STRONG DEGREE SPECTRA OF RELATIONS

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B.A. December 2001, State University of New York at Geneseo

A Dissertation submitted to

The Faculty of
The Columbian College of Arts and Sciences
of The George Washington University
in partial fulfillment of the requirements
for the degree of Doctor of Philosophy

August 31, 2008

Dissertation directed by

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The Columbian College of Arts and Sciences of The George Washington University certifies that Sarah Pingrey has passed the Final Examination for the degree of Doctor of Philosophy as of April 21, 2008. This is the final and approved form of the dissertation.

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Dedication

For my husband Andrew, and children Prudence, Jancis, and Rutherford.

Abstract

One of the main areas of study in computable model theory is examining how certain aspects of a computable structure may change under an isomorphism to another computable structure. Let $\mathcal{A}$ be a computable structure, and let R be an additional relation on the domain of $\mathcal{A}$, so it is not named in the language of $\mathcal{A}$. Harizanov defined the *Turing degree spectrum of R on $\mathcal{A}$* to be the set of all Turing degrees of the images of R under all isomorphisms from $\mathcal{A}$ onto computable structures. Similarly, we define this notion for strong degrees such as weak truth-table degrees and truth-table degrees.

We show that the conditions necessary for the Turing degree spectrum to contain all Turing degrees, found by Harizanov, are also enough to have the truth-table degree spectrum to contain all truth-table degrees.

We further study the degree-theoretic complexity of initial segments of computable linear orderings. In particular, let $\mathcal{L}$ be a computable linear ordering of order type $\omega+\omega^*$. Harizanov showed that the Turing degree spectrum of the ω-part of $\mathcal{L}$ is all of the limit computable Turing degrees. We describe the technique of *interval trees* to show that this result does not adapt to strong degrees. We can use interval trees to translate properties of trees onto equivalent ones on linear orderings. Branches of an interval tree correspond to initial

segments of the corresponding linear ordering. We define the notion of *limit branch*, and show that if an interval tree has a unique noncomputable limit branch, then its corresponding linear ordering must have order type $\omega + \omega^*$.

Using interval trees, we show the best possible positive result for strong degrees motivated by Harizanov's theorem: for every limit computable set A, there is a computable linear ordering $\mathcal{L}$ of order type $\omega + \omega^*$ such that A is Turing reducible to the ω-part of $\mathcal{L}$ which is truth-table reducible to A. In the above result, our second application of interval trees shows that we cannot replace Turing reducibility with a stronger reducibility.

Contents

Chapter 1

Introduction

Computable model theory combines computability theory with model theory to study the algorithmic content of mathematical constructions and theorems. In particular, one goal is to find algorithmic versions of classical results in model theory. If an algorithmic version does not exist, then an algorithmic counterexample shall be found. For example, consider König's Lemma, that every infinite binary tree has an infinite branch. An algorithmic approach to this theorem is to see whether every computable infinite binary tree has a computable infinite branch. Kleene in [22] found a counterexample to this algorithmic version of König's Lemma. There is even a computable infinite binary tree without a computably enumerable infinite branch. Jockusch and Soare in [21] found the best possible positive result by showing that every infinite computable binary tree has an infinite branch of low Turing degree.

Another goal of computable model theory is examining how the degree-theoretic properties of structures change under isomorphisms. In computable mathematics, two structures may be isomorphic but have different algorithmic properties. For example, Tennenbaum showed that there is a computable linear ordering of order type $\omega + \omega^*$ with no computable ascending or descending sequence. Therefore, instead of studying structures up to isomorphism, we study structures up to computable isomorphism.

Ash and Nerode in [2] began studying algorithmic properties of additional relations on structures such as being computably enumerable (abbreviated by c.e.) in all computable isomorphic copies. Harizanov in [14] defined the *Turing degree spectrum* of a relation on a computable structure to be the collection of Turing degrees of the images of the relation in the structure's computable copies. We will be examining this notion for strong degrees such as weak truth-table degrees and truth-table degrees. In particular, we will find the syntactic conditions equivalent for the truth-table degree spectrum to contain all truth-table degrees. We will also look at the strong degree spectrum of the ω-part of a computable linear ordering of order type $\omega + \omega^*$.

The syntactic conditions sufficient and necessary for the Turing degree spectrum to be as large as possible were found by Harizanov in [16]. The same conditions also obtain all of the truth-table degrees in the degree spectrum.

Degree spectra of specific linear orderings are also studied, and in particular we examine a computable linear ordering $\mathcal{L}$ of order type $\omega+\omega^*$. Harizanov in [16] shows that the Turing degree spectrum of the ω-part of $\mathcal{L}$, $\omega_{\mathcal{L}}$, is all of the Δ^0_2 degrees. We show that this is not the case for the weak truth-table degree spectrum (and therefore for the truth-table degree spectrum also). We show that there exists a c.e. set D that is not weak truth-table reducible to $\omega_{\mathcal{L}}$. In fact, we can show that D is not weak truth-table reducible to any initial segment of any computable linear ordering that does not contain a copy of $\mathbb{Q}$. We establish these results by proving the corresponding theorems about so-called *interval trees* and transferring the theorems to linear orderings. We describe interval trees in detail.

Interval trees are used to translate properties of trees into corresponding ones on linear orderings. Branches of an interval tree correspond to initial segments of the associated linear ordering. We define the notion of limit branch and show that if an interval tree has a unique noncomputable limit branch, then its corresponding linear ordering must have order type $\omega + \omega^*$. We show that there exists a c.e. set D such that whenever T is a computable binary tree with no terminal nodes having a unique limit branch, then D is not weak truth-table reducible to any branch of T. This theorem gives us the above mentioned result.

Our best possible positive result for strong degrees motivated by Harizanov's theorem is that for every Δ^0_2 set A, there is a computable linear ordering $\mathcal{L}$ of order type $\omega + \omega^*$ such that A is Turing reducible to $\omega_{\mathcal{L}}$ which is truth-table reducible to A. We obtain this result using interval trees.

1.1 Computable Structures

We are considering only countable structures for computable languages. We will denote structures by script letters, and their domains by the corresponding Latin letters. The *theory of a structure* $\mathcal{A}$, denoted by $\mathrm{Th}(\mathcal{A})$, is the collection of all sentences true in $\mathcal{A}$. Two structures $\mathcal{A}$ and $\mathcal{B}$ are *elementarily equivalent*, denoted by $\mathcal{A} \equiv \mathcal{B}$, if $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B})$.

Let L_A be the language L expanded by adding extra constant symbols for each element in the universe. So, $L_A = L \cup \{\mathbf{a} : a \in A\}$. The structure $\mathcal{A}_A$ is $\mathcal{A}$ expanded to the language L_A so that for every $a \in A$, $\mathbf{a}$ is interpreted by a. So, $\mathcal{A}_A = (\mathcal{A}, a)_{a \in A}$. We use $\Delta_{\mathcal{A}}$ to denote the set of all quantifier-free sentences of L_A which are true in $\mathcal{A}_A$. This set of sentences, $\Delta_{\mathcal{A}}$, is decidable, or computable, if the set of Gödel numbers of its sentences is computable.

A structure $\mathcal{A}$ is *computable* if its domain A is computable and the atomic diagram of $\mathcal{A}$, $\Delta_{\mathcal{A}}$, is decidable. In other words, $\mathcal{A}$ is computable if its domain A is computable, there is a computable enumeration $\{a_i\}_{i\in\omega}$ of its domain A, and there is an algorithm which determines for all sentences $\Theta(x_0, \ldots, x_{n-1})$ and sequences $(a_{i_0}, \ldots, a_{i_{n-1}}) \in A^n$, whether $\mathcal{A}_A \models \Theta(a_{i_0}, \ldots, a_{i_{n-1}})$.

We will be considering two kinds of computable structures that are commonly studied in computable model theory, computable trees and computable linear orderings.

Definition 1.1.1. *A* computable tree $\mathcal{T} = (T, \subseteq)$ *is a computable set* T *closed under initial segments and with a root, together with a computable binary relation* $\subseteq$ *which satisfies the axioms for a partial ordering. Therefore,* $\subseteq$ *is reflexive, antisymmetric, and transitive.*

The *root* of a tree is denoted by $<>$ (the empty sequence). We join sequences of numbers together using concatenation, denoted by $^\wedge$. Elements of a tree are called *nodes*, and a maximal linearly ordered subset of sequences under $\subseteq$ is called a *branch*. Branches may or may not have terminal nodes, and in the case when a branch has no terminal nodes, it is infinite. We will be considering trees that are subtrees of $\omega^{<\omega}$ or $2^{<\omega}$. The collection of infinite branches through a tree T is denoted by $[T]$. More specifically, $[T] = \{\beta : (\forall n)[\beta \upharpoonright n \in T]\}$.

Definition 1.1.2. *A* Π^0_1 class *is the set of all infinite branches of a computable tree.*

Definition 1.1.3. *A* computable linear ordering $\mathcal{L} = (L, \leq)$ *is a computable set* $L \subseteq \omega$ *together with a computable binary relation* $\leq$ *which satisfies the axioms for a linear ordering. Therefore,* $\leq$ *is reflexive, antisymmetric, transitive, and total.*

Two linear orderings $\mathcal{L}'$ and $\mathcal{L}$ have the same *order type* if $\mathcal{L} \cong \mathcal{L}'$. Having the same order type is an equivalence relation on the class of all linear orderings. Therefore, we say that a linear ordering has *order type* τ if τ is the order type of its equivalence class. For more on linear orderings, see [30].

1.2 The Arithmetical Hierarchy

We now introduce the arithmetical hierarchy of sets for the convenience of the reader. In the arithmetical hierarchy, the number of alternations of quantifiers in front of a computable relation measures how uncomputable a set is.

Definition 1.2.1. *1. A set B is such that $B \in \Sigma^0_0$ if B is computable (similarly for $B \in \Pi^0_0$).*

2. A set B is such that $B \in \Sigma^0_n$ if there exists a computable $(n+1)$-ary relation R such that

$$x \in B \Longleftrightarrow (\exists y_1)(\forall y_2)(\exists y_3)\dots(Qy_n)[R(x, y_1, y_2, \dots, y_n)]$$

where Q is $\exists$ if n is odd, and Q is $\forall$ if n is even.

3. A set B is such that $B \in \Pi^0_n$ if the alternations of quantifiers start with $\forall$. If $B \in \Sigma^0_n \cap \Pi^0_n$, then $B \in \Delta^0_n$.

Some important levels of the hierarchy that we will examine are as follows: the $\Sigma^0_0 = \Pi^0_0 = \Delta^0_0$ sets are the computable sets; the Σ^0_1 sets are the c.e. sets; and the Δ^0_2 sets are the sets that the halting set computes and are also called limit computable sets. Equivalently, a set B is *limit computable* if and only if there exists a computable binary function g such that $\chi_B(x) = \lim_{s\to\infty} g(x, s)$ where χ_B is the characteristic function of B.

1.3 The Ash-Nerode Program

Ash and Nerode in [2] began studying relations on computable structures. For a computable structure $\mathcal{A}$, and an additional computable relation R on its domain, they were interested in when there is a computable structure $\mathcal{B} \cong \mathcal{A}$ where the isomorphism takes R to a non-computable (or non-c.e.) relation on $\mathcal{B}$. This led to the following definition from [2].

Definition 1.3.1. *Let $\mathcal{A}$ be a computable structure, and let R be an additional computable relation on the domain of $\mathcal{A}$. The relation R is* intrinsically computable (intrinsically c.e.) *on $\mathcal{A}$ if every isomorphism of $\mathcal{A}$ to a computable structure $\mathcal{B}$ takes R to a computable (c.e.) relation.*

For computable relations R it is easy to see that R is intrinsically computable if for all computable structures $\mathcal{B} \cong \mathcal{A}$, every isomorphism from $\mathcal{B}$ to $\mathcal{A}$ is computable (so $\mathcal{A}$ is *computably stable*). Theorem 1.3.2 below gives a computable syntactic condition for $\mathcal{A}$ to be computable stable, provided $\mathcal{A}$ satisfies an additional decidability condition.

Theorem 1.3.2 (Ash and Nerode [2])**.** *Let $\mathcal{A}$ be a computable structure such that there is a computable procedure for determining, given an existential formula ψ and $\overrightarrow{a} \in A^{<\omega}$, whether $\mathcal{A} \models \psi(\overrightarrow{a})$ ($\mathcal{A}$ is called* 1-decidable *in this case). Then, $\mathcal{A}$ is computably stable if and only if there exists $\overrightarrow{c} \in A^{<\omega}$ and a computable sequence $\{\psi_n\}$ of existential formulas such that the sets $\{a \in A : \mathcal{A}_A \models \psi_n(a, \overrightarrow{c})\}$ form a family of singletons whose union is A.*

A relation R is *formally c.e. on* $\mathcal{A}$ if it is equivalent to an infinite disjunction of a computable sequence of existential (Σ^0_1) formulas with finitely many parameters.

