbol{0}^{(n+1)}\}$		Y	Y	Y(5.8)	Y(5.18)
$\{\boldsymbol{d} \geq \boldsymbol{c}\}$	$\boldsymbol{c} \neq \boldsymbol{0}$	Y	Y	N(5.1)	N(5.2)
$\{\boldsymbol{d}' \geq \boldsymbol{c}\}$	$\boldsymbol{c} > \boldsymbol{0}'$	Y	Y	N(5.4)	Y(5.3)
$\{\boldsymbol{d}^{(n)} \geq \boldsymbol{c}\}$	$\boldsymbol{c} > \boldsymbol{0}^{(n)}$	Y	Y	Y(5.7)	Y(5.17)

Clearly, this table still has some holes to be filled in, and it would be of interest to know the correct answers to any of the entries marked "??." For each such entry involving $\boldsymbol{c}$, the question is completely open: we have no proof of realizability or nonrealizability for any degree $\boldsymbol{c}$ (subject to the restrictions on $\boldsymbol{c}$ in the second column).

The columns for graphs and relations on $\mathcal{G}$ reflect the relative ease of coding information structurally into a graph. However, if we consider the same question for complements $\{\boldsymbol{d} : \boldsymbol{d} \not\leq \boldsymbol{c}\}$ of lower cones below arbitrary degrees $\boldsymbol{c}$, then some N's would appear even in those two columns. Of course, the nonlow$_n$ degrees can be defined by either of the two conditions $\boldsymbol{d}^{(n)} > \mathbf{0}^{(n)}$ and $\boldsymbol{d}^{(n)} \not\leq \mathbf{0}^{(n)}$, since these are equivalent. When $\mathbf{0}^{(n)}$ is replaced by an arbitrary $\boldsymbol{c}$, however, the conditions need no longer be equivalent. Indeed, in [18], Kalimullin has shown that there exists a degree $\boldsymbol{c}$ computable in $\mathbf{0}''$ such that no graph realizes the spectrum $\{\boldsymbol{d} : \boldsymbol{d} \not\leq \boldsymbol{c}\}$, the complement of the lower cone below $\boldsymbol{c}$. (Therefore, of course, all columns would have an N in this row, if we added such a row to the table.) On the other hand, he has also shown, in [19] and [17], that this degree $\boldsymbol{c}$ cannot be taken to be c.e., nor to be low.

Noting that the theory of graphs is complete (in the sense of [14]) and that the theory of linear orders is not, we have another question.

Question 6.1 *Let T be the theory of linear orders. How many jumps away from being complete (for spectra of structures) is T? More specifically:*

1. *Does there exist an n such that for every graph $\mathcal{H}$ with $Spec(\mathcal{H}) \subseteq \{\boldsymbol{d} : \mathbf{0}^{(n)} \leq \boldsymbol{d}\}$, there exists a linear order $\mathcal{L}$ with $\{\boldsymbol{d}^{(n)} : \boldsymbol{d} \in Spec(\mathcal{L})\} = Spec(\mathcal{H})$?*

2. *Does there exist an n such that for every graph $\mathcal{H}$, there is a linear order $\mathcal{L}$ with $\{\boldsymbol{d}^{(n)} : \boldsymbol{d} \in Spec(\mathcal{L})\} = \{\boldsymbol{d}^{(n)} : \boldsymbol{d} \in Spec(\mathcal{H})\}$?*

3. *If (2) fails, then reverse the quantifiers: is it true that for every graph $\mathcal{H}$, there exists an n and an $\mathcal{L}$ with $\{\boldsymbol{d}^{(n)} : \boldsymbol{d} \in Spec(\mathcal{L})\} = \{\boldsymbol{d}^{(n)} : \boldsymbol{d} \in Spec(\mathcal{H})\}$?*

Similar questions apply to relations on $\mathbb{Q}$ and on the random graph $\mathcal{G}$.

In light of Theorem 5.4, any positive answer to (1) would necessarily have $n > 1$. We tentatively conjecture that the answer to (1) is negative, based on the apparent difficulty of coding isomorphism types of graphs into linear order types.

Other possible questions involve models of other theories. Of course, graphs are included here mainly as a universal theory, as justified by the results in [14]. Boolean algebras are distinguished from linear orders and from graphs by their possible spectra: it has been known since [6] that if the spectrum of a Boolean algebra contains a low degree, then it contains the degree $\mathbf{0}$, and indeed the same holds for low_4 degrees, by results in [29] and [21]. In the table above, n is allowed to represent any value > 1, because as far as is known for linear orders and graphs, all answers are the same for all such values of n. Potentially, Boolean algebras could distinguish values of n greater than 1 under questions like these – for example, if the result on low_4 Boolean algebras fails to extend to low_n ones for $n > 4$.

Recently, three of us (Frolov, Kalimullin, and Miller) have announced the construction of a field whose spectrum contains exactly the high degrees. They conjecture that similar constructions can realize the high_n degrees for each $n > 1$, and [10] is expected to contain results along these lines. It was already known from [4] that the high_0 degrees, and any other upper cone, can be realized as the spectrum of an algebraic field extension of $\mathbb{Q}$. Algebraic fields are also known (from [9]) to have jump degrees, and every $\boldsymbol{c} \geq \mathbf{0}'$ is the jump degree of some algebraic field. However, algebraic fields are not capable of realizing the nonlow_0 degrees as a spectrum, and indeed spectra of algebraic fields are really just cones of enumeration degrees. These results generalize easily to spectra of fields of finite transcendence degree over the prime subfield, and Coles, Downey and Slaman showed in [5] that similar results hold for spectra of torsion-free abelian groups of rank 1.

We have focused here entirely on arithmetic jumps – that is, restricting to finitely many iterations of the jump operator. The results in [11] went further, considering the α-th jump $\mathbf{0}^{(\alpha)}$ for computable ordinals α. It would be natural to investigate the possibility of extending the above results, particularly for linear orders, to the hyperarithmetic degrees in this manner.

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