SEQUENCES OF n-DIAGRAMS

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1. Introduction

We consider only computable languages, and countable structures, with universe a subset of ω, which we think of as a set of constants. We identify sentences with their Gödel numbers. Thus, for a structure $\mathcal{A}$, the complete (elementary) diagram, $D^c(\mathcal{A})$, and the atomic diagram, $D(\mathcal{A})$, are subsets of ω. We classify formulas as usual. A formula is both Σ_0 and Π_0 if it is open. For $n > 0$, a formula, in prenex normal form, is Σ_n, or Π_n, if it has n blocks of like quantifiers, beginning with $\exists$, or $\forall$. For a formula θ, in prenex normal form, we let $neg(\theta)$ denote the dual formula that is logically equivalent to $\neg\theta$—if θ is Σ_n, then $neg(\theta)$ is Π_n, and vice versa.

Definition 1.1. *For a structure $\mathcal{A}$, the* n-diagram *is*

$$D_n(\mathcal{A}) = D^c(\mathcal{A}) \cap \Sigma_n.$$

We are interested in complexity, which we measure by Turing degree. We denote Turing reducibility by $\leq_T$, and Turing equivalence by $\equiv_T$. It is clear that for any structure $\mathcal{A}$, $D_0(\mathcal{A}) \equiv_T D(\mathcal{A})$. We show that for any $\mathcal{A}$, there exists $\mathcal{B} \cong \mathcal{A}$ such that $D^c(\mathcal{B}) \equiv_T D(\mathcal{B})$. If $\mathcal{A}$ is an algebraically closed field, a real closed field, or any other structure in which we have effective elimination of quantifiers, then this collapse is "intrinsic"; i.e., it happens in all copies. For models of PA, the collapse is not intrinsic. For the standard model of arithmetic, $\mathcal{N} = (\omega, +, \cdot, S, 0)$, we have $D_n(\mathcal{N}) \equiv_T \emptyset^{(n)}$, uniformly in n. In [10], it is shown that for any model $\mathcal{A}$ of PA, there exists $\mathcal{B} \cong \mathcal{A}$ such that $D_{n+1}(\mathcal{B}) \not\leq_T D_n(\mathcal{B})$.

We first consider the following problem.

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Problem 1. *Find syntactic conditions on $\mathcal{A}$ guaranteeing that for some n, for all $\mathcal{B} \cong \mathcal{A}$, $D^c(\mathcal{B}) \equiv_T D_n(\mathcal{B})$. In particular, for $n = 0$, find syntactic conditions guaranteeing that for all $\mathcal{B} \cong \mathcal{A}$, $D^c(\mathcal{B}) \equiv_T D(\mathcal{B})$.*

For structures $\mathcal{A}$ that do not exhibit intrinsic collapse of the complete diagram to the atomic diagram, we consider the sequences $(D_n(\mathcal{B}))_{n\in\omega}$ for $\mathcal{B} \cong \mathcal{A}$. We focus on the corresponding sequences of Turing degrees.

Definition 1.2. (*i*) *For sets X and Y, Y is* c.e. in and above X *if Y is c.e. relative to X, and $X \leq_T Y$.*

(*ii*) *An* ω-table *is a sequence of sets $(C_n)_{n\in\omega}$ such that C_{n+1} is c.e. in and above C_n, uniformly in n. Similarly, for $N \in \omega$, an* N-table *is a sequence $(C_n)_{n<N}$ such that for $n+1 < N$, C_{n+1} is c.e. in and above C_n.*

In [1], there is a definition of an *α-table*, where α is an arbitrary computable ordinal. These sequences are sometimes called *α-REA*, or *α-CEA sets*. The following proposition is clear.

Proposition 1.1. *For any structure $\mathcal{A}$, $(D_n(\mathcal{A}))_{n\in\omega}$ is an ω-table.*

We look for conditions guaranteeing that a structure will have copies in which the degrees of n-diagrams can be assigned arbitrarily, subject only to the constraints of Proposition 1.1.

Problem 2. *For finite N, give conditions on $\mathcal{A}$ guaranteeing that for every $(N+1)$-table $(C_n)_{n\leq N}$, there exists $\mathcal{B} \cong \mathcal{A}$ such that for all $n \leq N$, $D_n(\mathcal{B}) \equiv_T C_n$.*

Problem 3. *Give conditions guaranteeing that for every ω-table $(C_n)_{n\in\omega}$, there exists $\mathcal{B} \cong \mathcal{A}$ such that for all n, $D_n(\mathcal{B}) \equiv_T C_n$.*

In Section 2, we give results related to Problem 1. In Section 3, we describe some examples. In Section 4, we state the results on Problems 2 and 3. In Section 5, we give some applications. In Sections 6 and 7, we prove the results on Problems 2 and 3. In Section 8, we give some open problems.

2. Conditions for intrinsic collapse

A structure $\mathcal{A}$ is *trivial* if there is a tuple of elements $\overline{c}$ such that every permutation of the universe that fixes $\overline{c}$ pointwise is an automorphism. In [8], it is shown that for a non-trivial structure, the set of Turing degrees of copies is closed upwards, while for a trivial structure, all copies have the same Turing degree. The same ideas yield the result below.

Theorem 2.1. *For any structure $\mathcal{A}$, there exists $\mathcal{B} \cong \mathcal{A}$ such that $D(\mathcal{B}) \equiv_T D^c(\mathcal{B})$. Then for all n, $D_n(\mathcal{B}) \equiv_T D(\mathcal{B})$.*

Proof. We consider two cases.

Case 1. Suppose $\mathcal{A}$ is non-trivial.

In [8], it is shown that if $\mathcal{A}$ is non-trivial, then for any set X such that $D(\mathcal{A}) \leq_T X$, there is a permutation π of the universe such that for the induced structure $\mathcal{B}$ with $\mathcal{A} \cong_\pi \mathcal{B}$, we have $X \leq_T D(\mathcal{B})$ and $\pi \leq_T X$. Since $D(\mathcal{B})$ is computable in $\pi \oplus D(\mathcal{A})$, it follows that $D(\mathcal{B}) \leq_T X$. Here we choose X such that $D^c(\mathcal{A}) \leq_T X$. Then

$$X \leq_T D(\mathcal{B}) \leq_T D^c(\mathcal{B}) \leq_T \pi \oplus D^c(\mathcal{A}) \leq_T X.$$

Case 2. Suppose $\mathcal{A}$ is trivial.

Take a tuple $\overline{c}$ such that every permutation fixing $\overline{c}$ pointwise is an automorphism of $\mathcal{A}$. We can show, by induction on formulas, that for each formula $\theta(\overline{c}, \overline{x})$, there is a quantifier-free formula $\theta^*(\overline{c}, \overline{x})$, in the empty language (with just $=$), such that

$$\mathcal{A} \models \forall \overline{x}\, [\theta(\overline{c}, \overline{x}) \Leftrightarrow \theta^*(\overline{c}, \overline{x})].$$

Moreover, the process of passing from θ to θ^* is effective in $D(\mathcal{A})$.

The most interesting case is when $\theta = \theta(\overline{c}, \overline{x})$ is an atomic formula involving a relation symbol R, with some places filled by constants from $\overline{c}$ and others by variables from $\overline{x}$. We consider a family of quantifier-free formulas $\psi(\overline{c}, \overline{x})$, in the empty language, each specifying the equality relation $\overline{c}, \overline{x}$ completely, so that all possible equality relations are represented. For each formula $\psi(\overline{c}, \overline{x})$, we choose a sample tuple $\overline{a}$ such that $\mathcal{A} \models \psi(\overline{c}, \overline{a})$. We let θ^* be the disjunction of the formulas $\psi(\overline{c}, \overline{x})$ for which the corresponding sample tuple satisfies $\theta(\overline{c}, \overline{x})$, or $\perp$, if no sample tuple satisfies $\theta(\overline{c}, \overline{x})$.

The other clauses are as usual in arguments for effective elimination of quantifiers. For the existential clause, consider $\theta = \exists u \psi(\overline{c}, \overline{x}, u)$, where $\psi(\overline{c}, \overline{x}, u)$ is quantifier-free in the empty language. By the usual rearrangements, we may reduce to the case where $\psi(\overline{c}, \overline{x}, u)$ is a conjunction specifying the equality relations completely. If there is a conjunct $u = c$ or $u = x$, then we can drop the quantifier $\exists u$ and replace all occurrences of u by c, or x. If there are conjuncts $u \neq c$, $u \neq x$ for all $c \in \overline{c}$, $x \in \overline{x}$, then, assuming that we have a conjunct $u = u$, we can drop the quantifier $\exists u$ and all conjuncts mentioning u—if there is nothing left, then we replace θ by $\top$ (truth). If we have a conjunct $u \neq u$, then we replace θ by $\perp$ (falsity).

Here is our result on Problem 1, giving conditions for intrinsic collapse of the complete diagram to the n-diagram.

Theorem 2.2. *For any structure $\mathcal{A}$ and any n, the following are equivalent:*

1. *For all $\mathcal{B} \cong \mathcal{A}$, $D^c(\mathcal{B}) \equiv_T D_n(\mathcal{B})$,*
2. *For some tuple $\overline{c}$, there is a computable function d taking each (finitary) formula $\theta(\overline{x})$ to a formula $d_\theta(\overline{c}, \overline{x})$, a c.e. disjunction of (finitary) Σ_{n+1} formulas with parameters $\overline{c}$, such that*
$$\mathcal{A} \models \forall \overline{x}\, [\theta(\overline{x}) \Leftrightarrow d_\theta(\overline{c}, \overline{x})].$$

Proof. For $2 \Rightarrow 1$, let $\mathcal{B}$ be a copy of $\mathcal{A}$, with tuple $\overline{d}$ corresponding to $\overline{c}$. Given $D_n(\mathcal{B})$, we can enumerate $D_{n+1}(\mathcal{B})$. To determine whether a sentence $\theta(\overline{b})$ is in $D^c(\mathcal{B})$, using $D_n(\mathcal{B})$, we search simultaneously for a disjunct of $d_\theta(\overline{d}, \overline{b})$ and for a disjunct of $d_{\neg\theta}(\overline{d}, \overline{b})$. One of these will appear, and then we will have the answer.

We obtain $1 \Rightarrow 2$ by varying the following result from [3] and [5].

Proposition 2.1. *For a structure $\mathcal{A}$ and a relation R on $\mathcal{A}$, the following are equivalent:*

1. *For all isomorphisms F from $\mathcal{A}$ onto a copy $\mathcal{B}$, $F(R)$ is c.e. relative to $D_0(\mathcal{B})$,*
2. *For some $\overline{c}$, there is a c.e. set of finitary existential formulas, with parameters $\overline{c}$, whose disjunction defines R.*

We may vary Proposition 2.1, replacing the single relation by a family of relations.