The following theorem of Ash and Nerode shows the equivalence of a semantic condition on $\mathcal{A}$ and R (R being intrinsically c.e. on $\mathcal{A}$), and a syntactic condition (R being formally c.e. on $\mathcal{A}$), under an extra decidability condition D on $\mathcal{A}$. The Ash-Nerode decidability condition D is:

(D) There is a computable procedure for determining, given an existential formula $\psi(x, \overrightarrow{y})$ and a tuple $\overrightarrow{c} \in A^{<\omega}$ of the same length as $\overrightarrow{y}$, whether the following implication is true in $\mathcal{A}_A$ for every $a \in A$: $\psi(a, \overrightarrow{c}) \Longrightarrow R(a)$.

Theorem 1.3.3 (Ash and Nerode [2])**.** *Let $\mathcal{A}$ be a computable structure, and let R be an additional relation on the domain of $\mathcal{A}$ such that (D) holds. Then, R is intrinsically c.e. on $\mathcal{A}$ if and only if R is formally c.e. on $\mathcal{A}$.*

Now, we will look at an example of an additional relation on a linear ordering which is not intrinsically c.e. A natural example of a computable linear ordering is the natural numbers, with the usual computable ordering $\leq$. The (binary) successor relation S is obviously computable since we can tell for all $x, y \in \omega$, whether x is the successor of y. Ash and Nerode in [2] showed that S is not formally c.e. and that a linear ordering of order type ω can be chosen so that S satisfies condition (D). Therefore, by Thereom 1.3.3, S is not intrinsically c.e. This also gives an example of two computable structures which are isomorphic but not computably isomorphic (i.e. not computably categorical), since a computable isomorphism would preserve the successor relation being computable. It has also been shown (refer to [15]) that for every c.e. Turing degree $\mathbf{d}$, we can construct a computable linear ordering where the successor has degree $\mathbf{d}$.

We can generalize this notion of being intrinsically c.e. as follows.

Definition 1.3.4. *Let $\mathcal{A}$ be a computable structure, and let R be an additional relation on $\mathcal{A}$ which is in $\mathcal{P}$ (where $\mathcal{P}$ is a class of relations in the arithmetical or hyperarithmetical* *hierarchy) on the domain of $\mathcal{A}$. The relation R is* intrinsically $\mathcal{P}$ *if every isomorphism of $\mathcal{A}$ to a computable structure $\mathcal{B}$ takes R to a relation in $\mathcal{P}$.*

Barker in [3] studied the syntactic conditions necessary and sufficient for a relation to be intrinsically Σ^0_α, for a computable ordinal α. Barker extended Theorem 1.3.3 for these relations.

1.4 Turing Degrees

The Turing degree of a set measures its relative computability theoretic complexity. Two sets have the same degree if they are equally as difficult to compute. In the following definition, we use φ_i to denote the ith partial computable function. For other notation not described here, please refer to [31].

Definition 1.4.1. *A set A is* Turing reducible (T-reducible) *to a set B, $A \leq_T B$, if there exists an $e \in \omega$ such that $A(n) = \varphi_e^B(n)$ for all $n \in \omega$.*

Therefore, $A \leq_T B$ if there is an algorithm for computing A that can ask membership questions of B of the form,"Is $y \in B$?" We let $A <_T B$ if and only if $A \leq_T B$ and $B \nleq_T A$. $A \equiv_T B$ if and only if $A \leq_T B$ and $B \leq_T A$. Notice that $\leq_T$ is reflexive and transitive and therefore $\equiv_T$ is an equivalence relation. The equivalence classes with respect to $\equiv_T$ are called *Turing degrees (T-degrees)*. We denote the T-degree of a set A as $\mathbf{a} = \deg(A) = \{B : B \equiv_T A\}$.

The collection of all T-degrees is denoted by $\mathcal{D}$, and there is a partial ordering $\leq$ induced on the T-degrees by $\leq_T$. We will denote the structure $(\mathcal{D}, \leq)$ with just $\mathcal{D}$. It is an upper semmilattice with supremum $\deg(A) \vee \deg(B) = \deg(A \oplus B)$ where $A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\}$.

The T-degree of the computable sets is $\mathbf{0}$, and $\mathbf{0}'$ is the T-degree of the halting set K. Both of these T-degrees are c.e. T-degrees, since they contain c.e. sets. $\mathbf{0}$ is the least c.e. T-degree, and $\mathbf{0}'$ is the greatest c.e. T-degree. Define $K^A = \{x : \varphi_x^A(x) \downarrow\}$. K^A is the *jump* of A, denoted by A', and $\deg(A') = \mathbf{a}'$. $A^{(n)}$ is the *nth jump* of A, obtained by iterating the jump n times.

A set A has *low degree* if $A' \equiv_T \emptyset'$. A set A has *high degree* if $A' \equiv_T \emptyset''$. A set of low degree is close to being computable, while a set of high degree is close to being as complicated as the halting set.

1.5 Turing Degree Spectra of Relations

Harizanov examines the additional computable relations on a computable structure using a finer hierarchy than Ash and Nerode. In the following definition, we look at the T-degrees of these relations in the computable isomorphic copies of these computable structures. Definitions 1.5.1 and 1.5.2 were introduced by Harizanov in her doctoral dissertation [13], and in [14]. The definitions were subsequently examined for both wtt-degrees and tt-degrees in [5].

Definition 1.5.1. *Let $\mathcal{A}$ be a computable structure, and let R be an additional relation on the domain of $\mathcal{A}$. The* Turing degree spectrum of R on $\mathcal{A}$, *$DgSp_{\mathcal{A}}(R)$, is the set of all Turing degrees of the images of R under all isomorphisms from $\mathcal{A}$ to computable structures.*

In other words,

$$DgSp_{\mathcal{A}}(R) = \{\deg(X) : (\exists \mathcal{B})[\mathcal{B} \text{ is computable } \& \ (\exists f)[f : (\mathcal{A}, R) \cong (\mathcal{B}, X)]]\}.$$

Let $\mathcal{B}$ be a computable structure such that $\mathcal{B} \cong \mathcal{A}$. We can also look at the T-degrees of the relation in this specific copy of $\mathcal{A}$ under all isomorphisms from $\mathcal{A}$ to $\mathcal{B}$.

Definition 1.5.2. *Let $\mathcal{A}$ be a computable structure, and let R be an additional relation on the domain of $\mathcal{A}$. Let $\mathcal{B}$ be a computable structure such that $\mathcal{B} \cong \mathcal{A}$. The* Turing degree spectrum of R on $\mathcal{A}$ with respect to $\mathcal{B}$, *$DgSp_{\mathcal{A},\mathcal{B}}(R)$, is the set of all Turing degrees of the images of R under all isomorphisms f from $\mathcal{A}$ to $\mathcal{B}$. In other words,*

$$DgSp_{\mathcal{A},\mathcal{B}}(R) = \{\deg(X) : (\exists f)[f : (\mathcal{A}, R) \cong (\mathcal{B}, X)]\}.$$

Harizanov in [16] found a sufficient and necessary condition for the Turing degree spectrum to contain all Turing degrees. For a linear ordering $\mathcal{L}$ of order type $\omega+\omega^*$, we denote the ω-part of $\mathcal{L}$ by $\omega_{\mathcal{L}}$. Harizanov examined the Turing degree spectrum of $\omega_{\mathcal{L}}$ for a computable linear ordering $\omega + \omega^*$ and found that $DgSp_{\mathcal{L}}(\omega_{\mathcal{L}})$ is all of the Δ^0_2 degrees.

In Section 3.2 we will show that when the Turing degree spectrum is as large as possible, we also get the truth-table degree spectrum to be as large as possible. We find in Chapter 5 that we cannot get the weak truth-table degree spectrum of $\omega_{\mathcal{L}}$ on $\mathcal{L}$ to be all of the Δ^0_2 wtt-degrees, and we also show the best possible positive result for weak truth-table degrees.

Chapter 2

Strong Degrees

2.1 Weak Truth-Table and Truth-Table Degrees

Post's problem is to find a c.e. T-degree between $\mathbf{0}$ and $\mathbf{0}'$. Post first solved this problem for many-one degrees (m-degrees) by constructing a simple set. Note that for a set A, both A and χ_A are used to denote the characteristic function of A.

Definition 2.1.1. *A set A is* many-one reducible (m-reducible) *to a set B, $A \leq_m B$, if there exists a computable function f, such that $A(n) = B(f(n))$.*

In cases where $A \leq_m B$ via the function f and f is one-to-one, we say that $A \leq_1 B$.

Definition 2.1.2 (Post [28])**.** *1. An infinite set B is* immune *if it does not contain an infinite c.e. subset.*

2. A set A is simple *if it is c.e. and $\overline{A}$ is immune.*

The m-degree of a simple set is strictly between $\mathbf{0_m}$, the m-degree of the computable sets, and $\mathbf{0'_m}$, the m-degree of the halting set. (For more on m-degrees, see [27].)

When Post solved the problem for m-degrees, his solution did not solve the problem for T-degrees, therefore he relaxed m-reducibility and obtained truth-table reducibility. Post then solved the problem for truth-table degrees in 1944 by the construction of a hypersimple set, which will be defined below. The existence of a simple set does not immediately solve Post's problem for truth-table degrees, since there is a simple set which has truth-table degree $\mathbf{0}'_{\mathbf{tt}}$.

We let D_y be the finite set with canonical index y. That is, $D_y = \{x_0, x_1, \ldots, x_k\}$ where $x_0 < x_1 \cdots < x_k$ and $y = 2^{x_0} + 2^{x_1} + \cdots + 2^{x_k}$. $\{D_{f(x)}\}_{x \in \omega}$, where f is a computable function, is called a *strong array* of finite sets. An array is *disjoint* if its members are pairwise disjoint. The following definition is due to Post in [28].

Definition 2.1.3. 1. *An infinite set B is* hyperimmune *if there is no disjoint strong array with all of its members having a nonempty intersection with B.*

2. *A set A is* hypersimple *if it is c.e. and $\overline{A}$ is hyperimmune.*

The truth-table degree of a hypersimple set is strictly between $\mathbf{0}_{\mathbf{tt}}$ and $\mathbf{0}'_{\mathbf{tt}}$. Post constructs a hypersimple set, therefore solving the problem for truth-table degrees. A thorough discussion of Post's problem for truth-table degrees can also be found in [27]. Afterward, Friedberg and Rogers in [12] introduced weak truth-table degrees by relaxing truth-table degrees, and solved Post's problem for them in 1959.

Both truth-table and weak truth-table reducibilities are stronger than Turing reducibility, that is, they imply T-reducibility. We now give formal definitions of weak truth-table and truth-table reducibility, followed by an informal explanation of how they relate to T-reducibility.

Definition 2.1.4. *A set A is* weak truth-table reducible (wtt-reducible) *to a set B, $A \leq_{wtt} B$, if there exists an $e \in \omega$ and a computable function h such that $A(n) = \varphi_e^{(B \upharpoonright h(n))}(n)$ for all $n \in \omega$.*

Definition 2.1.5. *A set A is* truth-table reducible (tt-reducible) *to a set B, $A \leq_{tt} B$, if there exists an $e \in \omega$ and a computable function h such that $A(n) = \varphi_e^B(n)$, and for each $n \in \omega$, $\varphi_e^\sigma(n)$ converges for every $\sigma \in 2^{h(n)}$.*

When $A \leq_{wtt} B$ (respectively, $A \leq_{tt} B$), we say that $A \leq_{wtt} B$ (respectively, $A \leq_{tt} B$) via φ_e and h.

Note that in the definition of tt-reducible, we mention the function h just for comparison with the definition of wtt-reducible, but it can easily be seen that h is not necessary.

For T-reducibility, the questions asked of the oracle B may not be known ahead of time and there may be no computable bound on the number of questions asked. This type of questioning is called serial querying. With wtt-reducibility, there is a computable bound on the amount of the oracle you are allowed to use, and the elements on which the oracle will be queried can be seen ahead of time. The computable function h in the definition gives the computable bound. With tt-reducibility, the elements on which the oracle will be queried can also be seen ahead of time, as well as the outcome of all possible answers. Therefore, if we are considering $A \leq_{tt} B$ via φ_e and h, then φ_e must halt for all possible oracles. For wtt-reducibility and T-reducibility, φ_e may not halt for oracles other than B. Both wtt-reducibility and tt-reducibility use parallel querying when asking questions of the oracle. It is easy to see that $A \leq_{tt} B \Longrightarrow A \leq_{wtt} B \Longrightarrow A \leq_T B$.

As with T-degrees, we use similar terminology to talk about wtt-degrees and tt-degrees.

2.2 Bounded Truth-Table Reducibility

Post introduced bounded truth-table reducibility in [28] while trying to solve Post's problem for Turing reducibility.

Definition 2.2.1. *A set A is* bounded truth-table reducible (btt-reducible) *to a set B, $A \leq_{btt} B$, if there exists an $e \in \omega$, and computable functions h and $f_0, f_1, \ldots, f_{k-1}$, for some $k \in \omega$, such that $A = \varphi_e^B$, and for each $n \in \omega$, $\varphi_e^{(\sigma(f_0(n)) \cup \sigma(f_1(n)) \cup \ldots \cup \sigma(f_{k-1}(n)))}(n)$ converges for every $\sigma \in 2^{h(n)}$.*

We say that $A \leq_{btt(k)} B$ via φ_e, h, and $f_0, f_1, \ldots, f_{k-1}$. The number k is called the *norm* of the btt-reducibility. Notice that m-reducibility is a special case of btt-reducibility where the norm is 1.

It is easy to see that btt-reducibility is stronger than tt-reducibility, that is, $A \leq_{btt} B \implies A \leq_{tt} B$. The difference between the two reducibilities is that for tt-reducibility, to find out if $n \in A$, where $A \leq_{tt} B$ via φ_{e_0} and h_0, we can access B up to the first $h_0(n)$ elements, so $B \upharpoonright h_0(n)$. For btt-reducibility, to find out if $n \in A$, where $A \leq_{btt(k)} B$ via φ_{e_1}, h_1, and $f_0, f_1, \ldots, f_{k-1}$, we can access only k specific elements of B, namely, $B(f_0(n)), B(f_1(n)), \ldots, B(f_{k-1}(n))$.

2.3 Structure of the Strong Degrees

We will now examine the strong degrees more closely. The following proposition (Posner's Trick) shows that if $A \leq_{wtt} B$ ($A \leq_{tt} B$) via φ_{e_0} and h_0 and $B \leq_{wtt} A$ ($B \leq_{tt} A$) via φ_{e_1} and h_1, then we can find an $e \in \omega$ and computable function h such that $A \equiv_{wtt} B$ ($A \equiv_{tt} B$) via

φ_e and h. Therefore, we can find one partial computable function φ_e and one computable function h which witness both reducibilities.

Proposition 2.3.1 ([31]). *$A \equiv_{wtt} B$ ($A \equiv_{tt} B$) if and only if there exists an $e \in \omega$ and computable function h such that $A \leq_{wtt} B$ ($A \leq_{tt} B$) via φ_e and h and $B \leq_{wtt} A$ ($B \leq_{tt} A$) via φ_e and h.*

Proof. We will prove this proposition for truth-table reducibility, however, this same proof can be used for proving the proposition for weak truth-table reducibility by changing all instances of "truth-table" to "weak truth-table."

Assume that $A \equiv_{tt} B$. Suppose that $A \leq_{tt} B$ via φ_{e_0} and h_0 and $B \leq_{tt} A$ via φ_{e_1} and h_1. If $A = B$, then the proposition is trivial. Therefore, suppose that $A \neq B$ and that there exists an $a' \in A - B$. We want to find a program number $e \in \omega$ such that $A = \varphi_e^B$ and $B = \varphi_e^A$. To find this e, we will use the programs e_0 and e_1. Let $X \subseteq \omega$. Find $e \in \omega$ such that

$$\varphi_e^X(x) = \begin{cases} \varphi_{e_0}^X(x), & \text{if } a' \notin X, \\ \varphi_{e_1}^X(x), & \text{if } a' \in X. \end{cases}$$

If $X = A$, then since $a' \in A$, $\varphi_e^A(x) = \varphi_{e_1}^A(x) = B$. If $X = B$, then since $a' \notin B$, $\varphi_e^B(x) = \varphi_{e_0}^B(x) = A$. Therefore, when the oracle is A, φ_e^A computes B, and when the oracle is B, φ_e^B computes A.

To find a computable function h, let $h(x) = \max\{h_0(x), h_1(x)\}$. The function h is clearly computable since h_0 and h_1 are both computable. $\varphi_e^\sigma(n)$ converges for all $\sigma \in 2^{h(n)}$ since both $\varphi_{e_0}^\sigma(n)$ and $\varphi_{e_1}^\sigma(n)$ converge for all $\sigma \in 2^{h(n)}$.

Therefore, we have found an $e \in \omega$ and computable function h such that φ_e and h witness both truth-table reductions $A \leq_{tt} B$ and $B \leq_{tt} A$, and therefore witness that $A \equiv_{tt} B$. □

The following proposition gives a brief outline on why two sets being T-equivalent does not imply that the sets are wtt-equivalent, which in turn does not imply that the sets are tt-equivalent.

Proposition 2.3.2. *$A \leq_{tt} B \implies A \leq_{wtt} B \implies A \leq_T B$, and the reverse implications do not hold.*

Proof. It is obvious that $A \leq_{tt} B \implies A \leq_{wtt} B \implies A \leq_T B$. We will only show that the reverse implications do not hold.

First, we will show that the second reverse implication does not hold. Dekker showed in [7] that every T-degree which is noncomputable and c.e. contains a hypersimple set. Therefore, $\mathbf{0}'$ contains a hypersimple set, say H. Therefore, for the halting set K we have $K \leq_T H$. However, Friedberg and Rogers in [12] showed that no hypersimple set has wtt-degree $\mathbf{0}'_{\mathbf{wtt}}$. Therefore, since the degree of a hypersimple set is strictly between $\mathbf{0}_{\mathbf{wtt}}$ and $\mathbf{0}'_{\mathbf{wtt}}$, we have that $K \not\leq_{wtt} H$.

Lachlan showed in [25] that the first implication is not reversible by constructing a c.e. set A such that $K \leq_{wtt} A$, and then defining a c.e. set B such that $B \not\leq_{tt} A$. Since $B \leq_{tt} K$ and $B \not\leq_{tt} A$, we have that $K \not\leq_{tt} A$. □

By $\mathcal{D}^{\mathbf{tt}}$ we denote the structure of the tt-degrees, and by $\mathcal{D}^{\mathbf{wtt}}$ we denote the structure of the wtt-degrees.

Proposition 2.3.3. *$\mathcal{D}^{\mathbf{tt}}$ is an uppersemilattice, and has a least element and no greatest element.*

Proof. Kleene and Post in [23] proved this proposition for T-degrees with a similar proof.

In order to show that $\mathcal{D}^{\mathbf{tt}}$ is an uppersemilattice, we need to show that every two tt-degrees $\mathbf{a}$ and $\mathbf{b}$ have a least upper bound, denoted by $\mathbf{a} \vee \mathbf{b}$. Let $\mathbf{a} = \deg_{tt}(A)$ and $\mathbf{b} = \deg_{tt}(B)$. Define $A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\}$. We claim that $\deg_{tt}(A \oplus B) = \mathbf{a} \vee \mathbf{b}$.

First, we will show that $\deg_{tt}(A \oplus B)$ is an upper bound of $\deg_{tt}(A)$ and $\deg_{tt}(B)$. We will start by showing that $A \leq_{tt} A \oplus B$. Define $h_0(n) = 2n$ for all $n \in \omega$. Then, to find out if $n \in A$, we only need to access the oracle $A \oplus B$ up to the first $h_0(n)$ elements to see if $h_0(n)$ is an element of $A \oplus B$. If $h_0(n) \in A \oplus B$, then $n \in A$. Otherwise, $n \notin A$. Notice that no matter what oracle we choose, our reduction will halt.

Similarly, to show that $B \leq_{tt} A \oplus B$, define $h_1(n) = 2n+1$ for all $n \in \omega$.

Next, we will show that $\deg_{tt}(A \oplus B)$ is the least upper bound of $\deg_{tt}(A)$ and $\deg_{tt}(B)$. Suppose that $A \leq_{tt} C$ via φ_{e_0} and h_0, and $B \leq_{tt} C$ via φ_{e_1} and h_1. Find $e \in \omega$ such that

$$\varphi_e^C(z) = \begin{cases} \varphi_{e_0}^C(x), & \text{if } z = 2x \\ \\ \varphi_{e_1}^C(x), & \text{if } z = 2x+1. \end{cases}$$

Then, $A \oplus B \leq_{tt} C$ via φ_e and $h(n) = \max\{h_0(n), h_1(n)\}$ for all $n \in \omega$. Since φ_{e_0} and φ_{e_1} witness tt-reductions and therefore halt for all possible oracles, φ_e also halts for all possible oracles.

The tt-degree of the computable sets, $\mathbf{0_{tt}}$ is the least tt-degree. There is no greatest tt-degree since we can iterate the jump operator, and the tt-degree of the jump of a set is strictly greater than the tt-degree of the set. □

Proposition 2.3.3 also holds for wtt-degrees (and for T-degrees) with minor proof modifications. A good discussion of the structure of strong degrees can be found in [26], [27], and [33].

Now we will look at the structure inside degrees. When a degree is stronger than another degree, then the weaker degree contains the stronger degree. For example, T-degrees contain tt-degrees. One area of study is examining how many stronger degrees are inside weaker degrees. For example, Jockusch looked at tt-degrees inside T-degrees.

Theorem 2.3.4 (Jockusch [19])**.** *1. A T-degree contains either exactly one tt-degree (and is called an* irreducible *T-degree) or infinitely many tt-degrees.*

2. A T-degree is irreducible if and only if it does not contain a hyperimmune set.

Notice that Jockusch's result implies that T-degrees and tt-degrees coincide for hyperimmune-free degrees. Since wtt-reducibility is stronger than T-reducibility, but weaker than tt-reducibility, we have that wtt-degrees also coincide with T-degrees and tt-degrees for hyperimmune-free degrees.

We have thus far considered several degrees stronger than tt-degrees including btt-degrees, m-degrees, and 1-degrees. Next, we will look at these degrees inside tt-degrees.

Theorem 2.3.5 (Stephan [33])**.** *Every noncomputable tt-degree contains infinitely many btt-degrees.*

The above theorem is a more recent result from 2001 and implies the result of Jockusch in [17] from 1969 that every noncomputable tt-degree contains infinitely many m-degrees (since m-degrees are contained inside btt-degrees). This is also true for 1-degrees.

A natural question to ask is whether there is an infinite chain or antichain of stronger degrees inside a weaker degree. Recall that a set of degrees is called a *chain* if it is linearly ordered. A set of degrees is an *antichain* if it does not contain the computable degree, and all of its members are pairwise incomparable.

Stephan in [33] showed that every noncomputable tt-degree contains an infinite chain and an infinite antichain of btt-degrees. Again, Stephan's result implies Jockusch's result contained in [19]. Stephan's result also implies that every noncomputable tt-degree also contains an infinite antichain of m-degrees. (The same is true for 1-degrees since they are stronger than m-degrees.) Looking at tt-degrees inside T-degrees, Jockusch, in his proof of Theorem 2.3.4, actually showed that T-degrees that are not irreducible contain an infinite chain of tt-degrees. Degtev in [6] showed that T-degrees that are not irreducible contain an infinite antichain of tt-degrees.

Another natural question that is examined is whether there is a greatest or least stronger degree (with respect to the given reducibility) inside the weaker degree. Again, by the proof of Theorem 2.3.4, it can be shown that a T-degree has a greatest tt-degree if and only if the T-degree is irreducible. Rogers in [29] showed that every tt-degree contains a greatest m-degree and a greatest 1-degree, while Stephan in [33] showed that every tt-degree contains a greatest btt-degree. These results for degrees inside tt-degrees also hold for wtt-degrees.

Regarding least degrees, Jockusch showed that every noncomputable tt-degree does not contain a least m-degree. (So, every noncomputable tt-degree does not contain a least 1-degree also.) Stephan showed in [33] that some tt-degrees contain least btt-degrees, and some do not.

Chapter 3

Degree Spectra Containing All Truth-Table Degrees or Limit Computable Degrees

3.1 Related Results

We examine a computable structure $\mathcal{A}$ with an additional computable relation R. An area of study when examining degree spectra is trying to obtain the sufficient or necessary conditions to get the Turing degree spectrum to contain or equal certain classes of T-degrees. In [14], Harizanov asked what conditions are sufficient to have the Turing degree spectrum of R on $\mathcal{A}$ to be all of the T-degrees, $\mathcal{D}$. In order to state this theorem, we start with the following definition.