Proposition 2.2. *For a structure $\mathcal{A}$ and a sequence $(R_k)_{k\in\omega}$ of relations on $\mathcal{A}$, the following are equivalent:*

(1) *For all isomorphisms F from $\mathcal{A}$ onto a copy $\mathcal{B}$, $F(R_k)$ is c.e. relative to $D_0(\mathcal{B})$, uniformly in k,*

(2) *For some $\overline{c}$, for each k, we can effectively find an index for a c.e. set of finitary existential formulas, with parameters $\overline{c}$, whose disjunction defines R_k.*

We may vary Propositions 2.1 and 2.2, replacing the 0-diagram by the n-diagram.

Proposition 2.3. *For a structure $\mathcal{A}$, a relation R on $\mathcal{A}$, and $n \in \omega$, the following are equivalent:*

(1) *For all isomorphisms F from $\mathcal{A}$ onto a copy $\mathcal{B}$, $F(R)$ is c.e. relative to $D_n(\mathcal{B})$,*

(2) *For some $\overline{c}$, there is a c.e. set of finitary Σ_{n+1} formulas, with parameters $\overline{c}$, whose disjunction defines R.*

Proposition 2.4. *For a structure $\mathcal{A}$, a sequence $(R_k)_{k\in\omega}$ of relations on $\mathcal{A}$, and $n \in \omega$, the following are equivalent:*

(1) *For all isomorphisms F from $\mathcal{A}$ onto a copy $\mathcal{B}$, $F(R_k)$ is c.e. relative to $D_n(\mathcal{B})$, uniformly in k,*

(2) *For some $\overline{c}$, for each k, we can effectively find an index for a c.e. set of finitary Σ_{n+1} formulas, with parameters $\overline{c}$, whose disjunction defines R_k.*

We say just a little about the proofs of the propositions, emphasizing the (minor) differences. In all cases, $2 \Rightarrow 1$ is clear, and $1 \Rightarrow 2$ is obtained by producing a generic copy $\mathcal{B}$ of $\mathcal{A}$. We take the proof of Proposition 2.1 in [2] as our guide. Let B be an infinite computable set of constants, for the universe of $\mathcal{B}$. The forcing conditions are finite partial 1-1 functions from B to $\mathcal{A}$. As a forcing language, we use a computable infinitary propositional language. For Proposition 2.1, the propositional variables are the atomic sentences in the language $L(\mathcal{A}) \cup B$ and those in the language $\{R\} \cup B$. For Proposition 2.2, we replace $\{R\}$ by $\{R_k : k \in \omega\}$. For Propositions 2.3 and 2.4, we replace the atomic sentences in the language $L(\mathcal{A})$ by the finitary Σ_n sentences.

In all cases, we form a complete sequence of forcing conditions $(p_n)_{n\in\omega}$ whose union is a 1-1 function from B onto $\mathcal{A}$. We let F be the inverse function, and we let $\mathcal{B}$ be the structure induced by F on B. From $\mathcal{B}$, and $F(R)$, or $F(R_k)$ for $k \in \omega$, we obtain propositional structures in a natural way. For Proposition 2.1, we take the set of atomic sentences in $D(\mathcal{B})$ together with the sentences $R\overline{b}$, where $F(\overline{b}) \in R$. For Proposition 2.3, we take $D_n(\mathcal{B})$ together with the atomic sentences $R\overline{b}$, where $F(\overline{b}) \in R$. In each case, a sentence in the forcing language is true in the appropriate propositional structure just in case it is forced by some p_n. Let $(\varphi_i)_{i\in\omega}$ be a standard effective enumeration of all unary partial computable functions, and let $W_i = dom(\varphi_i)$ for $i \in \omega$.

For Proposition 2.1, the hypothesis says that $F(R) = W_e^{D_0(\mathcal{B})}$, for some e. The forcing language includes a sentence with this meaning, which must be forced by some p. Say p takes $\overline{d}$ to $\overline{c}$. Then R is defined by a formula $\theta(\overline{c}, \overline{x})$ saying

$$(\exists q \supseteq p) \bigvee_{\overline{b}} [q(\overline{b}) = \overline{x} \wedge q \Vdash \varphi_e^{D_0(\mathcal{B})}(\overline{b}) \downarrow],$$

where $\theta(\overline{c}, \overline{x})$ is a c.e. disjunction of finitary existential formulas.

For Proposition 2.2, the hypothesis says that there is some e such that for all k, $F(R_k) = W_{\varphi_e(k)}^{D_0(\mathcal{B})}$. The forcing language includes a sentence with this meaning, which must be forced by some p. Say p takes $\overline{d}$ to $\overline{c}$. Then for each k, R_k is defined by a formula $\theta_k(\overline{c}, \overline{x}_k)$ saying

$$(\exists q \supseteq p) \bigvee_{\overline{b}} [q(\overline{b}) = \overline{x}_k \wedge q \Vdash \varphi_{\varphi_e(k)}^{D_0(\mathcal{B})}(\overline{b}) \downarrow].$$

For each k, $\theta_k(\overline{c}, \overline{x}_k)$ is a c.e. disjunction of finitary existential formulas, with index determined effectively from k.

For Proposition 2.3, the hypothesis says that $F(R) = W_e^{D_n(\mathcal{B})}$, for some e. Again, the forcing language includes a sentence with this meaning, forced by some p. Say p has range $\overline{c}$. We arrive at a definition $\theta(\overline{c}, \overline{x})$ of R saying

$$(\exists q \supseteq p) \bigvee_{\overline{b}} [q(\overline{b}) = \overline{x} \wedge q \Vdash \varphi_e^{D_n(\mathcal{B})}(\overline{b}) \downarrow],$$

where $\theta(\overline{c}, \overline{x})$ is a c.e. disjunction of finitary Σ_{n+1} formulas.

For Proposition 2.4, the hypothesis says that there is some e such that for all k, $F(R_k) = W_{\varphi_e(k)}^{D_n(\mathcal{B})}$. The forcing language includes a sentence with this meaning, forced by some p. Say p has range $\overline{c}$. Then for each k, R_k is defined by a formula $\theta_k(\overline{c}, \overline{x}_k)$, saying

$$(\exists q \supseteq p) \bigvee_{\overline{b}} [q(\overline{b}) = \overline{x}_k \wedge q \Vdash \varphi_{\varphi_e(k)}^{D_n(\mathcal{B})}(\overline{b}) \downarrow],$$

where $\theta_k(\overline{c}, \overline{x}_k)$ is a c.e. disjunction of finitary Σ_{n+1} formulas, with index determined effectively from k.

Using Proposition 2.4, we can easily complete the proof of Theorem 2.2. We let R_k be the relation defined by the k^{th} formula $\psi(\overline{x})$.

We next give a result on intrinsic collapse at a single level.

Theorem 2.5. *For any structure $\mathcal{A}$ and any n, the following are equivalent:*

(1) *For all $\mathcal{B} \cong \mathcal{A}$, $D_{n+1}(\mathcal{B}) \equiv_T D_n(\mathcal{B})$,*

(2) *For some $\overline{c}$, there is a computable function d taking each (finitary) Π_{n+1} formula $\theta(\overline{x})$ to a formula $d_\theta(\overline{c}, \overline{x})$, a c.e. disjunction of finitary Σ_{n+1} formulas with parameters $\overline{c}$, such that*

$$\mathcal{A} \models \forall \overline{x} \, [\theta(\overline{x}) \Leftrightarrow d_\theta(\overline{c}, \overline{x})].$$

Proof. For $2 \Rightarrow 1$, let $\mathcal{B}$ be a copy of $\mathcal{A}$, with tuple $\overline{d}$ corresponding to $\overline{c}$. Given $D_n(\mathcal{B})$, we can enumerate $D_{n+1}(\mathcal{B})$. To determine whether a sentence $\theta(\overline{b})$ will appear, we search simultaneously for the sentence itself and a disjunct of $d_{\neg\theta}(\overline{d}, \overline{b})$. For $1 \Rightarrow 2$, we use Proposition 2.4 again, letting R_k be the relation defined by the k^{th} Π_{n+1} formula $\psi(\overline{x})$.

3. Examples illustrating Theorems 2.2 and 2.5

For an algebraically closed field, or a real closed field, where there is effective elimination of quantifiers, we have intrinsic collapse of the complete diagram to the atomic diagram. Below, we give examples, for each $n > 0$, for which there is intrinsic collapse of the complete diagram to the n-diagram, but not to the $(n-1)$-diagram.

The examples are linear orderings. We need some notation. If $\mathcal{C}$ and $\mathcal{D}$ are linear orderings, then their product $\mathcal{C}\cdot\mathcal{D}$ is the result of replacing each element of $\mathcal{D}$ by a copy of $\mathcal{C}$. We let η denote the order type of the rationals.

Example 3.1. *For $N \geq 1$, if $\mathcal{B}$ is a linear ordering of type ω^N, then*

$$D^c(\mathcal{B}) \equiv_T D_{2N-1}(\mathcal{B}).$$

(*This result is sharp. In Section 5, we shall return to this example and show that there exist linear orderings $\mathcal{B}$ of type ω^N such that for every $k < 2N-1$, $D_k(\mathcal{B}) <_T D_{k+1}(\mathcal{B})$.*)

We shall describe formulas d_θ. First, let $\mathcal{A}$ be a linear ordering of type ω^N such that $D^c(\mathcal{A})$ is computable. The existence of such $\mathcal{A}$ is well-known. We could give an inductive proof, staring with a decidable linear ordering of type ω, and using the fact that if linear orderings $\mathcal{C}$ and $\mathcal{D}$ have decidable copies, then so does $\mathcal{C}\cdot\mathcal{D}$ (a consequence of the Feferman-Vaught Theorem).

Claim 1. For each tuple $\overline{a}$, we can find a (finitary) Σ_{2N} formula $\psi_{\overline{a}}(\overline{x})$ that defines $\overline{a}$ in $\mathcal{A}$.

Proof. We give a brief description of the definitions in the case where $\overline{a}$ consists of a single element a. If $N = 1$, then $\psi_a(x)$ is a finitary Π_2 formula saying that for the appropriate k, there are exactly k elements to the left of x. If $N = 2$ and a does not lie on the first copy of ω, then $\psi_a(x)$ is a finitary Π_4 formula saying that for the appropriate m and k, there are exactly m limit points to the left of x, and there are exactly k elements between the last of these limit points and x. In general, if $N > 1$ and a does not lie on the first copy of ω^N, then $\psi_a(x)$ is a finitary Π_{2N} formula saying that for the appropriate m, there are exactly m $(N-1)$-limit points to the left of x, and that x has the position of the element a in the copy of ω^{N-1} that begins with the last of these limit points.

For each finitary formula $\theta(\overline{x})$, we let $d_\theta(\overline{x})$ be the disjunction of the formulas $\psi_{\overline{a}}(\overline{x})$ (as in Claim 1), where $\mathcal{A} \models \theta(\overline{a})$.

In Example 3.1, we described formulas d_θ that appear to be infinitary. However, we could carry out an effective elimination of quantifiers, and assign to each formula θ a formula d_θ (equivalent to θ in ω^N), where d_θ is finitary Σ_{2N}. In the next example, the formulas d_θ are necessarily infinitary.