Definition 3.1.1. *A partial function p is a* finite partial isomorphism *from $\mathcal{A}$ to $\mathcal{B}$ if*

1. *p is one-to-one,*

2. dom(p) *is finite, and*

3. *for all atomic formulas* $\psi(x_0, \ldots, x_{n-1})$ *and every tuple* $(a_0, \ldots, a_{n-1}) \in \mathrm{dom}(p)$*, we have*

$$\mathcal{A} \models \psi[a_0, \ldots, a_{n-1}] \Longleftrightarrow \mathcal{B} \models \psi[b_0, \ldots, b_{n-1}],$$

where $b_0 = p(a_0), \ldots, b_{n-1} = p(a_{n-1})$.

Let $I_{fin}(\mathcal{A}, \mathcal{B})$ be the set of all finite partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$. Let $q, r \in I_{fin}(\mathcal{A}, \mathcal{B})$. We will define a 2-ary relation $\sim_R$ as follows:

$$q \sim_R r \Longleftrightarrow (\forall b \in ran(q) \cap ran(r))\ [q^{-1}(b) \in R \Leftrightarrow r^{-1}(b) \in R]$$
$$\Longleftrightarrow (\forall b \in ran(q) \cap ran(r))\ [b \in q(R) \Leftrightarrow b \in r(R)].$$

Part (1) of the following theorem gives the conditions for having the Turing degree spectrum of R on $\mathcal{A}$ be uncountable, so it is unbounded in the T-degrees. In part (2), we get the sufficient conditions for having the Turing degree spectrum of R on $\mathcal{A}$ with respect to $\mathcal{B}$, where $\mathcal{B}$ is a computable copy of $\mathcal{A}$, be all of the T-degrees.

Theorem 3.1.2 (Harizanov [14]). *Let* $\mathcal{A}$ *be a computable structure and* R *an additional computable relation on* $\mathcal{A}$.

1. *The following are equivalent:*

   (a) $DgSp_{\mathcal{A}}(R)$ *is uncountable.*

(b) *For every computable model* $\mathcal{B}$ *isomorphic to* $\mathcal{A}$*, the degree spectrum* $DgSp_{\mathcal{A},\mathcal{B}}(R)$ *is uncountable.*

(c) *For every computable model* $\mathcal{B}$ *isomorphic to* $\mathcal{A}$*, the degree spectrum* $DgSp_{\mathcal{A},\mathcal{B}}(R)$ *has cardinality* 2^ω.

(d) *For every computable model* $\mathcal{B}$ *isomorphic to* $\mathcal{A}$*, there is a nonempty set* $\mathbb{S} \subseteq I_{fin}(\mathcal{A}, \mathcal{B})$ *such that the following two conditions are satisfied:*

i. $(\forall p \in \mathbb{S})(\forall a \in A)(\forall b \in B)(\exists q \in \mathbb{S})\ [q \supseteq p \wedge a \in \mathrm{dom}(q) \wedge b \in \mathrm{ran}(q)]$, *and*

ii. $(\forall p \in \mathbb{S})(\exists q, r \in \mathbb{S})\ [q \supseteq p \wedge r \supseteq p \wedge \neg(q \sim_R r)]$.

2. *Let a family* $\mathbb{S} \subseteq I_{fin}(\mathcal{A}, \mathcal{B})$ *satisfy conditions* (i) *and* (ii) *of part* (d)*. Then for every set* $C \subseteq \omega$ *such that* $C \geq_T \mathbb{S}$*, there is an isomorphism* f *from* $\mathcal{A}$ *to* $\mathcal{B}$ *for which we have*

$$C \equiv_T f(R) \oplus \mathbb{S} \equiv_T f \oplus \mathbb{S}.$$

In particular, if $\mathbb{S}$ *is computable (or even just c.e.), then we have* $DgSp_{\mathcal{A},\mathcal{B}}(R) = \mathcal{D}$ *and, moreover, for every set* $C \subseteq \omega$*, there is an isomorphism* f *from* $\mathcal{A}$ *to* $\mathcal{B}$ *such that*

$$C \equiv_T f(R) \equiv_T f.$$

Independently, Ash, Cholak and Knight, and Harizanov, showed in part (1) of the following theorem that the sufficient conditions equivalent to Theorem 3.1.2 (d) turned out to also be necessary, provided that the isomorphism f is of the same T-degree as the image of R. In part (2), they showed that if there is a T-degree not in the Turing degree spectrum of R on $\mathcal{A}$ with respect to $\mathcal{B}$, where $\mathcal{B}$ is a computable copy of $\mathcal{A}$, and provided that the isomorphism f has the same T-degree as the image of R, then there is a T-degree that is Δ^0_3.

Theorem 3.1.3 (Ash, Cholak, and Knight [1]; Harizanov [16])**.** *Let $\mathcal{A}$ be a computable structure with an additional computable relation R.*

1. *The following are equivalent:*

   (a) *$DgSp_{\mathcal{A}}(R) = \mathcal{D}$ and, moreover, for every set $C \subseteq \omega$, there is an isomorphism f from $\mathcal{A}$ onto a computable $\mathcal{B}$ such that*

   $$C \equiv_T f(R) \equiv_T f.$$

   (b) *For every set $C \subseteq \omega$, there is an automorphism f of $\mathcal{A}$ such that*

   $$C \equiv_T f(R) \equiv_T f.$$

   (c) *There is a nonempty computable (or even c.e.) family $\mathbb{S} \subseteq I_{fin}(\mathcal{A}, \mathcal{A})$ such that the conditions* (i) *and* (ii) *from part* (d) *of Theorem 3.1.2 are satisfied.*

2. *Let $\mathcal{B}$ be a computable structure isomorphic to $\mathcal{A}$. If there is a T-degree $\mathbf{d}$ that cannot be obtained in $DgSp_{\mathcal{A},\mathcal{B}}(R)$ via an isomorphism of degree $\mathbf{d}$, then there is such a T-degree that is Δ^0_3.*

Harizanov shows in the next theorem that the Ash-Nerode decidability condition (D) (from Section 1.3) is necessary to have a non-intrinsically c.e. T-degree have an infinite

Turing degree spectrum.

Theorem 3.1.4 (Harizanov [14]). *1. Let $\mathcal{A}$ be a computable structure with an additional non-intrinsically c.e. relation R such that (D) holds. Then, $DgSp_{\mathcal{A}}(R)$ is infinite.*

2. There is a computable structure $\mathcal{A}$ with an additional non-intrinsically c.e. relation R such that (D) does not hold and $DgSp_{\mathcal{A}}(R)$ contains exactly two T-degrees.

Now, we will consider a particular computable structure $\mathcal{L}$, a linear ordering of order type $\omega+\omega^*$. Let $\omega_{\mathcal{L}}$ be the ω-part of $\mathcal{L}$. The following illustrates an example of such a linear ordering.

Example 3.1.5. *Let $|\mathcal{L}_0| = \omega$ and $\prec$ be the ordering relation defined on $\mathcal{L}_0$ as follows:*

$$0 \prec 2 \prec 4 \prec \cdots \prec 5 \prec 3 \prec 1.$$

So, $\mathcal{L}_0 \cong \omega + \omega^*$ and $\omega_{\mathcal{L}_0} = \{0, 2, 4, \ldots\}$.

Theorem 3.1.9 says that the Turing degree spectrum of $\omega_{\mathcal{L}}$ on $\mathcal{L}$ are all of the Δ^0_2 T-degrees (the T-degrees that are computable in $\emptyset'$). In order to prove this theorem, we give the following definition and lemmas.

Definition 3.1.6 (Jockusch [17]). *A set $A \subseteq \omega$ is* semirecursive *if there is a computable function $f(x, y)$ such that*

1. $(\forall x)(\forall y)[f(x, y) = x \vee f(x, y) = y]$*; and*

2. $(\forall x)(\forall y)[(x \in A \vee y \in A) \Rightarrow f(x, y) \in A]$.

Lemma 3.1.7 (Appel and McLaughlin [17]). *The semirecursive sets are precisely the initial segments of computable linear orderings.*

Lemma 3.1.8 (Jockusch [17]). *Let $A \subseteq \omega$ be a set such that $A \leq_T \emptyset'$. Then A contains a semirecursive set which is both immune and coimmune.*

Combining the results of Jockusch, and Appel and McLaughlin, we have that every nonzero T-degree $\mathbf{a} \leq \mathbf{0}'$ contains an immune, coimmune, initial segment of a computable linear ordering. We use this fact when proving the next theorem.

Theorem 3.1.9 (Harizanov [16]). *Let $\mathcal{L}$ be a computable linear ordering of order type $\omega+\omega^*$. Then, $DgSp_{\mathcal{L}}(\omega_{\mathcal{L}})$ are all of the Δ^0_2 degrees.*

Proof. First, we will show that $DgSp_{\mathcal{L}}(\omega_{\mathcal{L}}) \subseteq \{\mathbf{a} : \mathbf{a} \leq \mathbf{0}'\}$. If an element $x \in \mathcal{L}$ is in $\omega_{\mathcal{L}}$, then x has only finitely many predecessors. Therefore,

$$x \in \omega_{\mathcal{L}} \iff \bigvee\nolimits_{n\in\omega} (\exists x_0)\ldots(\exists x_n)[x_0 \prec x_1 \prec \cdots \prec x_n \wedge x = x_n \wedge$$

$$(\forall y)[\neg(y \prec x_0) \wedge \neg(x_0 \prec y \prec x_1) \wedge \ldots \wedge \neg(x_{n-1} \prec y \prec x_n)]].$$

If an element $x \in \mathcal{L}$ is not in $\omega_{\mathcal{L}}$, then x has only finitely many successors. Therefore,

$$x \notin \omega_{\mathcal{L}} \iff \bigvee\nolimits_{n\in\omega} (\exists x_0)\ldots(\exists x_n)[x_0 \succ x_1 \succ \cdots \succ x_n \wedge x = x_n \wedge$$

$$(\forall y)[\neg(y \succ x_0) \wedge \neg(x_0 \succ y \succ x_1) \wedge \ldots \wedge \neg(x_{n-1} \succ y \succ x_n)]].$$

Thus, $\omega_{\mathcal{L}} \leq_T \emptyset'$.

Next, we will show that $\{\mathbf{a} : \mathbf{a} \leq \mathbf{0}'\} \subseteq DgSp_{\mathcal{L}}(\omega_{\mathcal{L}})$. Let $\mathbf{0} < \mathbf{d} \leq \mathbf{0}'$. By our observation above, **d** contains an immune, coimmune, initial segment of a computable linear ordering. Let $\mathcal{L}_0$ be this computable linear ordering, and let A be its immune, coimmune, initial segment. Since A is immune, no element of A can have infinitely many predecessors, and since it is coimmune, no element of $|\mathcal{L}_0| - A$ can have infinitely many successors. Therefore, it must be the case that $\mathcal{L}_0 \cong \omega + \omega^*$ and $\omega_{\mathcal{L}_0} = A$. □

3.2 Degree Spectra Containing All Truth-Table Degrees

We extend the notion of Turing degree spectrum to wtt-degrees and tt-degrees, and will study $DgSp^{wtt}_{\mathcal{A}}(R)$ and $DgSp^{tt}_{\mathcal{A}}(R)$, respectively. We will first investigate what conditions are sufficient to have the truth-table degree spectrum be all of the tt-degrees, $\mathcal{D}^{\mathbf{tt}}$. We found that the conditions Harizanov and Ash, Cholak, and Knight found in Theorem 3.1.3 are sufficient to not only get the Turing degree spectrum to be as large as possible, but to also get the truth-table degree spectrum to be as large as possible. This, of course, implies that we also get the weak truth-table degree spectrum to be as large as possible. (Notice in the following theorem that we are assuming condition (1)(c) from Theorem 3.1.3.)

Theorem 3.2.1 (Chisholm et al. [5]). *Let $\mathcal{A}$ be a computable structure and R an additional computable relation on $\mathcal{A}$. Assume that there is a nonempty computable family $\mathbb{S} \subseteq I_{fin}(\mathcal{A}, \mathcal{A})$ such that the following two conditions hold.*

1. $(\forall p \in \mathbb{S})(\forall a, b \in A)(\exists q \in \mathbb{S})\ [q \supseteq p \wedge a \in \mathrm{dom}(q) \wedge b \in \mathrm{ran}(q)]$, *and*

2. $(\forall p \in \mathbb{S})(\exists q, r \in \mathbb{S})\ [q \supseteq p \wedge r \supseteq p \wedge \neg(q \sim_R r)]$.

Then, $DgSp^{tt}_{\mathcal{A}}(R) = \mathcal{D}^{\mathbf{tt}}$.

Proof. We will show that for all $X \in 2^\omega$, there exists an automorphism $f_X : \mathcal{A} \to \mathcal{A}$ such that $X \equiv_{tt} f_X(R)$. If we can show this, since $\{\deg(X) : X \in 2^\omega\} = \mathcal{D}^{\mathbf{tt}}$, we will have $\{\deg(f_X(R)) : X \in 2^\omega\} = \mathcal{D}^{\mathbf{tt}}$ and so $DgSp^{tt}_{\mathcal{A}}(R) = \mathcal{D}^{\mathbf{tt}}$.

First, we will construct two computable maps, $\mathcal{T} : 2^{<\omega} \to \mathbb{S}$ and $\mathcal{N} : 2^{<\omega} \to \omega$. We will define these maps in stages. In stage $s > 0$, we first define $\mathcal{N}(\sigma)$ for $|\sigma| = s - 1$, and then define $\mathcal{T}(\sigma^\wedge 0)$ and $\mathcal{T}(\sigma^\wedge 1)$.

Construction:

Stage $s = 0$: Consider all $\sigma \in 2^{<\omega}$ such that $|\sigma| = 0$. Therefore, we will only consider $<>$, the empty sequence.

Define $\mathcal{T}(<>) = \emptyset$, where $\emptyset$ represents the empty finite partial automorphism.

Stage $s + 1$: Assume that $\mathcal{T}$ is defined for all $\alpha \in 2^{<\omega}$ where $|\alpha| \leq s$, and that $\mathcal{N}$ is defined for all $\beta \in 2^{<\omega}$ where $|\beta| < s$. Suppose that $\mathcal{T}(\sigma) = r$ for some $\sigma \in 2^{<\omega}$ where $|\sigma| = s$.

To define $\mathcal{N}(\sigma)$, search for the least $n \in \omega$ and first $r_0, r_1 \in \mathbb{S}$ such that $r_0 \supseteq r$, $r_1 \supseteq r$, $r_1^{-1}(n) \in R$, and $r_0^{-1}(n) \notin R$. We know that we can find r_0 and r_1 because of assumption (2) in the statement of the theorem. Define $\mathcal{N}(\sigma) = n$.

To define $\mathcal{T}(\sigma^\wedge i)$ for $i = 0, 1$, find the first $q_i \in \mathbb{S}$ such that $q_i \supseteq r_i$ and $\mathrm{dom}(q_i)$ contains the least element of $\mathcal{A}$ not in the $\mathrm{dom}(r)$, and also the $\mathrm{ran}(q_i)$ contains the least element of $\mathcal{A}$ not in the $\mathrm{ran}(r)$. We know that we can find each q_i because of assumption (1) in the statement of the theorem. Define $\mathcal{T}(\sigma^\wedge i) = q_i$.

End of construction.

Let $X \in 2^\omega$. We will define the corresponding automorphism of $\mathcal{A}$ as follows: $f_X = \cup_{l\in\omega}\mathcal{T}(X \upharpoonright l)$. f_X is an automorphism by construction. Therefore, each infinite branch on the full binary tree corresponds to an automorphism of $\mathcal{A}$.

Lemma 3.2.2. *For every $X \in 2^\omega$, $f_X(R) \leq_{tt} X$.*

Proof. We need to use the oracle X to decide if $n \in \omega$ is an element of $f_X(R)$. First notice that because of how we defined $\mathcal{T}$, there will be some level l on the full binary tree where $n \in \operatorname{ran}(\mathcal{T}(\sigma))$ for all $\sigma \in 2^l$. We find this l by going through the construction of $\mathcal{T}$, which is a computable procedure (since $\mathbb{S}$ is computable) that does not depend on X. Therefore, $n \in \operatorname{ran}(\mathcal{T}(X \upharpoonright l))$. So,

$$n \in f_X(R) \iff (\mathcal{T}(X \upharpoonright l))^{-1}(n) \in R.$$

Notice that this algorithm will halt no matter what oracle we choose. Specifically, define the computable function $h_0(n)$ to be the least level of the full binary tree where $n \in \operatorname{ran}(\mathcal{T}(\sigma))$ for all $|\sigma| = h(n)$. Let the above computable procedure be φ_{e_0}. Then, $f_X(R) \leq_{tt} X$ via φ_{e_0} and h_0 and for all $n \in \omega$, $\varphi_{e_0}^\sigma(n)\downarrow$ for all $\sigma \in 2^{h_0(n)}$. □

Lemma 3.2.3. *For every $X \in 2^\omega$, $X \leq_{tt} f_X(R)$.*

Proof. Suppose we know that $X \upharpoonright l = \sigma$ for some $l \in \omega$. We want to know whether $X \upharpoonright (l+1) = \sigma^\wedge 1$ or $X \upharpoonright (l+1) = \sigma^\wedge 0$.

Observe that $X \upharpoonright (l+1) = \sigma^\wedge 1$ if and only if $\mathcal{T}(\sigma^\wedge 1) \subset f_X$, by definition of f_X. After defining $\mathcal{T}(\sigma)$, we search for the least $n \in \omega$ and first $r_0, r_1 \in \mathbb{S}$ such that $r_0 \supseteq r$, $r_1 \supseteq r$, $r_1^{-1}(n) \in R$, and $r_0^{-1}(n) \notin R$ so that we can define $\mathcal{N}(\sigma)$. Define $\mathcal{N}(\sigma) = n$.

$\mathcal{T}(\sigma^\wedge 1) = q_1$ where $q_1 \supseteq r_1$, and $\text{dom}(q_i)$ contains the least element of $\mathcal{A}$ not in the $\text{dom}(r)$, and the $\text{ran}(q_i)$ contains the least element of $\mathcal{A}$ not in the $\text{ran}(r)$. Therefore, $r_1^{-1}(n) \in R$, so $n \in r_1(R)$, and therefore $\mathcal{N}(\sigma) \in r_1(R)$, and thus $\mathcal{N}(\sigma) \in f_X(R)$. We have that

$$X \upharpoonright (l+1) = \sigma^\wedge 1 \Longleftrightarrow \mathcal{T}(\sigma^\wedge 1) \subset f_X \Longleftrightarrow \mathcal{N}(\sigma) \in f_X(R).$$

Similarly, we have that

$$X \upharpoonright (l+1) = \sigma^\wedge 0 \Longleftrightarrow \mathcal{T}(\sigma^\wedge 0) \subset f_X \Longleftrightarrow \mathcal{N}(\sigma) \notin f_X(R).$$

Notice that the above procedure is computable, which we will let be φ_{e_1}. Also, the procedure will halt no matter what oracle we choose. We can define the computable function $h_1(n) = n$ (trivially). Then, $X \leq_{tt} f_X(R)$ via φ_{e_1} and h_1 and for all $n \in \omega$, $\varphi_{e_1}^{\sigma}(n) \downarrow$ for all $\sigma \in 2^{h_1(n)}$. $\square$

By the lemmas above, we have that $f_X(R) \equiv_{tt} X$, and by our argument above we have that $DgSp_{\mathcal{A}}^{tt}(R) = \mathcal{D}^{\mathbf{tt}}$. $\square$

Chapter 4

Interval Trees

4.1 Introduction

Our main results are about the initial segments of computable linear orderings. The technique of constructing an interval tree of a linear ordering is a way for us to relate initial segments of the linear ordering to infinite branches of a computable binary tree. An *interval tree* is a partial function from a tree $T \subset \omega^{<\omega}$ to intervals of a linear ordering with certain properties. This function gives us a way to view linear orderings as trees, and therefore we can translate properties of trees onto equivalent ones on linear orderings. A rigorous definition of an interval tree is given in Section 4.2. The linear ordering we will be working with is of order type $\omega + \omega^*$. The following illustrates an example of an interval tree of this linear ordering.

Let $\mathcal{L} = (\omega, \prec)$ be a linear ordering of order type $\omega + \omega^*$ with domain ω, and such that $\prec$ is defined as follows:

$$0 \prec 2 \prec 4 \prec \cdots \prec 5 \prec 3 \prec 1.$$

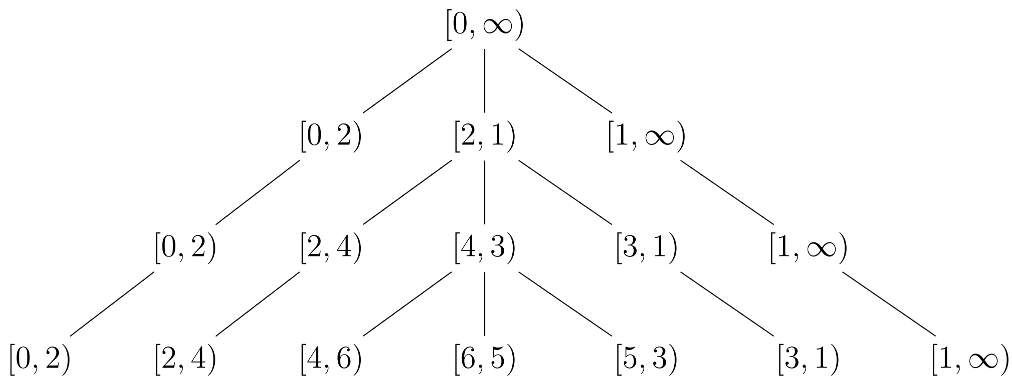

Figure 4.1: An example of an interval tree of order type $\omega + \omega^*$.

Figure 4.1 is an example of an interval tree of $\mathcal{L}$ generated to the 3rd level (the first level is the 0th level), and with $g(n) = 2n$, where g is a function such that $g(n) + 1$ is the number of nodes on the nth level of the interval tree. Please refer to the discussion after Definition 4.2.2 for the definition of g and how to generate the tree. It can be clearly seen in the figure that an interval tree is composed of half open intervals of the linear ordering. As we generate more of the tree, the interior of all of the intervals is going to be empty (in the limit). A branch of this tree is a sequence of intervals that corresponds to a maximal linearly ordered subset of sequences in $\omega^{<\omega}$ under $\subseteq$. We will show in Section 4.3 that branches of the interval tree correspond to initial segments of the linear ordering (by picking the right most element in each of the intervals along the branch). After making this correspondence, we prove a proposition and a theorem in this section, which will give us some interesting applications.

In Section 4.4 we show that if the interval tree of a countable linear ordering has countably many branches, then it is equivalent to the linear ordering having countably many initial segments. We also show that if this linear ordering has countably many initial segments, then it is equivalent to the linear ordering being *scattered* (not containing a copy of $\mathbb{Q}$). Now, notice in the figure that the interval on each level of the tree that contains an even and an odd element of $\mathcal{L}$ will keep splitting into subsequent intervals. This type of branching is called a *limit branch*. In the same section, we show that if a countable linear ordering has a unique noncomputable limit branch, then it must have order type $\omega+\omega^*$. These observations are used in the subsequent sections to prove our main theorems.

4.2 The Technique of Interval Trees

In this section, we will develop the technique of constructing an interval tree from a given linear ordering. First, we will introduce some terminology. We will let $<_{lex}$ be the lexicographical order on the sequences of $\omega^{<\omega}$. For $\sigma \in \omega^{<\omega}$, $|\sigma|$ is the length of σ. For $T \subset \omega^{<\omega}$, T is *leftward-closed* if whenever $\rho^\wedge(i+1) \in T$, we have that $\rho^\wedge i \in T$, for all $i \in \omega$. A computable tree is *computably bounded* if the number of nodes of length n on the tree are bounded by $f(n)$ where f is a computable function. We say that a tree is *finitely branching* if it has a finite number of nodes of length n, for all $n \in \omega$.

By the following proposition, we can assume that computable trees that are computably bounded are leftward-closed. (We want our trees to be leftward-closed so that we can clearly and rigorously define the technique of interval trees.)

Proposition 4.2.1. *Let $T \subset \omega^{<\omega}$ be a computably bounded, computable tree. Then, there exists a unique leftward-closed, computably bounded, computable tree $\overleftarrow{T} \subset \omega^{<\omega}$ such that $(T, <_{lex}) \cong (\overleftarrow{T}, <_{lex})$ and a unique isomorphism $f : (T, <_{lex}) \to (\overleftarrow{T}, <_{lex})$.*

In addition, let β be a branch of T and let $f(\beta) = \overleftarrow{\beta} \in \overleftarrow{T}$. Then, the graph of β is 1-equivalent to the graph of $\overleftarrow{\beta}$.