Example 3.2. *If $\mathcal{B}$ is a linear ordering of type $\omega^N \cdot \eta$, then*

$$D^c(\mathcal{B}) \equiv_T D_{2N}(\mathcal{B}).$$

(*In Section 5, we shall return to this example and show that there exist linear orderings $\mathcal{B}$ of type $\omega^N \cdot \eta$ such that for every $k < 2N$, $D_k(\mathcal{B}) <_T D_{k+1}(\mathcal{B})$.*)

We shall describe formulas d_θ. First, let $\mathcal{A}$ be a linear ordering of type $\omega^N \cdot \eta$ such that $D^c(\mathcal{A})$ is computable. The existence of such $\mathcal{A}$ follows from the fact that there are decidable linear orderings of types ω^N and η (using the Feferman-Vaught Theorem).

Claim 2. For each tuple $\overline{a}$, we can find a (finitary) Σ_{2N+1} formula $\psi_{\overline{a}}(\overline{x})$ that defines the orbit of $\overline{a}$ under the automorphisms of $\mathcal{A}$.

Proof. The formula $\psi_{\overline{a}}(\overline{x})$ says how the elements of the tuple are ordered, gives the position of each element in its copy of ω^N, and for each successive pair in the tuple, says whether there is an N-limit point between them.

For each formula $\theta(\overline{x})$, we let $d_\theta(\overline{x})$ be the disjunction of the formulas $\psi_{\overline{a}}(\overline{x})$ (as in Claim 2), where $\mathcal{A} \models \theta(\overline{a})$.

We show that in the case where $N = 1$, there is a formula θ such that d_θ cannot be made finitary.

Claim 3. *Let $\theta(x, y)$ be the formula saying that $x < y$ and there is no limit point between x and y. There is no finitary Σ_3 formula that is equivalent to $\theta(x, y)$ in $\omega \cdot \eta$.*

Proof. Consider the formula $\theta(x, y) = \exists\overline{z}\,\forall\overline{u}\,\exists\overline{v}\,\delta(x, y, \overline{z}, \overline{u}, \overline{v})$, where δ is open. We show that if $\theta(x, y)$ is satisfied by pairs sufficiently far apart on a single copy of ω, then it is satisfied by pairs on different copies of ω. Let $m > length(\overline{u})$, $n > length(\overline{v})$, and $r > length(\overline{z})$. Choose x and y such that the interval between them is finite, but of size at least rmn. Take $\overline{z}$ such that $\forall \overline{u}\, \exists \overline{v}\, \delta(x, y, \overline{z}, \overline{u}, \overline{v})$ holds. Look at the way $\overline{z}$ partitions the interval between x and y. There must be at least one sub-interval of size at least mn. We refer to this sub-interval as "large".

Now, choose x^*, y^* and $\overline{z}^*$, ordered in the same way as x, y and $\overline{z}$, such that x^* and y^* lie on different copies of ω, and the sizes of the corresponding intervals match, except that the interval corresponding to the large finite sub-interval is infinite. To show that $\theta(x^*, y^*)$ holds, it is enough to show that for any $\overline{u}^*$, there exists $\overline{v}^*$ such that $\delta(x^*, y^*, \overline{z}^*, \overline{u}^*, \overline{v}^*)$ holds. We look at the way $\overline{u}^*$ partitions the infinite sub-interval. We choose $\overline{u}$ matching any sub-sub-intervals of size less than n, and letting others have size at least n. We have $\overline{v}$ such that $\delta(x, y, \overline{z}, \overline{u}, \overline{v})$ holds. Then we can choose $\overline{v}^*$ so that the orderings match. It follows that $\delta(x^*, y^*, \overline{z}^*, \overline{u}^*, \overline{v}^*)$ holds. This proves the claim.

We show that Theorem 2.5 applies to some structures to which Theorem 2.2 does not apply.

Example 3.3. *Moses* [12] *showed that for each n, there is a linear ordering $\mathcal{A}$ such that $D_n(\mathcal{A})$ is computable, but $D_{n+1}(\mathcal{A})$ is not computable. The structure $\mathcal{A}^* = (\mathcal{A}, (a)_{a\in\mathcal{A}})$ is relatively computably stable—for any copy $\mathcal{B}$, the unique isomorphism is computable relative to $D_0(\mathcal{B})$. It follows that for all $\mathcal{B} \cong \mathcal{A}^*$, $D_n(\mathcal{B}) \leq_T D_0(\mathcal{B})$, but $D_{n+1}(\mathcal{B}) \not\leq_T D_n(\mathcal{B})$.*

Chisholm and Moses [6] *also gave an example of a linear ordering $\mathcal{A}$ such that for all n, $D_n(\mathcal{A})$ is computable, but $D^c(\mathcal{A})$ is not computable. Again $\mathcal{A}^* = (\mathcal{A}, (a)_{a\in\mathcal{A}})$ is relatively computably stable. For all $\mathcal{B} \cong \mathcal{A}^*$ and all n, $D_n(\mathcal{B}) \leq_T D_0(\mathcal{B})$, but $D^c(\mathcal{B}) \not\leq_T D_0(\mathcal{B})$.*[1]

4. Results on Problems 2 and 3

Here we give conditions on a structure $\mathcal{A}$ guaranteeing that we can represent Turing degrees of an arbitrary N-table, or ω-table by the degrees of n-diagrams of copies of $\mathcal{A}$. For simplicity, we consider only structures for a finite relational language. In addition, we adopt the convention that if $\overline{c}$ denotes a tuple from $\mathcal{A}$, then the elements of $\overline{c}$ are distinct. If $\overline{c}, \overline{a}$ denotes the concatenation of tuples $\overline{c}$ and $\overline{a}$, then the elements of $\overline{c}$ and $\overline{a}$ are all assumed to be distinct.

[1] J. Knight and R. Shore have similar examples in a finite relational language.

The standard *back-and-forth relations* $\leq_\alpha$ are defined in [4] and [2] for arbitrary countable ordinals α. Below, we give a definition in the case where α is finite. The fact that we have a finite relational language simplifies the definition of $\leq_0$.

Definition 4.1. *Let $\overline{a}$, $\overline{b}$ be tuples from $\mathcal{A}$ with length($\overline{a}$) $\leq$ length($\overline{b}$).*

(1) *$\overline{a} \leq_0 \overline{b}$ if the open formulas true of $\overline{a}$ are all true of the corresponding elements of $\overline{b}$.*
(2) *$\overline{a} \leq_{n+1} \overline{b}$ if for each $\overline{d}$, there exists $\overline{c}$ such that $\overline{b}, \overline{d} \leq_n \overline{a}, \overline{c}$.*

Next, we define a notion of independence. We consider formulas in the language of $\mathcal{A}$. We say that a tuple $\overline{a}$ in $\mathcal{A}$ *corresponds* to a tuple of variables $\overline{u}$ if the tuples have the same length, so that $\overline{a}$ is appropriate to substitute for $\overline{u}$.

Definition 4.2. *Let $\overline{u}$ be a tuple of variables.*

(1) *The formula $\theta(\overline{u}, \overline{x})$ is* 0-independent *over $\overline{u}$ if it is open, and for each $\overline{c}$ in $\mathcal{A}$, corresponding to $\overline{u}$, there exist $\overline{a}$ and $\overline{a}'$, corresponding to $\overline{x}$, such that $\mathcal{A} \models \theta(\overline{c}, \overline{a})$ and $\mathcal{A} \models \neg\,\theta(\overline{c}, \overline{a}')$. (The assumption of distinctness means that $=$ is preserved between $\overline{c}, \overline{a}$ and $\overline{c}, \overline{a}'$.)*
(2) *For $n > 0$, the formula $\theta(\overline{u}, \overline{x})$ is* n-independent *over $\overline{u}$ if it is Π_n and for each $\overline{c}$, corresponding to $\overline{u}$,*

(i) there exists $\overline{a}$ such that $\mathcal{A} \models \theta(\overline{c}, \overline{a})$, and

(ii) for any $\overline{a}$ such that $\mathcal{A} \models \theta(\overline{c}, \overline{a})$, and any $\overline{a}_1$, there exist $\overline{a}'$ and $\overline{a}'_1$ such that $\mathcal{A} \models \neg\,\theta(\overline{c}, \overline{a}')$ and $\overline{c}, \overline{a}, \overline{a}_1 \leq_{n-1} \overline{c}, \overline{a}', \overline{a}'_1$.

Here are the results on Problems 2 and 3.

Theorem 4.1. *Suppose $D_N(\mathcal{A})$ is computable, and the relations $\leq_n$ are c.e., for $n < N$. Suppose also that for each tuple $\overline{u}$ of variables, and each $n \leq N$, we can effectively find a formula that is n-independent over $\overline{u}$. Then for any $(N+1)$-table $(C_n)_{n \leq N}$, there exists $\mathcal{B} \cong \mathcal{A}$ such that $D_n(\mathcal{B}) \equiv_T C_n$ for all $n \leq N$.*

Theorem 4.2. *Suppose $D^c(\mathcal{A})$ is computable, and the relations $\leq_n$ are c.e., uniformly in n, for $n \in \omega$. Suppose also that for each tuple $\overline{u}$ of distinct variables, and each n, we can effectively find a Π_n formula that is n-independent over $\overline{u}$. Then for any ω-table $(C_n)_{n<\omega}$, there exists $\mathcal{B} \cong \mathcal{A}$ such that $D_n(\mathcal{B}) \equiv_T C_n$, uniformly in n. The uniformity implies that $D^c(\mathcal{B}) \equiv_T \oplus_n C_n$.*[2]

Before proving Theorems 4.1 and 4.2, we consider some examples.

5. Examples illustrating Theorems 4.1 and 4.2

In Section 3, we showed that for any linear ordering $\mathcal{B}$ of type ω^N,

$$D^c(\mathcal{B}) \equiv_T D_{2N-1}(\mathcal{B}).$$

We also showed that for any linear ordering $\mathcal{B}$ of type $\omega^N \cdot \eta$,

$$D^c(\mathcal{B}) \equiv_T D_{2N}(\mathcal{B}).$$

Now, using Theorem 4.1, we show that for any $2N$-table $(C_n)_{n\leq 2N-1}$, there is a linear ordering $\mathcal{B}$ of type ω^N such that for all $n \leq 2N-1$, $D_n(\mathcal{B}) \equiv_T C_n$. We also show that for any $(2N+1)$-table $(C_n)_{n\leq 2N}$, there is a linear ordering $\mathcal{B}$ of type $\omega^N \cdot \eta$ such that for all $n \leq 2N$, $D_n(\mathcal{B}) \equiv_T C_n$. We begin with the case where $N = 1$.