Proof. We will first define the isomorphism $f : (T, <_{lex}) \to (\overleftarrow{T}, <_{lex})$. Let $n = |\sigma|$ for all

$\sigma \in T$.

For $n = 0$: $f(<>) = <>$.

For $n > 0$: Let $\rho_0, \ldots, \rho_k$ be all of the elements of T, for some $k \in \omega$, such that $|\rho_i| = n-1$, for $0 \leq i \leq k$, and $f(\rho_i)$ is defined. Then, $f(\rho_i{}^\wedge m) = f(\rho_i)^\wedge n$ if and only if m is the nth natural number such that $\rho_i{}^\wedge m \in T$.

Now we need to show that f is an isomorphism. To show that f is one-to-one, suppose that $f(\rho) = f(\tau)$, for some $\rho, \tau \in T$ of length k. We will show that $\rho = \tau$. By definition of f,

$$f(\rho) = f(\rho_0{}^\wedge m_0) = f(\rho_0)^\wedge n_0, \text{ and}$$

$$f(\tau) = f(\tau_0{}^\wedge m_0') = f(\tau_0)^\wedge n_0'.$$

So, $f(\rho_0)^\wedge n_0 = f(\tau_0)^\wedge n_0'$, by assumption, and therefore $n_0 = n_0'$. If we continue in this manner, we will have that

$$f(\rho) = f(<>{}^\wedge m_{k-1})^\wedge n_{k-2}{}^\wedge \cdots {}^\wedge n_0 = f(<>)^\wedge n_{k-1}{}^\wedge \cdots {}^\wedge n_0, \text{ and}$$

$$f(\tau) = f(<>{}^\wedge m_{k-1}')^\wedge n_{k-2}{}^\wedge \cdots {}^\wedge n_0 = f(<>)^\wedge n_{k-1}{}^\wedge \cdots {}^\wedge n_0.$$

So, m_{k-1} is the n_{k-1}th natural number such that $<>{}^\wedge m_{k-1} \in T$ and m_{k-1}' is the n_{k-1}th natural number such that $<>{}^\wedge m_{k-1}' \in T$. Therefore $<>{}^\wedge m_{k-1} \in T = <>{}^\wedge m_{k-1}' \in T$, and $m_{k-1} = m_{k-1}'$. Continuing this argument gives us that $\rho = \tau$. Therefore, f is one-to-one.

Next, we will show that f is onto. Let $\overleftarrow{\rho} \in \overleftarrow{T}$ of length k. So, $\overleftarrow{\rho} = <>{}^\wedge n_{k-1}{}^\wedge \cdots {}^\wedge n_0$ for some $n_0, \ldots, n_{k-1} \in \omega$. Let m_0 be the n_{k-1}th number such that $<>{}^\wedge m_0 \in T$. Then

$f(<>\,^\wedge m_0) = f(<>)^\wedge n_{k-1} = n_{k-1}$. If we continue in this manner, we will let m_{k-1} be the n_0th natural number such that $<>\,^\wedge m_0{}^\wedge \cdots {}^\wedge m_{k-1} \in T$. Then, $f(<>\,^\wedge m_0{}^\wedge \cdots {}^\wedge m_{k-1}) = f(<>\,^\wedge m_0{}^\wedge \cdots {}^\wedge m_{k-2})^\wedge n_0 = \overleftarrow{\rho}$. Therefore f is onto, and so is an isomorphism. Since T is computable and computably bounded, so is $\overleftarrow{T}$.

Let β be a branch of T and let $f(\beta) = \overleftarrow{\beta}$. To show that the graph of β is 1-equivalent to the graph of $\overleftarrow{\beta}$, we need to ignore the labels on the nodes of T, and instead label these nodes with the labels from $\overleftarrow{T}$ (this is the graph of β). Let $\sigma \in \omega^{<\omega}$. σ is in the graph of β if and only if σ is in the graph of $\overleftarrow{\beta}$. Therefore, the graph of β is 1-equivalent to the graph of $\overleftarrow{\beta}$. □

Definition 4.2.2. *Let $\mathcal{L}$ be a linear ordering. An* interval tree *on $\mathcal{L}$ is a partial function $\mathcal{F}$ from a leftward-closed, finitely branching, tree $T \subset \omega^{<\omega}$ with no terminal nodes, to the set of (nonempty) intervals of $\mathcal{L}$ such that*

1. *$\mathcal{F}(<>) = |\mathcal{L}|$,*
2. *$\mathcal{F}(\rho)$ is the disjoint union of all intervals $\mathcal{F}(\rho^\wedge m)$ such that $\rho^\wedge m \in T$, and*
3. *for every infinite branch β of T, $\cap_{n\in\omega}\text{interior}(\mathcal{F}(\beta \upharpoonright n)) = \emptyset$.*

We will only define an interval tree for linear orderings with a left endpoint (and possibly a right endpoint) since we will be mostly concerned with the linear ordering $\omega + \omega^*$ in our results. We could similarly define an interval tree for linear orderings with a right endpoint (and possibly a left endpoint), or for linear orderings with no endpoints, by varying the following definition.

For the remainder of the section (unless otherwise stated), we will assume that $\mathcal{L}$ is a linear ordering with a left endpoint a_0 such that $|\mathcal{L}| = \{a_i\}_{i\in\omega}$. Notice that the left endpoint is the first element enumerated into $|\mathcal{L}|$. Also, we will let $g : \omega \to \omega$ be a nondecreasing computable function where $g(0) = 0$. We will define the interval tree of $\mathcal{L}$, $\mathcal{F} = \mathcal{F}^g_{\mathcal{L}}$, as follows. Let $n = |\sigma|$, for $\sigma \in \omega^{<\omega}$.

For $n = 0$: Enumerate $<>$ into the dom($\mathcal{F}$). Let $\mathcal{F}(<>) = [a_0, \infty)$.

For $n > 0$: Let $\{b_0, b_1, \ldots, b_{g(n)}\}$ be the first $g(n) + 1$ elements enumerated into $\mathcal{L}$ in $<_{\mathcal{L}}$-order. So, $\{b_0, b_1, \ldots, b_{g(n)}\} = \{a_0, a_1, \ldots, a_{g(n)}\}$ where $b_0 <_{\mathcal{L}} b_1 <_{\mathcal{L}} \cdots <_{\mathcal{L}} b_{g(n)}$. Find $\rho \in \text{dom}(\mathcal{F})$ such that $|\rho| = n - 1$. Since $\rho \in \text{dom}(\mathcal{F})$, $\mathcal{F}(\rho) = [b_j, b_{j+m})$ for some $j, m \in \omega$. (Note that $b_{g(n)+1} = \infty$.) Define $\mathcal{F}(\rho^\wedge i) = [b_{j+i}, b_{j+i+1})$ for $0 \leq i \leq m - 1$. For all $i \geq m$, $\rho^\wedge i \notin \text{dom}(\mathcal{F})$.

Though the rightmost interval at each level n of the interval tree is of the form $[b_{g(n)}, \infty)$, if our linear ordering has a right endpoint, then we consider $b_{g(n)}$ as the rightmost element of the linear ordering so far at level n.

All of the sequences $\sigma \in \omega^{<\omega}$ that are also in the dom($\mathcal{F}$) form a leftward-closed, finitely branching, tree T with no terminal nodes. We think of the interval tree $\mathcal{F}$ as a labeling of the nodes of T with the corresponding intervals. For different functions $g(n)$ we get different interval trees $\mathcal{F}$. Primarily, $g(n)$ gives us a way to index the elements on each level of the tree. In fact, we have the following observation:

$$\text{For all } n \in \omega, g(n) = |\omega^n \cap \text{dom}(\mathcal{F})| - 1. \tag{4.1}$$

For convenience of notation, let $L^g_n = \{a_i : i \leq g(n)\}$ for all $n \in \omega$. So, L^g_n contains the first $g(n) + 1$ elements enumerated into $\mathcal{L}$ for each n. In addition, the nth level of the interval tree $\mathcal{F}$ contains intervals constructed by the elements in L^g_n. Therefore, it is easy to see that L_n^g is computable in $\mathcal{F}$.

The following proposition shows that interval trees are well-defined. If the domains of two interval trees are the same, then their corresponding linear orderings are isomorphic. If these linear orderings are computable and both of the corresponding nondecreasing functions on ω are computable, then the linear orderings are computably isomorphic. On the other hand, Proposition 4.2.4 says that every computably bounded, computable, leftward-closed, tree $T \subset \omega^{<\omega}$ with no terminal nodes, is the domain of some interval tree of some unique (up to computable isomorphism) computable linear ordering.

Proposition 4.2.3. *Let $\mathcal{L}$ and $\tilde{\mathcal{L}}$ be computable linear orderings with left endpoints, and let $g, \tilde{g}$ be nondecreasing computable functions on ω such that $g(0) = \tilde{g}(0) = 0$. Assume that $\operatorname{dom}(\mathcal{F}_{\mathcal{L}}^g) = \operatorname{dom}(\mathcal{F}_{\tilde{\mathcal{L}}}^{\tilde{g}})$. Then,*

1. *$g = \tilde{g}$, and*

2. *$\mathcal{L} \cong_c \tilde{\mathcal{L}}$.*

Proof. To prove 1, we will use observation 4.1. By this observation, $g(n) = |\omega^n \cap \operatorname{dom}(\mathcal{F}_{\mathcal{L}}^g)| - 1$ and $\tilde{g}(n) = |\omega^n \cap \operatorname{dom}(\mathcal{F}_{\tilde{\mathcal{L}}}^{\tilde{g}})| - 1$. Since $\operatorname{dom}(\mathcal{F}_{\mathcal{L}}^g) = \operatorname{dom}(\mathcal{F}_{\tilde{\mathcal{L}}}^{\tilde{g}})$, we have that $\tilde{g}(n) = g(n)$ for all $n \in \omega$.

To prove 2, we will define a computable isomorphism $f : \mathcal{L} \to \tilde{\mathcal{L}}$. For each $a \in |\mathcal{L}|$, let n be such that $a \in L_n^g$. Let m be such that a is the mth greatest element of L_n^g. Define $f(a)$ to be the mth greatest element of $\tilde{L}_n^g$.

Now, we need to show that f is well-defined. Suppose now that a is the $(m+k)$th greatest element of L_{n+1}^g. So, L_{n+1}^g has k new points to the left of a. Since $\operatorname{dom}(\mathcal{F}_{\mathcal{L}}^g) = \operatorname{dom}(\mathcal{F}_{\tilde{\mathcal{L}}}^g)$, it must be the case that k new points are to the left of $f(a)$. By induction, for all $N \geq n$ and for all $K \leq g(N)$, a is the Kth greatest element of L_N^g if and only if $f(a)$ is the Kth greatest element of $\tilde{L}_N^g$.

Next, we need to show that f is an order-preserving isomorphism. To show that f is order-preserving, let $a, b \in |\mathcal{L}|$ such that $a <_{\mathcal{L}} b$. We need to show that $f(a) <_{\tilde{\mathcal{L}}} f(b)$. Let n be such that $a, b \in L_n^g$. Let m be such that a is the mth greatest element of L_n^g, and let m' be such that b is the m'th greatest element of L_n^g. So, $f(a)$ is the mth greatest element of $\tilde{L}_n^g$ and $f(b)$ is the m'th greatest element of $\tilde{L}_n^g$. Since $a <_{\mathcal{L}} b$, we have that $m < m'$. So, $f(a) <_{\tilde{\mathcal{L}}} f(b)$.

In order to show that f is an isomorphism, we will first show that f is one-to-one. Let $a, b \in |\mathcal{L}|$. Suppose that $f(a) = f(b)$. We must show that $a = b$. Let n be such that $a, b \in L_n^g$. Suppose that $f(a)$ is the mth greatest element of $\tilde{L}_n^g$. Then, a and b are both the mth greatest element of L_n^g. So, $a = b$. To show that f is onto, let $b \in |\tilde{\mathcal{L}}|$. We must show that there exists an $a \in |\mathcal{L}|$ such that $f(a) = b$. Let n be such that $b \in \tilde{L}_n^g$. Let m be such that b is the mth greatest element of $\tilde{L}_n^g$. It is also the case that L_n^g contains an mth greatest element, say a. By definition of f, $f(a) = b$.