Proposition 5.1. *For any 2-table $(C_n)_{n\leq 1}$, there is an ordering $\mathcal{B}$ of type ω such that for every $n \leq 1$, $D_n(\mathcal{B}) \equiv_T C_n$.*

Proof. We shall apply Theorem 4.1 with $N = 1$. We have $\overline{a} \leq_0 \overline{b}$ if the ordering of $\overline{a}$ is the same as that of the corresponding elements of $\overline{b}$. We need n-independent formulas for $n = 0, 1$. As a 0-independent formula over $\overline{u}$, we take $\theta(\overline{u}, x, y)$ saying that x is greater than anything in $\overline{u}$, and $x < y$. As a 1-independent formula over $\overline{u}$, we take $\theta(\overline{u}, x, y)$ saying that x is greater than anything in $\overline{u}$, and y is the successor of x. We satisfy the effectiveness conditions by taking an ordering $\mathcal{A}$ of type ω such that $D_1(\mathcal{A})$ is computable. We are in a position to apply Theorem 4.1. We get $\mathcal{B} \cong \mathcal{A}$ such that $D_n(\mathcal{B}) \equiv_T C_n$ for $n = 0, 1$.

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[2]R. Shore pointed out the uniformity and its consequence for the complete diagram.

Proposition 5.2. *For any 3-table $(C_n)_{n\leq 2}$, there is an ordering $\mathcal{B}$ of type $\omega\cdot\eta$ such that for every $n\leq 2$, $D_n(\mathcal{B})\equiv_T C_n$.*

Proof. We apply Theorem 4.1 with $N=2$. As above, $\overline{a}\leq_0\overline{b}$ if the ordering is preserved. We have $\overline{a}\leq_1\overline{b}$ if the ordering is preserved, and, in addition, the intervals determined by $\overline{a}$ are at least as large as the intervals determined by the corresponding elements of $\overline{b}$. We need n-independent formulas for $n=0,1,2$. For $n=0,1$, we use the same formulas as above. As a 2-independent formula over $\overline{u}$, we take $\theta(\overline{u},x)$ saying that x is greater than anything in $\overline{u}$, and it is first in its copy of ω. We satisfy the effectiveness conditions by taking an ordering $\mathcal{A}$ of type $\omega\cdot\eta$ such that $D_2(\mathcal{A})$ is computable, and $\leq_1$ is c.e.[3] We are in a position to apply Theorem 4.1. We get $\mathcal{B}\cong\mathcal{A}$ such that $D_n(\mathcal{B})\equiv_T C_n$ for $n=0,1,2$.

Here are the extensions of Propositions 5.1 and 5.2.

Proposition 5.3. *For any $2N$-table $(C_n)_{n\leq 2N-1}$, there is an ordering $\mathcal{B}$ of type ω^N such that for all $n\leq 2N-1$, $D_n(\mathcal{B})\equiv_T C_n$.*

Proposition 5.4. *For any $(2N+1)$-table $(C_n)_{n\leq 2N}$, there is an ordering $\mathcal{B}$ of type $\omega^N\cdot\eta$ such that for all $n\leq 2N$, $D_n(\mathcal{B})\equiv_T C_n$.*

For orderings of type ω^ω or $\omega^\omega\cdot\eta$, we can represent the sequence of Turing degrees of an arbitrary ω-table.

Proposition 5.5. *For any ω-table $(C_n)_{n\in\omega}$, there is an ordering $\mathcal{B}$ of type ω^ω such that for all n, $D_n(\mathcal{B})\equiv_T C_n$, uniformly in n. There is also an ordering $\mathcal{B}$ of type $\omega^\omega\cdot\eta$ such that for all n, $D_n(\mathcal{B})\equiv_T C_n$, uniformly in n.*

Propositions 5.3 and 5.4 are proved using Theorem 4.1. Proposition 5.5 uses Theorem 4.2. The n-independent formulas are taken from Moses [12]. For $n=0,1,2$, they are the same as for Propositions 5.1 and 5.2. To make the pattern clear, we also give the formulas for $n=3,4,5$.

(1) The 0-independent formula over $\overline{u}$ says that x is greater than anything in $\overline{u}$, and $x<y$.

[3]In a computable structure for a finite relational language, $\leq_0$ is automatically c.e.

(2) The 1-independent formula over $\overline{u}$ says that x is greater than anything in $\overline{u}$, and y is the successor of x.

(3) The 2-independent formula over $\overline{u}$ says that x is greater than anything in $\overline{u}$, and x is a 1-limit (i.e., first in its copy of ω).

(4) The 3-independent formula over $\overline{u}$ says that x is greater than anything in $\overline{u}$, and x and y are 1-limits, where y is the next one after x.

(5) The 4-independent formula over $\overline{u}$ says that x is greater than anything in $\overline{u}$, and x is a 2-limit (i.e., first in its copy of ω^2).

(6) The 5-independent formula over $\overline{u}$ says that x is greater than anything in $\overline{u}$, and x and y are 2-limits, where y is the next one after x.

Remarks

(1) Using Proposition 5.3 or Proposition 5.4, we may obtain the result of Moses [12] saying that for each n, there is a linear ordering $\mathcal{B}$ such that $D_n(\mathcal{B})$ is computable, but $D_{n+1}(\mathcal{B})$ is not. We take N such that $2N-1$, or $2N$, is greater than n, and we choose sets C_k such that for $k \leq n$, C_k is computable, but C_{n+1} is not computable. We get orderings $\mathcal{B}$ of types ω^N and $\omega^N \cdot \eta$ such that $D_n(\mathcal{B})$ is computable, but $D_{n+1}(\mathcal{B})$ is not.

(2) Using Proposition 5.5, we may obtain the result of Chisholm and Moses [6] saying that there is a linear ordering $\mathcal{B}$ such that $D_n(\mathcal{B})$ is computable for all n, but $D^c(\mathcal{B})$ is not computable. We choose an ω-table $(C_n)_{n\in\omega}$ such that C_n is computable for all n, but $\oplus_n C_n$ is not computable. For example, we may let $C_n = K \cap (n+1)$, where K is the halting set. We get orderings $\mathcal{B}$ of types ω^ω and $\omega^\omega \cdot \eta$ such that for all n, $D_n(\mathcal{B})$ is computable, but $D^c(\mathcal{B}) \equiv_T K$.

6. A special case of Theorem 4.1

We begin by proving Theorem 4.1 in the case where $N = 3$. Here is the statement.

Theorem 6.1. *Let $\mathcal{A}$ be a structure for a finite relational language. Suppose $D_2(\mathcal{A})$ is computable, $\leq_1$ is c.e., and for each tuple of variables $\overline{u}$, and each $n < 3$, we can effectively find a formula that is n-independent over $\overline{u}$. Then for any 3-table $(C_n)_{n<3}$, there exists $\mathcal{B} \cong \mathcal{A}$ such that for $n < 3$, $D_n(\mathcal{B}) \equiv_T C_n$.*

Proof. We use a tree construction here, in hopes of making as much as possible look familiar. The requirements are at three different levels, with information coming from the sets C_0, C_1 and C_2. We determine stage s approximations of these sets in such a way that there are "true" stages—in which the information in the approximations is all correct. One feature of the construction is that requirements at lower levels are allowed to crowd in front of those at higher levels, so the priorities change as the construction proceeds.

Let B be an infinite computable set of constants, for the universe of $\mathcal{B}$. For $n \in \{0, 1\}$, let $(\psi_k^n)_{k\in\omega}$ be a computable list of the Σ_n sentences in the language $L(\mathcal{A}) \cup B$. We may suppose that the constants of ψ_k^n are among the first k. We shall determine a 1-1 function F from B onto $\mathcal{A}$, and let $\mathcal{B}$ be the structure induced on B so that $\mathcal{B} \cong_F \mathcal{A}$.

Requirements

We group the requirements as follows.

R_{3k}: Code $\chi_{C_0}(k)$ and decide whether $\psi_k^0 \in D_0(\mathcal{B})$.

R_{3k+1}: Code $\chi_{C_1}(k)$ and decide whether $\psi_k^1 \in D_1(\mathcal{B})$.

R_{3k+2}: Code $\chi_{C_2}(k)$ and put into $ran(F)$, $dom(F)$ the k^{th} element of $\mathcal{A}$, B, respectively.

For each m, requirement R_m corresponds to the pair (n, k), where $m = 3k + n$ for $n < 3$ and $k \in \omega$.

Labels

Let $\mathcal{F}$ be the set of finite partial 1-1 functions from B to $\mathcal{A}$. At each stage in the construction, we determine a *label* $\ell = (p, e, w)$, where $p \in \mathcal{F}$, e is a finite set of sentences in the language $L(\mathcal{A}) \cup B$, each of which is Σ_n or Π_n for some $n < 3$, w is a function defined on a finite set of pairs (n, k), where if $(n, k) \in dom(w)$ and $k' < k$, then $(n, k') \in dom(w)$, and the following conditions are satisfied:

(1) If $\phi \in e$ is a Σ_n sentence for $n \in \{1,2\}$, then e also includes a Π_{n-1} sentence witnessing the truth of ϕ,

(2) For each pair $(n,k) \in dom(w)$, $w(n,k) = (\theta, u)$, where θ, called the *coding sentence*, is Π_n, and u, called the *action*, is 0 or 1; moreover, if the action is 0 then $\theta \in e$, and if the action is 1 then $neg(\theta) \in e$,

(3) If $(n,k) \in dom(w)$, then e includes ψ^n_k or $neg(\psi^n_k)$,

(4) $dom(p)$ includes the constants appearing in the sentences of e, and p makes these sentences true in $\mathcal{A}$.

For a label $\ell = (p,e,w)$, we may write $p(\ell)$, $E(\ell)$, $w(\ell)$ for p, e, w, respectively. We may write $E_n(\ell)$ for the restriction of $E(\ell)$ to Σ_n and Π_n sentences, and we may write $w_n(\ell)$ for the restriction of $w(\ell)$ to pairs (n,k).

We say that $\ell = (p,e,w)$ *satisfies* R_{3k+n} *using information* u, where u is 0 or 1, provided that:

(1) $w(\ell)(n,k) \downarrow$ with action u,

(2) If $n = 2$, then $p(\ell)$ includes the first k constants from B in its domain, and the first k elements of $\mathcal{A}$ in its range.

Relations on labels

Let L be the set of labels. We have binary relations $\leq_0$, $\leq_1$, $\subseteq$ on tuples from $\mathcal{A}$. We extend the relations first to $\mathcal{F}$, and then to L. For $p,q \in \mathcal{F}$, we have $p \leq_0 q$, $p \leq_1 q$, or $p \subseteq q$ if $dom(p) \subseteq dom(q)$ and the relation $\leq_0$, $\leq_1$, or $\subseteq$ holds between $ran(p)$ and $ran(q)$. For $\ell, \ell' \in L$ and $n \in \{0,1\}$, we let $\ell \leq_n \ell'$ if $p(\ell) \leq_n p(\ell')$, $E_j(\ell) \subseteq E_j(\ell')$ for $j \leq n$, and $w(\ell)(j,k) = w(\ell')(j,k)$, whenever $w(j,k) \downarrow$, for $j \leq n$. We let $\ell \subseteq \ell'$ if $p(\ell) \subseteq p(\ell')$, $E(\ell) \subseteq E(\ell')$, and $w(\ell) \subseteq w(\ell')$.