Therefore, f is an order-preserving bijection from L_n^g to $\tilde{L}_n^g$. It is a computable isomorphism since $\mathcal{L}$, $\tilde{\mathcal{L}}$, and g are computable and we have an algorithm which computes f. □

Proposition 4.2.4. *Let $T \subset \omega^{<\omega}$ be computably bounded, computable, leftward-closed, and have no terminal nodes. Let $g(n) = |T \cap \omega^n| - 1$, for all $n \in \omega$. Then, there exists a computable linear ordering $\mathcal{L}$ such that* $\mathrm{dom}(\mathcal{F}_{\mathcal{L}}^g) = T$. *In addition, by Proposition 4.2.3, $\mathcal{L}$ is unique up to computable isomorphism.*

Proof. We will define $\mathcal{L}$ and $\mathcal{F} = \mathcal{F}^g_{\mathcal{L}}$ simultaneously. Let $n = |\sigma|$, for $\sigma \in \omega^{<\omega}$.

For $n = 0$: Let $\mathcal{F}(<>) = [c_0, d_0)$, and enumerate c_0 into $|\mathcal{L}|$, the left endpoint of $\mathcal{L}$. Here, $d_0 = \infty$.

For $n > 0$: Let $\rho_0, \rho_1, \ldots, \rho_{g(n-1)}$ be all of the elements of $T \cap \omega^{n-1}$ in $<_{\text{lex}}$-order such that they are also in $\text{dom}(\mathcal{F})$. For $0 \leq k \leq g(n-1)$, let $\mathcal{F}(\rho_k) = [c_k, d_k)$. (Note that $d_{g(n-1)} = \infty$. Though $\mathcal{F}(\rho_k)$ is already defined, we will let $\mathcal{F}(\rho_k)$ be the interval $[c_k, d_k)$ for convenience of notation.) For all k, we will let $m_k \in \omega$ be greatest such that $\rho_k{}^\wedge m_k \in T$. We will divide $[c_k, d_k)$ into $m_k + 1$ subintervals by inserting m_k new elements between c_k and d_k. So, we will define $\mathcal{F}(\rho_k{}^\wedge j)$ to be the $(j+1)$st interval for $0 \leq j \leq m_k$. Enumerate these new elements into $|\mathcal{L}|$, and keep the induced order on these elements by the dividing of the intervals into subintervals. This induced order is the order in $\mathcal{L}$.

More rigourously, let $M_k = g(n-1) + \sum_{k'<k} m_{k'}$ for $0 \leq k \leq g(n-1)$. M_k keeps track of the index numbers of all of the new elements we are adding to $\mathcal{L}$. We want to make sure that all of the elements we add to $\mathcal{L}$, as we go through the tree level by level, are different. Therefore, our numbering continues throughout the levels, which is the reason for the $g(n-1)$ in the defintion. We also keep track of all of the elements added on the level we are currently in, which is the reason for the summation.

Now, let

$$\mathcal{F}(\rho_k{}^\wedge j) = \begin{cases} [a_{M_k+j}, a_{M_k+j+1}), & \text{if } 0 < j < m_k, \\ [c_k, a_{M_k+1}), & \text{if } 0 = j < m_k, \\ [a_{M_k+m_k}, d_k), & \text{if } 0 < j = m_k, \\ [c_k, d_k), & \text{if } 0 = j = m_k. \end{cases}$$

At each level, we enumerate these a_i's into $|\mathcal{L}|$, for $M_k+1 \leq i \leq M_k+m_k$. As mentioned above, the induced ordering on these elements is the ordering in $\mathcal{L}$. So, we have constructed a computable linear ordering $\mathcal{L}$ such that $\mathrm{dom}(\mathcal{F}^g_{\mathcal{L}}) = T$. □

4.3 Branches of Interval Trees

We are mostly concerned with initial segments of computable linear orderings, which, as we show, correspond to branches of the interval tree. Notice that all branches of an interval tree are infinite. A branch of the interval tree is a sequence of intervals obtained by restricting the domain of the interval tree to just a branch. We show that both of these branches are tt-equivalent. Then, in Theorem 4.3.3, we go on to define a bijection from the initial segments of the linear ordering to branches of the interval tree.

Definition 4.3.1. *Let $\mathcal{F}$ be an interval tree for some linear ordering $\mathcal{L}$. A* branch of $\mathcal{F}$ *is the sequence $b = \{b(0), b(1), \ldots\}$ of intervals of $\mathcal{F}$ obtained along some branch $\beta \subset \mathrm{dom}(\mathcal{F})$, such that for all n, $b(n) = \mathcal{F}(\beta \upharpoonright n)$.*

Proposition 4.3.2. *Let $\mathcal{L}$ be a linear ordering, and $\mathcal{F}$ be a computable interval tree of $\mathcal{L}$. Let b be a branch of $\mathcal{F}$ and let β be the corresponding branch in $\mathrm{dom}(\mathcal{F})$. Then, $b \equiv_{tt} \beta$.*

Proof. First, we will show that $b \leq_{tt} \beta$.

Case 1: We want to find out if an interval of the form $[a_k, \infty)$ is in b. First, find the least $n \in \omega$ such that $a_k \in L^g_n$. (So, $a_k \notin L^g_{n-1}$.) Therefore, if $[a_k, \infty)$ is in $\mathcal{F}$, it must be on level n of $\mathcal{F}$. If $\mathcal{F}(\beta \upharpoonright n) = [a_k, \infty)$, then $[a_k, \infty) \in b$. Otherwise, $[a_k, \infty) \notin b$.

Case 2: We want to find out if an interval of the form $[a_k, a_l)$ is in b, where $a_l \neq \infty$.

First, find the least $n \in \omega$ such that $a_k, a_l \in L_n^g$. Similarly, if $[a_k, a_l)$ is in $\mathcal{F}$, then it must be on level n of $\mathcal{F}$. If $\mathcal{F}(\beta \upharpoonright n) = [a_k, a_l)$, then $[a_k, a_l) \in b$. Otherwise, $[a_k, a_l) \notin b$.

So, we have shown that $b \leq_{tt} \beta \oplus \mathcal{F}$, and since $\mathcal{F}$ is computable and this algorithm halts for any oracle β, it is a tt-reduction. Therefore, we have that $b \leq_{tt} \beta$.

Next, we will show that $\beta \leq_{tt} b$. Let $\alpha \in \omega^n$. We want to find out if $\alpha \in \beta$. The sequence α is of length n, therefore if it is in β, then $b(n) = \mathcal{F}(\alpha)$. If $\mathcal{F}(\alpha) = b(n)$, then $\alpha \in \beta$. Otherwise, $\alpha \notin \beta$.

Similarly, we have shown that $\beta \leq_{tt} b \oplus \mathcal{F}$, and since $\mathcal{F}$ is computable and this algorithm halts for any oracle b, it is a tt-reduction. Therefore, we have that $\beta \leq_{tt} b$. Actually, since we only need to know the nth interval of b, we have that $\beta \leq_{btt(1)} b$ (and therefore $\beta \leq_1 b$). □

Theorem 4.3.3. *Let $\mathcal{F} = \mathcal{F}_{\mathcal{L}}^g$ be a computable interval tree of a computable linear ordering $\mathcal{L}$. Let $\mathcal{I}$ be the set of nonempty initial segments of $\mathcal{L}$. Let $\mathcal{B} : \mathcal{I} \to [\mathcal{F}]$ be defined as follows: $(\mathcal{B}(I))(n) = [c_n, d_n)$, where $c_n <_{\mathcal{L}} d_n$ are adjacent elements of $L_n^g \cup \{\infty\}$ with $c_n \in I$ and $d_n \notin I$.*

Then,

1. *$\mathcal{B}$ is a bijection.*

2. *Every initial segment $I \in \mathcal{I}$ is tt-equivalent to its image branch $\mathcal{B}(I)$, so $I \equiv_{tt} \mathcal{B}(I)$.*

3. *Let $I_1, I_2 \in \mathcal{I}$. For $i = 1, 2$, let β_i be the branch of $\operatorname{dom}(\mathcal{F})$ corresponding to $\mathcal{B}(I_i)$, so that if $\mathcal{B}(I) = \{[c_0, d_0), [c_1, d_1), \ldots\}$ where $(\mathcal{B}(I))(n) = [c_n, d_n)$ for all $n \in \omega$, then $\beta \upharpoonright n = [c_n, d_n)$.*

   Then, $I_1 \subset I_2$ if and only if $\beta_1 <_{lex} \beta_2$.

Proof. To prove 1, we will define a bijection $\mathcal{J} : [\mathcal{F}] \to \mathcal{I}$ given by:

$$\text{For } 0 \leq k \leq g(n),\ a_k \in \mathcal{J}(b(n)) \text{ if and only if } a_k \leq_{\mathcal{L}} c_n,\ \text{where } b(n) = [c_n, d_n).$$

If $\mathcal{J}$ is the inverse of $\mathcal{B}$, then both $\mathcal{B}$ and $\mathcal{J}$ are bijections. Let $I \in \mathcal{I}$, and $b = \mathcal{B}(I) = \{[c_0, d_0), [c_1, d_1), \ldots\}$. Notice that I is the initial segment of $\mathcal{L}$ with elements $\leq_{\mathcal{L}} c_n$, for all $n \in \omega$. We must show that $\mathcal{J}(b) = I$. Notice that $\mathcal{J}(b(n))$ are all of the elements a_k, for $0 \leq k \leq g(n)$, such that $a_k \leq_{\mathcal{L}} c_n$. As $n \to \infty$, we can see that $\mathcal{J}(b(n)) = I$. So, $\mathcal{J}$ is a bijection and therefore $\mathcal{B}$ is also a bijection.

To prove 2, we will first show that $I \leq_{tt} \mathcal{B}(I)$. We want to find out if an element a of the linear ordering $\mathcal{L}$, is in the initial segment I. Find the least n such that $a \in L_n^g$. Let $(\mathcal{B}(I))(n) = [c_n, d_n)$. Then, it is easy to see that if $a \leq_{\mathcal{L}} c_n$ then $a \in I$. Otherwise, $a \notin I$.

We have shown that $I \leq_{tt} \mathcal{B}(I) \oplus L_n^g \oplus \leq_{\mathcal{L}}$, and since L_n^g and $\leq_{\mathcal{L}}$ are computable and this algorithm halts for any oracle, it is a tt-reduction. Therefore, we have that $I \leq_{tt} \mathcal{B}(I)$. Actually, since we only need to know the nth interval of $\mathcal{B}(I)$, we have that $I \leq_{btt(1)} \mathcal{B}(I)$.

Next, we will show that $\mathcal{B}(I) \leq_{tt} I$. We want to find out if an interval $[c, d)$ is an element of $\mathcal{B}(I)$. First of all, it must be the case that $c \in I$ and $d \notin I$. If this is not the case, then $[c, d) \notin \mathcal{B}(I)$.

Case 1: $d = \infty$. Let n be least such that $c \in L_n^g$. If c is the $(g(n))$th element of L_n^g, with respect to the order in $\mathcal{L}$, then $[c, d) \in \mathcal{B}(I)$. Otherwise, $[c, d) \notin \mathcal{B}(I)$.

Case 2: $d \neq \infty$. Let n be least such that $c, d \in L_n^g$. If c and d are adjacent elements of L_n^g, then $[c, d) \in \mathcal{B}(I)$. Otherwise, $[c, d) \notin \mathcal{B}(I)$.

We have shown that $\mathcal{B}(I) \leq_{tt} I \oplus L_n^g \oplus \leq_{\mathcal{L}}$, and since L_n^g and $\leq_{\mathcal{L}}$ are computable and this algorithm halts for any oracle, it is a tt-reduction. Therefore, we have that $\mathcal{B}(I) \leq_{tt} I$. □

4.4 Core Propositions

The following proposition is true for all countable linear orderings (regardless if they have left endpoints or not). In this proposition, we further relate linear orderings with their corresponding interval trees and notice that if a linear ordering has only countably many initial segments, then its corresponding interval tree has only countably many branches. Recall that a linear ordering $\mathcal{L}$ is *scattered* if it does not contain a copy of $\mathbb{Q}$. Therefore, $\mathcal{L}$ contains no subset $S \subseteq |\mathcal{L}|$ such that $(S, <_{\mathcal{L}}) \cong (\mathbb{Q}, <)$.