Approximations

Let i be an index such that $C_1 = W_i^{C_0}$ and $C_2 = W_i^{C_1}$ (Posner's trick). We describe a construction in stages, using oracle C_0. We must say what information is used at stage s. First, we define, for arbitrary $\sigma \in 2^{<\omega}$, a specific restriction of W_i^σ, denoted by σ^+. The idea is that if σ_s is the restriction of the characteristic function of a set X to s, then for each n, there should be a unique s such that σ_s^+ is the restriction of W_i^X to n. We define σ^+ by induction on the length of σ.

Definition 6.1. *For length* 0, *we let* $\emptyset^+ = \emptyset$. *Let* σ *have length* s *and let* ρ *be an extension of length* $(s+1)$. *Suppose that we have determined* σ^+, *which has length* n, *and we need to determine* ρ^+.

(1) *If there exists* $k < n$ *such that* $k \in W^{\rho}_{i,s+1} - W^{\sigma}_{i,s}$, *then* $\rho^+ = (\sigma^+|k)1$.
(2) *If there is no such* k, *then* ρ^+ *is the restriction to* $(n+1)$ *of the characteristic function of* $W^{\rho}_{i,s+1}$.

Definition 6.2. *The* stage s version *of* C_n, *denoted by* $C_{n,s}$, *is as follows for* $n = 0, 1, 2$*:*

(1) $C_{0,s} = \chi_{C_0}|s$,
(2) $C_{1,s} = C^+_{0,s}$,
(3) $C_{2,s} = C^+_{1,s}$.

Note that the lengths of $C_{0,s}$, $C_{1,s}$, and $C_{2,s}$ need not match. In general, $C_{0,s}$ will be longer than $C_{1,s}$, which will be longer than $C_{2,s}$.

Information tree

As usual in a tree construction, we define a tree of sequences, each carrying information for a finite set of requirements, and also assigning priorities to those requirements. We let T consist of all finite sequences

$$(n_0, k_0)u_0(n_1, k_1)u_1 \ldots (n_r, k_r)u_r,$$

where for $t \leq r$, u_t is 0 or 1, $n_t < 3$, and

(1) if $t \neq t'$, then $(n_t, k_t) \neq (n_{t'}, k_{t'})$,
(2) if $k < k_t$, then there exists $t' < t$ with $n_{t'} = n_t$ and $k_{t'} = k$.

Step-by-step construction

For each stage s, we determine $\delta_s \in T$, recording the stage s information, as follows. We let $\delta_0 = \emptyset$. We suppose that for $s' \leq s$, $\delta_{s'}$ records, in some order, the triples carrying information from $C_{0,s'}$, $C_{1,s'}$ and $C_{2,s'}$. Let $r \leq s$ be greatest such that $C_{1,r} \subseteq C_{1,s+1}$ and $C_{2,r} \subseteq C_{2,s+1}$, say $C_{2,r}$ has length m. Let $t \leq s$ be greatest such that $C_{1,t} \subseteq C_{1,s+1}$, say $C_{1,t}$ has length n. It should be the case that $C_{0,s+1}$ extends $C_{0,s}$ to s, $C_{1,s+1}$ extends $C_{1,t}$ to n, and $C_{2,s+1}$ extends $C_{2,r}$ to m. We let δ_r be an initial segment of δ_{s+1}. Next, we put any further triples of form $(1,x)u$ or $(0,x)u$ from δ_t, in order. After that, we put any further triples of the form $(0,x)u$ from δ_s, in order. Finally, we put $(0,s)C_{0,s+1}(s)$, $(1,n)C_{1,s+1}(n)$, and $(2,m)C_{2,s+1}(m)$, in this order.

For each s, we determine a label ℓ_s, satisfying the requirements with information in δ_s. This is done in such a way that the following conditions are maintained.

Preservation

Let $s' < s$.

(1) Then $\ell_{s'} \leq_0 \ell_s$. Moreover, for $k < s'$, the coding sentence for $(0,k)$ depends only on $C_0|k$, not on $C_0(k)$.
(2) If $C_{1,s'} \subseteq C_{1,s}$, then $\ell_{s'} \leq_1 \ell_s$. If $C_{1,s'}|k \subseteq C_{1,s}$ and $w(\ell_{s'})(1,k)\downarrow$, then $w(\ell_s)(1,k)\downarrow$, and the coding sentence is the same.
(3) If $C_{1,s'} \subseteq C_{1,s}$ and $C_{2,s'} \subseteq C_{2,s}$, then $\ell_{s'} \subseteq \ell_s$. If $C_{2,s'}|k \subseteq C_{2,s}$ and $w(\ell_{s'})(2,k)\downarrow$, then $w(\ell_s)(2,k)\downarrow$, and the coding sentence is the same.

Now, we define ℓ_s.

Construction of ℓ_s

Stage 0. Let $\ell_0 = (p_0, e_0, w_0)$, where $p = \emptyset$, $e_0 = \emptyset$, and $w_0 = \emptyset$.

Stage $(\mathbf{s+1})$. We pass from stage s to stage $(s+1)$ as follows. Let r, t, m and n be as above—$r \leq s$ is greatest such that $C_{1,r} \subseteq C_{1,s+1}$ and $C_{2,r} \subseteq C_{2,s+1}$, and m is the length of $C_{2,r}$; $t \leq s$ is greatest such that $C_{1,t} \subseteq C_{1,s+1}$, and n is the length of $C_{1,t}$. Let $r' \geq r$ be first, if any, such that $C_{1,r'} \subseteq C_{1,s+1}$ and $w(\ell_{r'})(2,m)\downarrow$, and let $t' \geq t$ be first, if any, such that $w(\ell_{t'})(1,n)\downarrow$.

Case 1. Suppose t' and r' are both undefined. By the preservation conditions, we have
$$\ell_r \subseteq \ell_t \leq_1 \ell_s.$$
Say p_s maps $\overline{d}$ to $\overline{c}$. Let $\overline{u}$ correspond to $\overline{c}$, and let $\theta_0(\overline{u},\overline{x})$ be 0-independent over $\overline{u}$. Let $\overline{b}$ be a tuple of new constants corresponding to $\overline{x}$. Then $\theta_0(\overline{d},\overline{b})$ is the coding sentence for $(0,s)$. This sentence depends only on $C_{0,s}$, not on $C_{0,s+1}(s)$. We let $w_{0,s+1}$ map $(0,s)$ to $(\theta_0(\overline{d},\overline{b}),u)$, for $u = C_{0,s+1}(s)$. We extend p_s to q, taking $\overline{b}$ to $\overline{a}$, where $\mathcal{A} \models \theta_0(\overline{c},\overline{a})$ just in case $u = 0$. We let $E_{0,s+1}$ extend $E_{0,s}$, adding $\theta(\overline{d},\overline{b})$ or $neg(\theta(\overline{d},\overline{b}))$, and ψ^0_s or $neg(\psi^0_s)$, where q makes these true in $\mathcal{A}$.

Since $p_t \leq_1 q$, there exists $q' \supseteq p_t$ such that $q \leq_0 q'$. Say q' maps $\overline{d}'$ to $\overline{c}'$. Let $\overline{u}'$ correspond to $\overline{c}'$, and let $\theta_1(\overline{u}', \overline{x}')$ be 1-independent over $\overline{u}'$. Let $\overline{b}'$ be a tuple of new constants corresponding to $\overline{x}'$. Then $\theta_1(\overline{d}', \overline{b}')$ is the coding sentence for $(1, n)$. We let $w_1(\ell_{s+1})$ extend $w_1(\ell_t)$, taking $(1, n)$ to $(\theta_1(\overline{d}', \overline{b}'), u')$, where $u' = C_{1,s+1}(n)$. We extend the current q', taking $\overline{b}'$ to some $\overline{a}'$, where $\mathcal{A} \models \theta_1(\overline{c}', \overline{a}')$ just in case $u' = 0$. We let $E_1(\ell_{s+1})$ extend $E_1(\ell_t)$, adding $\theta(\overline{d}', \overline{b}')$ or $neg(\theta(\overline{d}', \overline{b}'))$, and ψ^1_s or $neg(\psi^1_s)$, where q' makes these true in $\mathcal{A}$. If $\mathcal{A} \models neg(\theta_1(\overline{c}', \overline{a}'))$, then we extend q' further to include in the range elements witnessing the truth of $neg(\theta_1(\overline{c}', \overline{a}'))$.

Say at this point q' maps $\overline{d}''$ to $\overline{c}''$. Let $\overline{u}''$ correspond to $\overline{c}''$, and let $\theta_2(\overline{u}'', \overline{x})$ be 2-independent over $\overline{u}''$. Let $\overline{b}''$ be a tuple of new constants corresponding to $\overline{x}$. Then $\theta_2(\overline{d}'', \overline{b}'')$ is the coding sentence for $(2, m)$. We let $w_2(\ell_{s+1})$ extend $w_2(\ell_r)$, mapping $(2, m)$ to $(\theta_2(\overline{d}'', \overline{b}''), u'')$, for $u'' = C_{2,s+1}(m)$. We extend the current q', mapping $\overline{b}''$ to some $\overline{a}''$, where $\mathcal{A} \models \theta_2(\overline{c}'', \overline{a}'')$ just in case $u'' = 0$. If $\mathcal{A} \models neg(\theta_2(\overline{d}'', \overline{b}''))$, then we extend q' further to include in the range elements witnessing the truth of $neg(\theta_2(\overline{d}'', \overline{b}''))$. Finally, we extend q' to include the first $(s+1)$ elements of $\mathcal{A}$ in the range and the first $(s+1)$ elements of B in the domain. We now have ℓ_{s+1}, where p_{s+1} is the final version of q', and the other components are as described.

Case 2. Suppose t' and r' are defined, where $C_{1,t'}$ and $C_{1,s+1}$ differ on n, and $C_{2,r'}$ and $C_{2,s+1}$ differ on m. We have

$$C_{2,r} \subseteq C_{2,r'} \subseteq C_{2,t} \text{ and } C_{1,t} \subseteq C_{1,t'} \subseteq C_{1,s}.$$

By the preservation conditions, we have

$$\ell_r \subseteq \ell_{r'} \subseteq \ell_t \leq_1 \ell_{t'} \leq_1 \ell_s.$$

As in Case 1, we extend p_s to q, determine a coding sentence for $(0, s)$, and extend $E_{0,s}$ to $E_{0,s+1}$.

Since $p_{t'} \leq_1 q$, there exists $q' \supseteq p_{t'}$ such that $q \leq_0 q'$. Say

$$w(\ell_{t'})(1, n) = (\theta_1(\overline{d}', \overline{b}'), 0),$$

where $p_{t'}$ and q' make $\theta_1(\overline{d}', \overline{b}')$ true in $\mathcal{A}$. We may suppose that $dom(p_t) \subseteq \overline{d}$. By 1-independence, there exists $q'' \supseteq q'|\overline{d}$ such that $q' \leq_0 q''$ and q'' makes $neg(\theta_1(\overline{d'}, \overline{b'}))$ true in $\mathcal{A}$. We extend q'' to include witnesses for the existential sentence. Since $p_t \leq_1 q''$, there exists $q''' \supseteq p_t$ such that $q'' \leq_0 q'''$. Then q''' makes $neg(\theta_1(\overline{d}', \overline{b}'))$ true in $\mathcal{A}$. We let $w_1(\ell_{s+1})$ extend $w_1(\ell_t)$, taking $(1, n)$ to $(\theta_1(\overline{d}', \overline{b}'), 1)$. We let $E_{1,s+1}$ extend $E_{1,t}$, adding $neg(\theta_1(\overline{d}',\overline{b}'))$, and also adding ψ^1_n or $neg(\psi^1_n)$, whichever is made true in $\mathcal{A}$ by q'''.

Since $p_{r'} \leq_1 q'''$, there exists $q'''' \supseteq p_{r'}$ such that $q''' \leq_1 q''''$. Say

$$w(\ell_{r'})(2,m) = (\theta_2(\overline{d}'',\overline{b}''),0),$$

where $p_{r'}$ and q'''' make $\theta_2(\overline{d}'',\overline{b}'')$ true in $\mathcal{A}$. We may suppose that $dom(p_r) \subseteq \overline{d}''$. By 2-independence, there exists $q''''' \supseteq q'''|\overline{d}''$ such that $q''' \leq_1 q'''''$ and q''''' makes $neg(\theta_2(\overline{d}'',\overline{b}''))$ true in $\mathcal{A}$. We extend q''''' to include witnesses for the Σ_2 sentence. Finally, we extend it to include the first $(s+1)$ elements in the domain and range. The resulting function is p_{s+1}. We let $w_2(\ell_{s+1})$ extend $w_2(\ell_r)$, taking $(2,m)$ to $(\theta_2(\overline{d}'',\overline{b}''),1)$. We let $E_{2,s+1}$ extend $E_{2,r}$, adding $neg(\theta_2(\overline{d}'',\overline{b}''))$, and also adding ψ^2_m or $neg(\psi^2_m)$, whichever is made true in $\mathcal{A}$ by p_{s+1}. We have determined all of ℓ_{s+1}.

There are two other possible cases, in which just one of t' and r' is defined. We omit the discussion of these cases, as the ideas are given in the first two cases.

We have described the sequence of steps, computable in C_0. For each n, let $s(n)$ be the first s such that $C_{1,s} = \chi_{C_1}|n$. For each k, let $r(k)$ be the first r such that $C_{2,r} = \chi_{C_2}|k$ and $C_{1,r} \subseteq \chi_{C_1}$. Let p_k be the first component of $\ell_{r(k)}$. The functions p_k, $k \in \omega$, form a chain, and $F =_{def} \cup_k p_k$ is a 1-1 function from B onto $\mathcal{A}$. Let $\mathcal{B}$ be the induced structure.

Lemma 6.2. $D_0(\mathcal{B}) \leq_T C_0$

Proof. We have

$$D_0(\mathcal{B}) \leq_T D^c(\mathcal{B}) \cap (\Sigma_0 \cup \Pi_0) = \cup_{k\in\omega} E_{0,r(k)} = \cup_{s\in\omega} E_{0,s}.$$

The last of these is computable in C_0.

Lemma 6.3. $C_0 \leq_T D_0(\mathcal{B})$

Proof. We describe an inductive procedure for determining $C_0 \cap s$, using $D_0(\mathcal{B})$. Given $C_0 \cap s$, we can find ℓ_s and the coding sentence for $(0,s)$, which will never change. Using $D_0(\mathcal{B})$, we determine whether the sentence is true, and then we know $C_0 \cap (s+1)$.

Lemma 6.4. $D_1(\mathcal{B}) \leq_T C_1$

Proof. Using C_1, we can find $s(n)$ for every n. We have

$$D_1(\mathcal{B}) \leq_T D^c(\mathcal{B}) \cap (\Sigma_1 \cup \Pi_1) = \cup_{k\in\omega} E_{1,r(k)} = \cup_{n\in\omega} E_{1,s(n)}.$$

Since $(s(n))_{n\in\omega}$ is computable in C_1, the last of these is c.e. relative to C_1.

Lemma 6.5. $C_1 \leq_T D_1(\mathcal{B})$

Proof. We describe an inductive procedure for determining $C_1 \cap n$, using C_0 and $D_1(\mathcal{B})$. Using C_0 and $C_1 \cap n$, we can find $s(n)$ and the coding sentence for $(1, n)$. After $C_1 \cap n$ is correct, and a coding sentence for $(1, n)$ has been chosen, the coding sentence will never change. Using $D_1(\mathcal{B})$, we determine whether the sentence is true, and then we know $C_1 \cap (n + 1)$.

Lemma 6.6. *We have $F \leq_T C_2$. Hence, $D_2(\mathcal{B}) \leq_T C_2$.*

Proof. Using C_1 and C_2, we can find $r(k)$ for every k, and from this, we can determine F. Given F and using $D_2(\mathcal{A})$, which is computable, we can determine $D_2(\mathcal{B})$.

Lemma 6.7. $C_2 \leq_T D_2(\mathcal{B})$

Proof. We describe an inductive procedure for determining $C_2 \cap k$, using $D_2(\mathcal{B})$ and C_1, where $C_1 \leq_T D_1(\mathcal{B}) \leq_T D_2(\mathcal{B})$. Using C_1, and knowing $C_2 \cap k$, we can find $r(k)$ and the coding sentence for $(2, k)$. After $C_2 \cap k$ is correct and a coding sentence for $(2, k)$ has been chosen, at a stage where the information about C_1 is correct, the coding sentence for $(2, k)$ will remain the same at all stages where the information about C_1 is correct. Using $D_2(\mathcal{B})$, we determine whether the sentence is true, and then we know $C_2 \cap (k + 1)$.

This completes the proof of Theorem 6.1.

7. Proofs of Theorems 4.1 and 4.2

We shall use some general machinery. We begin by describing the basic setting. By a *tree*, we mean a set P of nonempty finite sequences, closed under nonempty initial segments. We say that the tree is *on* L if the terms in the finite sequences come from L. A *path* through P is a function π on ω such that for all $n \geq 1$, $\pi|n \in P$. We consider a tree P on a set L. Let $\alpha \leq \omega$.

Definition 7.1. *An* enumeration function *on* $\alpha \times L$ *is a function* E *mapping elements of* $\alpha \times L$ *to finite subsets of* ω. *We write* $E_n(\ell)$ *for* $E(n, \ell)$.

Definition 7.2. *Let* D *be the set of all finite sets* $d \subseteq \alpha \times \omega$ *such that if* $(n, x) \in d$ *and* $y < x$, *then* $(n, y) \in d$. *A* coding function *on* L *is a function* w *such that* $w(\ell)$ *is a function from some* $d \in D$ *to* $\omega \times \{0, 1\}$. *If* $w(\ell)(n, x) = (b, u)$, *then we call* b *the* coding witness *and we call* u *the* action. *We write* $w_n(\ell)$ *for the restriction of* $w(\ell)$ *to pairs with first component* n.

Let $\hat{\ell} \in L$, where $w(\hat{\ell}) = \emptyset$. We consider a tree P of finite sequences from L, all with first term $\hat{\ell}$. For a path $\pi = \hat{\ell}\ell_1\ell_2\ldots$, and $n \in \omega$, let $E_n(\pi) = \cup_k E_n(\ell_k)$, and let $w(\pi) = \cup_k w(\ell_k)$.

Object of the construction

Let L, w, $\hat{\ell}$, E and P be as described above, where L and P are c.e. Recall that $\alpha \leq \omega$. Let $(C_n)_{n<\alpha}$ be an α-table. We want to produce a path π such that:

(1) $E_n(\pi)$ is c.e. in C_n, uniformly in n, for $n < \alpha$,

(2) For $n < \alpha$ and $x \in \omega$, $w(\pi)(n, x) \downarrow$ with action $C_n(x)$,

(3) For $x \in \omega$ and $n < \alpha$, we can effectively determine the coding witness in $w(\pi)(n, x)$, using $C_n \cap x$ and C_{n-1} if $n > 0$, or just $C_n \cap x$ if $n = 0$.

The conditions guaranteeing success of the construction involve a further function $b : L \to \omega$, and binary relations $\leq_n$ on L, for $n < \alpha$. The function b chooses coding witnesses for pairs $(0, x)$. The relations $\leq_n$ are used in "preservation conditions". We assume that if $\ell \leq_n \ell'$, then $w_m(\ell) \subseteq w_m(\ell')$ and $E_m(\ell) \subseteq E_m(\ell')$ for all $m \leq n$. For finite α, say $\alpha = N+1$, we may write $\subseteq$ for $\leq_N$. For $\alpha = \omega$, we may write $\subseteq$ for the intersection of all $\leq_n$. We suppose that if $\sigma = \ell_0\ell_1 \ldots$ is in P, then $\ell_k \subseteq \ell_{k+1}$ for all $k \in \omega$. We shall make a further assumption involving the relations $\leq_n$—saying that "pictures" can be "completed".

For $n < \alpha$, the pairs (n, x) represent the requirements for coding the set C_n. If $n > 0$, then the strategy for requirement (n, x)—viewed at level n, where we have oracle C_{n-1}—is to designate a coding witness b, and take action 0 until/unless x appears in C_n, then switch to action 1, dropping any coding witnesses that have been designated for requirements (n, y), where $y > x$. At level $(n + 1)$, where we have oracle C_n, there is no change in the action. If $n = 0$, and x is first such that $w(\ell)(0, x) \uparrow$ for the current ℓ, then the strategy for requirement $(0, x)$, at level 1, is to use $b(\ell)$ as a coding witness, and take action $C_0(x)$. The following definition indicates the strategy for requirements (n, x) for successive steps at levels n and $(n + 1)$.

Definition 7.3. *Suppose $\ell, \ell^* \in L$. We say that ℓ^** follows ℓ with action u on (n, x) *provided that:*

(1) *$w(\ell^*)$ and $w(\ell)$ agree on pairs (n, y) for $y < x$,*

(2) *if $y > x$, then $w(\ell^*)(n, y) \uparrow$,*

(3) *$w(\ell^*)(n, x)$ has form (b, u), where one of the following holds*

   (*a*) *x is first such that $w(\ell)(n, x) \uparrow$, and if $n = 0$, then $b = b(\ell)$,*

   (*b*) *$n \geq 1$, and for some b,*

   $$w(\ell)(n, x) = (b, 0), \text{ while } w(\ell^*)(n, x) = (b, 1).$$

   (Alternative (b) may happen at level n, not at level $(n + 1)$.)

The next definition represents information passed down through a sequence of levels.

Definition 7.4. *A* picture *is a triple $c = (\sigma\ell^0, \tau, a)$, where $\sigma\ell^0 \in P$, and τ and a satisfy one of the following:*

(1) *$\tau = \emptyset$, and $a : \{(n, x)\} \to \{0, 1\}$, where $n < \alpha$ and x is first such that $w(\ell^0)(n, x) \uparrow$,*

(2) *τ has form $n_0\ell^1 \dots n_{k-1}\ell^k$ and*

$$a : \{(n_0, x_0), \dots, (n_{k-1}, x_{k-1}), (n_k, x_k)\} \to \{0, 1\},$$

where

(a) $n_k < \dots < n_0 < \alpha$,

(b) $\ell^0 \leq_{n_0} \dots \leq_{n_{k-1}} \ell^k$,

(c) *for $i \leq k$, x_i is first such that $w(\ell^i)(n_i, x_i) \uparrow$,*

(d) *for $i < k$, either*

(i) $w(\ell^{i+1})(n_i, x_i) \uparrow$, *or*

(ii) *$w(\ell^{i+1})(n_i, x_i)$ has action 0, while $a(n_i, x_i) = 1$.*

Next, we say how the information in a picture is used.

Definition 7.5. *Let $c = (\sigma\ell^0, \tau, a)$ be a picture.*

***Case* 1**. *Suppose $\tau = \emptyset$, and $a(n, x) = u$. Then ℓ* completes *c provided that:*

(1) $\sigma\ell^0\ell \in P$,

(2) *if $m > n$, then $w_m(\ell) = w_m(\ell^0)$,*

(3) *ℓ follows ℓ^0 with action u on (n, x).*

***Case* 2**. *Suppose τ has form $n_0\ell^1 \dots n_{k-1}\ell^k$, and for $i \leq k$, $a(n_i, x_i) = u_i$. Then ℓ* completes *c provided that:*

(1) $\sigma\ell^0\ell \in P$,

(2) *if $0 < i \leq k$, then $\ell^i \leq_{n_i} \ell$,*

(3) *if $i \leq k$, then ℓ follows ℓ^i with action u_i on (n_i, x_i),*

(4) *if $i < k$, then ℓ follows ℓ^{i+1} with action u_i on (n_i, x_i),*

(5) *if $m > n_0$, then $w_m(\ell) = w_m(\ell^0)$,*

(6) *if $n_{i+1} < m < n_i$, then $w_m(\ell) = w_m(\ell^i)$.*

Definition 7.6. *Suppose $\alpha \leq \omega$. A* coding and enumerating α-system *is a structure*

$$(L, w, E, b, \hat{\ell}, P, (\leq_n)_{n<\alpha}),$$

where w is a coding function on L, E is an enumeration function on $\alpha \times L$, $\hat{\ell} \in L$ with $w(\hat{\ell}) = \emptyset$, $(\leq_n)_{n<\alpha}$ is a family of binary relations on L, P is a tree on L, consisting of finite sequences $\ell_0\ell_1\ell_2 \ldots \ell_r$ such that $\ell_0 = \hat{\ell}$ and for $i < r$, $\ell_i \subseteq \ell_{i+1}$ (where $\subseteq$ is the intersection of all $\leq_n$), and the following conditions hold:

(1) *$\leq_n$ is reflexive and transitive,*
(2) *if $\ell \leq_{n+1} \ell'$, then $\ell \leq_n \ell'$,*
(3) *if $\ell \leq_n \ell'$, then $E_n(\ell) \subseteq E_n(\ell')$ and $w_n(\ell) \subseteq w_n(\ell')$,*
(4) *every picture has a completion, where these are defined above.*

Note. In the presence of Condition 2, Condition 3 says that if $\ell \leq_n \ell'$, then for $m \leq n$, $E_m(\ell) \subseteq E_m(\ell')$ and $w_m(\ell) \subseteq w_m(\ell')$.

Here is the metatheorem for coding and enumerating α-systems, where $\alpha \leq \omega$. For the proof, see [11].

Metatheorem. *Suppose $(L, w, E, b, \hat{\ell}, P, (\leq_n)_{n<\alpha})$ is a coding and enumerating α-system, where L and P are c.e., w and b are partial computable, and the relations $\leq_n$ are c.e., uniformly in n. Let $(C_n)_{n<\alpha}$ be an α-table. Then P has a path π such that for $n < \alpha$, $E_n(\pi)$ is c.e. relative to C_n, uniformly in n, and for $(n, x) \in \alpha \times \omega$, $w(\pi)(n, x) \downarrow$ with action $\chi_{C_n}(x)$, and we have an effective procedure for determining the coding witness in $w(\pi)(n, x)$, using $C_n \cap x$ and (if $n \geq 1$) C_{n-1}.*

We are ready to prove Theorems 4.1 and 4.2. Recall the statements.

Theorem 4.1. *Suppose $D_N(\mathcal{A})$ is computable, and the relations $\leq_n$ on tuples (standard back-and-forth relations) are c.e. for $n < N$. Suppose also that for each tuple $\overline{u}$ of variables, and each $n \leq N$, we can effectively find a formula that is n-independent over $\overline{u}$. Then for any $(N+1)$-table $(C_n)_{n\leq N}$, there exists $\mathcal{B} \cong \mathcal{A}$ such that $D_n(\mathcal{B}) \equiv_T C_n$ for all $n \leq N$.*

Theorem 4.2. *Suppose $D^c(\mathcal{A})$ is computable, and the relations $\leq_n$ (standard back-and-forth relations again) are c.e., uniformly in n, for $n \in \omega$. Suppose also that for each tuple $\overline{u}$, and each n, we can effectively find a formula that is n-independent over $\overline{u}$. Then for any ω-table*

$(C_n)_{n<\omega}$, there exists $\mathcal{B} \cong \mathcal{A}$ such that $D_n(\mathcal{B}) \equiv_T C_n$, uniformly in n, for $n \in \omega$.

Proofs of Theorems 4.1 *and* 4.2. For simplicity, we suppose $\mathcal{A}$ is a structure for a finite relational language. We define a coding and enumerating α-system $(L, w, E, b, \hat{\ell}, P, (\leq_n)_{n<\alpha})$, where for Theorem 4.1, $\alpha = N + 1$, and for Theorem 4.2, $\alpha = \omega$. Let B be an infinite computable set of constants, for the universe of $\mathcal{B}$. Let ψ_k^n be the k^{th} Π_n sentence in the language $L(\mathcal{A}) \cup B$.

Let $\mathcal{F}$ be the set of finite 1-1 partial functions from B to $\mathcal{A}$. Let D be the set of all finite sets $d \subseteq \alpha \times \omega$ such that if $(n, x) \in d$ and $y < x$, then $(n, y) \in d$. Let L be the set of triples (p, e, w) with the following features:

(1) $p \in \mathcal{F}$,

(2) w is a function on some $d \in D$, such that for all $(n, x) \in d$, $w(n, x)$ is a pair (θ, u), where the action u is 0 or 1, and the coding sentence θ involves an initial set of constants, different for distinct pairs in d,

(3) The coding sentence of $w(n, x)$ has the form $\theta(\overline{d}, \overline{b})$, where $\overline{d}$ consists of the first few constants, $\overline{b}$ consists of the next few constants, and for any $(m, y) \in d$, if the coding sentence of $w(m, y)$ has fewer constants, these are included in $\overline{d}$, and $\theta(\overline{u}, \overline{x})$ is effectively chosen to be n-independent over a tuple of variables $\overline{u}$, corresponding to $\overline{d}$,

(4) e is a finite set of sentences made true by p in $\mathcal{A}$, such that any Σ_{n+1} sentence in e is witnessed by a Π_n sentence,

(5) If $w(n, x) = (\theta, u)$, then $\theta \in e$ if $u = 0$, and $neg(\theta) \in e$ if $u = 1$,

(6) If $(n, x) \in d$ and ψ is the x^{th} Π_n sentence, then the constants from ψ are in $dom(p)$, and $\psi \in e$ if p makes ψ true, while $neg(\psi) \in e$, if p makes $neg(\psi)$ true.

Let w be the function on L such that if $\ell = (p, e, w)$, then $w(\ell) = w$. Let E be the function on $\alpha \times L$ such that if $\ell = (p, e, w)$, then $E(n, \ell)$ is the set of Σ_n and Π_n sentences in e. Let b be the function on L such that if $\ell = (p, e, w)$, where $dom(p) = \overline{d}$, and $\overline{u}$ is the initial tuple of variables, corresponding to $\overline{d}$, then $b(\ell)$ is $\theta(\overline{d}, \overline{b})$, where $\theta(\overline{u}, \overline{x})$ is the effectively given formula that is 0-independent over $\overline{u}$, and $\overline{b}$ consists of the next few constants after those in $\overline{d}$.

Suppose $\ell = (p, e, w)$ and $\ell' = (q, e', w')$. Let $\ell \leq_n \ell'$ if the following hold:

(1) $p \leq_n q$, except if $\alpha = N+1$ and $n = N$, where we require $p \subseteq q$,

(2) for $m \leq n$, $w_m(\ell) \subseteq w_m(\ell')$ and $E_m(\ell) \subseteq E_m(\ell')$,

(3) if $(m, x) \in dom(w') - dom(w)$, where $m \leq n$, then the coding sentence in $w'(m, x)$ has the form $\theta(\overline{d}, \overline{b})$, where $dom(p) \subseteq \overline{d}$.

Let $\hat{\ell} = (\emptyset, \emptyset, \emptyset)$. Let P consist of the finite sequences $\ell_0\ell_1 \ldots \ell_r$, where $\ell_0 = \hat{\ell}$ and for $i \leq r$,

(1) if $\ell_i = (p_i, e_i, w_i)$, then $dom(p_i)$ includes the first i elements of B, and $ran(p_i)$ includes the first i elements of $\mathcal{A}$,

(2) $\ell_1 \subseteq \ldots \subseteq \ell_r$.

The following lemma is clear from the definitions.

Lemma 7.1. *Let $\pi = \ell_0\ell_1 \ldots$ be a path through P such that $w(\pi)$ is defined on all pairs (n, x) in $\alpha \times \omega$. Say $\ell_i = (p_i, e_i, w_i)$. Then $F = \cup_i p_i$ is a* 1-1 *function from B onto $\mathcal{A}$, and if $\mathcal{B}$ is the structure such that $\mathcal{B} \cong_F \mathcal{A}$, then $E_n(\pi) = D^c(\mathcal{B}) \cap (\Pi_n \cup \Sigma_n)$ for $n \in \omega$.*

We should check the conditions for a coding and enumerating α-system. This is done in the next four lemmas. The first three are clear from the definitions.

Lemma 7.2. *The relation $\leq_n$ is transitive and reflexive.*

Lemma 7.3. *If $\ell \leq_{n+1} \ell'$, then $\ell \leq_n \ell'$.*

Lemma 7.4. *If $\ell \leq_n \ell'$, then $E_n(\ell) \subseteq E_n(\ell')$ and $w_n(\ell) \subseteq w_n(\ell')$.*

Lemma 7.5. *Every picture has a completion.*

Proof. Let $c = (\sigma\ell^0, \tau, a)$ be a picture. We will show that c has a completion ℓ^*. We consider two cases.

Case 1. Suppose $\tau = \emptyset$ and $a(n, x) = u$.

Say $\ell^0 = (p, e, w)$, where p maps $\overline{d}$ to $\overline{c}$. Let $\overline{u}$ be the initial sequence of variables corresponding to $\overline{c}$, and let $\theta(\overline{u}, \overline{x})$ be effectively chosen to be n-independent over $\overline{u}$. Let $\overline{b}$ consist of the next few constants after $\overline{d}$. We let q be an extension of p taking $\overline{b}$ to some $\overline{a}$ such that $\mathcal{A} \models \theta(\overline{c}, \overline{a})$ iff $u = 0$. We extend the current q, if necessary, to include the required elements in the range and domain. Next, we let w' be the extension of w taking (n, x) to $(\theta(\overline{d}, \overline{b}), u)$. Finally, we extend e to e', including $\theta(\overline{d}, \overline{b})$ or $neg(\theta(\overline{d}, \overline{b}))$, and ψ^n_x or $neg(\psi^n_x)$. We extend e' further, if necessary, so that for any Σ_n sentence in e' for $n > 0$, there is a Π_{n-1} sentence witnessing its truth. We choose sentences and extend the current q further, if necessary, so that the final q makes the sentences of e' true in $\mathcal{A}$. Then $\ell^* =_{def} (q, e', w')$ completes c.

Case 2. Suppose $\tau = n_0\ell^1 \ldots n_{k-1}\ell^k$, and for $i \leq k$, $a(n_i, x_i) = u_i$.

Say $\ell^i = (p_i, e_i, w_i)$. Working our way from $i = k$ back to $i = 0$, we determine $q_i \supseteq p_i$, such that for $i < k$, $q_{i+1} \leq_{n_{i+1}} q_i$. We also determine w'_i and e'_i such that for $m \leq n_i$, w'_i is defined on pairs (m, x) the way we want $w(\ell^*)$ to be, and e_i has the Σ_m and Π_m sentences that we want in $E_m(\ell^*)$.

We begin with $i = k$. Suppose p_k maps $\overline{d}$ to $\overline{c}$. Let $\overline{u}$ be the initial sequence of variables, corresponding to $\overline{c}$. Let $\theta(\overline{u}, \overline{x})$ be effectively chosen to be n_k-independent over $\overline{u}$, and let $\overline{b}$ be a sequence of constants appropriate for $\overline{x}$, the next few after $\overline{d}$. Note that if $n_k = 0$, then $\theta(\overline{d}, \overline{b})$ is $b(\ell^k)$. Take $\overline{a}$ such that $\mathcal{A} \models \theta(\overline{c}, \overline{a})$ iff $u_k = 0$. Let q_k extend p_k, taking $\overline{b}$ to $\overline{a}$. We let w'_k agree with w_k on pairs (m, x) for $m \leq n_k$, but we extend it, taking (n_k, x_k) to $(\theta(\overline{d}, \overline{b}), u_k)$. We put into e'_k the Σ_m and Π_m sentences in e_k for $m \leq n_k$. We add $\theta(\overline{d}, \overline{b})$ or $neg(\theta(\overline{d}, \overline{b}))$, and $\psi^{n_k}_{x_k}$ or $neg(\psi^{n_k}_{x_k})$, and for any Σ_m sentence that we have added for $m > 0$, we add a Π_{m-1} sentence witnessing it. We do all of this in such a way that q_k makes the sentences of e'_k true in $\mathcal{A}$.

We next consider $i = n_{k-1}$. There are two possibilities, depending on whether $(n_{k-1}, x_{k-1}) \in dom(w_k)$.

(a) Suppose that $w_k(n_{k-1}, x_{k-1}) \uparrow$.

We proceed much as in Case 1. Since $p_{k-1} \leq_{n_{k-1}} p_k \subseteq q_k$, there exists $q \supseteq p_{k-1}$ such that $q_k \leq_{n_k} q$. (We could take q such that $q_k \leq_{n_{k-1}-1} q$, but we do not need it here.) Say q maps $\overline{d}'$ to $\overline{c}'$. Let $\overline{u}'$ be the initial sequence of variables, corresponding to $\overline{d}'$. Let $\theta'(\overline{u}', \overline{x}')$ be effectively chosen to be n_{k-1}-independent over $\overline{u}'$, and let $\overline{b}'$ be a sequence appropriate for $\overline{x}'$, the next few constants after $\overline{d}'$. Take $\overline{a}'$ such that $\mathcal{A} \models \theta'(\overline{c}', \overline{a}')$ iff $u_{k-1} = 0$. Let q_{k-1} extend q, taking $\overline{b}'$ to $\overline{a}'$. We let w'_{k-1} agree with w'_k on pairs (m, x) for $m \leq n_k$, and with w_{k-1} on pairs (m, x), where $n_k < m \leq n_{k-1}$, but we extend it, taking (n_{k-1}, x_{k-1}) to $(\theta'(\overline{d}', \overline{b}), u_{k-1})$. Let e'_{k-1} contain the Σ_m and Π_m sentences in e'_k for $m \leq n_k$, and further Σ_m and Π_m sentences in e_{k-1} for $m \leq n_{k-1}$. We add $\theta'(\overline{d}', \overline{b}')$ or $neg(\theta'(\overline{d}', \overline{b}'))$, and $\psi^{n_{k-1}}_{x_{k-1}}$ or $neg(\psi^{n_{k-1}}_{x_{k-1}})$, plus witnessing sentences. We do all of this in such a way that q_{k-1} makes the sentences of e'_{k-1} true in $\mathcal{A}$.

(b) Suppose that $w_k(n_{k-1}, x_{k-1}) \downarrow$.

Let $\theta'(\overline{d}', \overline{b}')$ be the coding sentence in $w_k(n_{k-1}, x_{k-1})$. The action is 0, while $a(n_{k-1}, x_{k-1}) = 1$. We have $dom(p_{k-1}) \subseteq \overline{d}'$, and q_k makes $\theta'(\overline{d}', \overline{b}')$ true in $\mathcal{A}$. Take q' agreeing with q_k on $\overline{d}'$, such that q' makes $neg(\theta'(\overline{d}', \overline{b}'))$ true in $\mathcal{A}$, and $q_k \leq_{n_k} q'$. (We could take q' such that $q_k \leq_{n_{k-1}-1} q'$, but we do not need it here.) We extend the current q', if necessary, to include witnesses for $neg(\theta'(\overline{d}', \overline{b}'))$. Since $p_{k-1} \leq_{n_{k-1}} q'$, there exists $q_{k-1} \supseteq p_{k-1}$ such that $q' \leq_{n_{k-1}-1} q_{k-1}$. Using this, we obtain that q_{k-1} makes $neg(\theta'(\overline{d}', \overline{b}'))$ true. We determine w_{k-1} and e_{k-1} as in (a).

We continue in this way until we arrive at q_0, w'_0 and e'_0. We let $q^* \supseteq q_0$, including the required elements in the domain and range. We let $w^* \supseteq w'_0$, agreeing with w_0 on any pair (m, x) for $m > n_0$. We let $e^* \supseteq e'_0$, adding any further Σ_m and Π_m sentences, for $m > n_0$, from e_0. Then $\ell^* =_{def} (q^*, e^*, w^*)$ is the desired completion.

We are in a position to apply the metatheorem. We get a special path $\pi = \hat{\ell}\ell_1\ell_2\ldots$. For all $n < \alpha$ and $x \in \omega$, $w(\pi)(n, x) \downarrow$. Say $\ell_i = (p_i, e_i, w_i)$, and let $F = \cup_i p_i$. By Lemma 7.1, F is a 1-1 mapping from B onto $\mathcal{A}$. Let $\mathcal{B}$ be the induced structure, with $\mathcal{B} \cong_F \mathcal{A}$. Again, by Lemma 7.1, for all $n < \alpha$, $E_n(\pi) = D^c(\mathcal{B}) \cap (\Pi_n \cup \Sigma_n)$. Since $E_n(\pi)$ is c.e. relative to C_n, and ψ^n_k or $neg(\psi^n_k)$ must appear, $D_n(\mathcal{B}) \leq_T C_n$. For all $n < \alpha$ and all $x \in \omega$, $w(\pi)(n, x)$ has action $C_n(x)$, and we can determine the coding sentence, using C_{n-1} and $C_n \cap x$ if $n > 0$, or using $C_0 \cap x$ if $n = 0$. Knowing the coding sentence, we can determine whether $x \in C_n$, using $D_n(\mathcal{B})$. Therefore, for all $n < \alpha$, $C_n \leq_T D_n(\mathcal{B})$. The procedures are all uniform in n.

8. Open problems

Here, we mention some open problems. First, we would like versions of Theorems 4.1 and 4.2 that apply to a larger collection of structures.

Problem 4. *Weaken the definition of n-independent formulas so that Theorems 4.1 and 4.2 still hold, but can be applied to Boolean algebras and other structures. Goncharov* [7] *showed that for every n, there is a Boolean algebra that is n-decidable, but not $(n+1)$-decidable. Goncharov* [7] *also showed that there is a Boolean algebra that is n-decidable for every n, but not decidable.*

The next three problems concern models of arithmetic. See [10], [11] for background.

Problem 5. *What are the possible sequences of Turing degrees $(deg(D_n(\mathcal{B})))_{n\in\omega}$, where $\mathcal{B} \cong \mathcal{N}$?*

Problem 6. *If $\mathcal{A}$ is a nonstandard model of PA, must there be $\mathcal{B} \cong \mathcal{A}$ such that $D_1(\mathcal{A}) \leq_T D_1(\mathcal{B})$ and $D_0(\mathcal{B}) <_T D_0(\mathcal{A})$?*

Related to Problem 6, it was shown in [9] that for any nonstandard model of PA, there is an isomorphic copy whose atomic diagram has strictly lower Turing degree. What is difficult is to produce such a copy without decreasing the degree of the 1-diagram.

In [10], it is shown that for any model of PA, there is a copy $\mathcal{B}$ such that the sequence $(deg(D_n(\mathcal{B})))_{n\in\omega}$ is strictly increasing. In [11], there is a characterization of the sequences $(deg(D_n(\mathcal{B})))_{n\in\omega}$, where $\mathcal{B}$ ranges over *all* nonstandard models of PA. They are the sequences $(deg(C_n))_{n\in\omega}$, where $(C_n)_{n\in\omega}$ is an ω-table and C_0 is a completion of PA.

Problem 7. *Characterize the sequences of Turing degrees of n-diagrams for models of a given completion of PA.*

There are some known necessary conditions. The following conjecture says that these conditions are sufficient.

Conjecture. Let S be a completion of PA and let $(C_n)_{n\in\omega}$ be an ω-table. Suppose that there exist an enumeration R of a Scott set and a family of functions $(t_n)_{n\geq 1}$ such that:

(1) $R \leq_T C_0$,

(2) $t_n \leq_T C_{n-1}$, uniformly in n,

(3) $\lim_s t_n(s)$ is an R-index for $S \cap \Sigma_n$,

(4) for all s, $t_n(s)$ is an R-index for a subset of $S \cap \Sigma_n$.

Then there is a (nonstandard) model $\mathcal{A}$ of S such that $D_n(\mathcal{A}) \equiv_T C_n$, uniformly in n.

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