Proposition 4.4.1. *Let $\mathcal{L}$ be a computable linear ordering with $|\mathcal{L}| = \{a_i\}_{i\in\omega}$. Then, the following are equivalent:*

1. *$\mathcal{L}$ is scattered.*
2. *$\mathcal{L}$ has only countably many initial segments.*
3. *The interval tree $\mathcal{F}$ of $\mathcal{L}$ has only countably many branches.*

Proof. First we will prove $2 \Rightarrow 1$. We will assume that $\mathcal{L}$ is not scattered and show that $\mathcal{L}$ has uncountably many initial segments. Since $\mathcal{L}$ is not scattered, there exists a subset $S \subseteq |\mathcal{L}|$ such that $(S, <_{\mathcal{L}}) \cong (\mathbb{Q}, <_{\mathcal{L}})$. Since $\mathbb{Q}$ has uncountably many initial segments, S also has uncountably many initial segments. Let I be an initial segment of S. Let $D(I)$ be the downward closure of I in $\mathcal{L}$. In other words, $D(I) = \{a : a \leq_{\mathcal{L}} s, \text{ for all } s \in I\}$. Then, $D(I)$ is an initial segment of $\mathcal{L}$. Since all of the downward closures of all of the initial segments of S are different, D is an injection. So, $\mathcal{L}$ has uncountably many initial segments.

Next, we will prove $1 \Rightarrow 2$. We will assume that $\mathcal{L}$ has uncountably many initial segments and show that $\mathcal{L}$ is not scattered. Let $\mathcal{I}$ be the set of all of the initial segments of $\mathcal{L}$. By assumption, $\mathcal{I}$ is uncountable. Therefore, $\mathcal{I}$ is a closed set in Cantor space, 2^{ω}. So, you can view $\mathcal{I}$ on a tree in $2^{<\omega}$ in the following way.

The empty node, $<>$, is on level 0 of the tree. Now, let $I \in \mathcal{I}$. On level n of the tree (sequences of length n), for $n > 0$, the tree branches right if $a_{n-1} \in I$, and branches left if $a_{n-1} \notin I$. Therefore, this initial segment I corresponds to a branch on a tree containing only branches representing each initial segment in $\mathcal{I}$. Since $\mathcal{I}$ is a closed set in Cantor space, it has a perfect subset $\mathcal{J} \subseteq \mathcal{I}$. Let $T \subseteq 2^{<\omega}$ be a perfect tree such that $[T] = \mathcal{J}$. We will define a map $\mathcal{U} : 2^{<\omega} \to T$ by recursion as follows. Let $n = |\sigma|$, for $\sigma \in 2^{<\omega}$.

For $n = 0$: Let $\mathcal{U}(<>) =<>$.

For $n > 0$: Given $\sigma \in \text{dom}(\mathcal{U})$ for some $|\sigma| = n-1, \mathcal{U}(\sigma^\wedge 0)$ and $\mathcal{U}(\sigma^\wedge 1)$ are incompatible extensions in T of $\mathcal{U}(\sigma)$. More explicitly, given $\mathcal{U}(\sigma) = \rho$,

$$\mathcal{U}(\sigma^\wedge 0) = \rho^\wedge \tau \text{ where } \tau \text{ ends in } 0,$$

$$\mathcal{U}(\sigma^\wedge 1) = \rho^\wedge \gamma \text{ where } \gamma \text{ ends in } 1, \text{ and where}$$

$$\rho^\wedge \tau \nsubseteq \rho^\wedge \gamma, \ \rho^\wedge \gamma \nsubseteq \rho^\wedge \tau, \text{ and } \rho^\wedge \tau, \rho^\wedge \gamma \in T.$$

For each $\sigma \in 2^{<\omega}$, let $d(\sigma)$ be the element in $|\mathcal{L}|$ on which $\mathcal{U}(\sigma^\wedge 0)$ and $\mathcal{U}(\sigma^\wedge 1)$ are defined and disagree. Let $S = \{d(\sigma) : \sigma \in 2^{<\omega}\}$. We will show that S has the order type of $\mathbb{Q}$ under $\mathcal{L}$. First, we will show that S has no least or greatest element. For every σ, there is a family of initial segments that contain $d(\sigma)$. This family contains initial segments that may contain elements to the right of $d(\sigma)$ (with respect to the linear ordering $\mathcal{L}$). There is also a family of initial segments that do not contain $d(\sigma)$. This family contains initial segments which contain elements only to the left of $d(\sigma)$. Therefore, $d(\sigma)$ is between $d(\sigma^\wedge 0)$ and $d(\sigma^\wedge 1)$. Therefore, S has no least or greatest element.

Next, we will show that S is dense. Let $\rho, \tau \in 2^{<\omega}$ with $\rho \neq \tau$. If ρ and τ are incompatible (so $\rho \nsubseteq \tau$ and $\tau \nsubseteq \rho$), then let γ be the longest string such that $\rho \supseteq \gamma$ and $\tau \supseteq \gamma$. According to our previous argument, $d(\gamma)$ is between $d(\gamma^\wedge 0)$ and $d(\gamma^\wedge 1)$. Suppose that $\rho \supseteq \gamma^\wedge 0$ (there is a similar argument if you assume that $\rho \supseteq \gamma^\wedge 1$). So ρ does not contain $d(\gamma)$ and τ does contain $d(\gamma)$. Now $d(\rho)$ is the element in $\mathcal{L}$ on which $\mathcal{U}(\rho^\wedge 0)$ and $\mathcal{U}(\rho^\wedge 1)$ are defined and disagree. Again, according to the above argument, there is a family of initial segments which contain initial segments which do not contain $d(\gamma)$ and do not contain $d(\rho)$, so they contain elements to the left of these two elements. There is also a family of initial segments which contain initial segments which do not contain $d(\gamma)$ and do contain $d(\rho)$. So, $d(\rho)$ is to the left of $d(\gamma)$. Similarly, $d(\tau)$ is to the right of $d(\gamma)$. So, S is dense since $d(\gamma)$ is strictly between $d(\rho)$ and $d(\tau)$.

Suppose now that ρ and τ are compatible and that $\rho \supset \tau$ (there is a similar argument if you assume that $\tau \supset \rho$). Like the above argument, $d(\rho)$ will either be to the left or to the right of $d(\tau)$. $d(\rho^\wedge 0)$ will be either to the left or to the right of $d(\rho)$. $d(\rho^\wedge 1)$ will be on the left of $d(\rho)$ if $d(\rho^\wedge 0)$ is to the right of $d(\rho)$, and $d(\rho^\wedge 1)$ will be on the right of $d(\rho)$ otherwise. So, we have that either $d(\rho^\wedge 0)$ or $d(\rho^\wedge 1)$ is in between $d(\rho)$ and $d(\tau)$.

Next, we will prove $2 \Rightarrow 3$. We will assume that $\mathcal{L}$ has only countably many initial segments and show that the corresponding interval tree $\mathcal{F}$ has only countably many branches. By Theorem 4.3.3, there is a bijection $\mathcal{B} : \mathcal{I} \to [\mathcal{F}]$, where $\mathcal{I}$ is the set of all of the initial segments of $\mathcal{L}$. Since $\mathcal{I}$ is countable, $[\mathcal{F}]$ must also be countable.

Finally, we will prove $3 \Rightarrow 2$. We will assume that $\mathcal{F}$ has only countably many branches, and show that $\mathcal{L}$ has only countably many initial segments. Similarly to the proof of $2 \Rightarrow 3$, by Theorem 4.3.3, we are done. □

Definition 4.4.2. *Let $T \subset \omega^{<\omega}$, and let $\beta \in [T]$. The branch β is a* limit branch *of T if for all $\sigma \subset \beta$, there exist σ_0 and σ_1 such that $\sigma_0 \supset \sigma$, $\sigma_1 \supset \sigma$, $\sigma_0 \nsubseteq \sigma_1$, and $\sigma_1 \nsubseteq \sigma_0$.*

If β is a limit branch of T, then there are an infinite number of branches which share an initial segment (in $\omega^{<\omega}$) with β. We will let β_0 be the branch which shares the shortest initial segment, say σ_0, with β. So, if $\sigma_0{}^\wedge 0 \subset \beta$, then $\sigma_0{}^\wedge 1 \subset \beta_0$, and visa-versa. Similarly, we will let β_n be the branch which shares the nth shortest initial segment with β. So, we have a collection of branches $\{\beta_n\}_{n\in\omega}$ of T, where $\beta_n \neq \beta$ for all n, and such that $\lim_{n\to\infty} \beta_n = \beta$.

Proposition 4.4.3. *Let $\mathcal{L}$ be a countable, non-finite, linear ordering, and $\mathcal{F}$ the corresponding interval tree. Let $T = \mathrm{dom}(\mathcal{F})$. Then, T has a unique limit branch if and only if $\mathcal{L}$ is isomorphic to a non-finite substructure of $\omega + \omega^*$. In particular, $\mathcal{L} \cong \omega + \omega^*$, or $\mathcal{L} \cong \omega + n$, or $\mathcal{L} \cong n + \omega^*$, where $n \in \omega$. If the limit branch of T is noncomputable, then $\mathcal{L} \cong \omega + \omega^*$.*

Proof. Assume that T has a unique limit branch β. We will associate β with its corresponding branch on the interval tree, so $\beta = \{[c_0, d_0), [c_1, d_1), \ldots\}$ where $[c_i, d_i)$ is an interval of elements from $\mathcal{L}$ for every $i \in \omega$. Let $\{\beta_n\}_{n\in\omega}$ be a collection of infinite branches of T such that $\beta_n \neq \beta$ for all n and $\lim_{n\to\infty} \beta_n = \beta$. Let β_0 be such that $\sigma \subset \beta_0$, $\sigma \subset \beta$ for some $\sigma \in \omega^{<\omega}$ where $|\sigma| = l$, and there is no $\gamma \in \omega^{<\omega}$ such that $\sigma \subset \gamma$, $\gamma \subset \beta_0$ and $\gamma \subset \beta$ (so σ is the largest initial segment extended by both β_0 and β). Then, β_0 and β share the interval $[c_l, d_l)$ in common, however, at level $l+1$ a new element is introduced between the elements c_l and d_l (with respect to the linear ordering $\mathcal{L}$). Since $\cap_{i\in\omega}\mathrm{interior}(\beta_0 \upharpoonright i) = \emptyset$ by definition of interval tree, since $\mathcal{L}$ is countable and β is unique, along β_0 there are only a finite number of elements that are inserted between c_l and d_l. Similarly, every element $\beta_k \in \{\beta_n\}_{n\in\omega}$ inserts only a finite number of elements in the interval in β which corresponds to the longest initial segment that β_k and β both extend. Since β is a limit branch, we have an infinite number of different branches in $\{\beta_n\}_{n\in\omega}$ which add at least one new element to the interval in β which corresponds to the longest initial segment that the branch shares with β. Again, since $\mathcal{L}$ is countable and β is unique, we have that $\mathcal{L} \cong \omega + \omega^*$, or $\mathcal{L} \cong \omega + n$, or $\mathcal{L} \cong n + \omega^*$, where $n \in \omega$.

Assume that $\mathcal{L}$ is isomorphic to a non-finite substructure of $\omega + \omega^*$. We will show that T has a unique limit branch. Suppose that $\mathcal{L} \cong \omega + n$ for some $n \in \omega$. On the right side of T, there will be a level where there are n infinite branches (and no more than n branches from that level on), representing each of the n elements of the right side of the linear ordering. The limit branch of T is the branch immediately to the left of these n branches, which contains only infinite intervals. This limit branch is unique. Note that since we can find this limit branch computably, it is computable. We can use a similar argument for when $\mathcal{L} \cong n + \omega^*$.

Assume that $\mathcal{L} \cong \omega + \omega^*$. The limit branch of T must contain the infinite interval on each level of the tree. An infinite interval has the form that one element is from $\omega_{\mathcal{L}}$ and one element is from $\omega^*_{\mathcal{L}}$. Therefore, by the definition of an interval tree, each level of T has only one infinite interval. Therefore, there is only one unique limit branch. By our argument in the above paragraph, if the limit branch is noncomputable, then $\mathcal{L} \cong \omega + \omega^*$. □

Chapter 5

Applications of Interval Trees

5.1 Limit Branch Inside Limit Computable Turing Degrees

In this section, we are trying to extend Theorem 3.1.9 for tt-degrees. Therefore, we are trying to show that the truth-table degree spectrum of $\omega_{\mathcal{L}}$ on $\mathcal{L}$, where $\mathcal{L}$ is a linear ordering of order type $\omega + \omega^*$, is all of the Δ^0_2 tt-degrees (so all of the tt-degrees computable in $\emptyset'$). This is not the case, and we cannot even extend Theorem 3.1.9 for wtt-degrees.

Corollary 5.1.2 gives us the best possible result, that we can find a computable linear ordering of order type $\omega+\omega^*$ such that for any Δ^0_2 set A, $\omega_{\mathcal{L}} \leq_{tt} A$, however, we only have that $A \leq_T \omega_{\mathcal{L}}$. Corollary 5.3.2 says that we cannot replace T-reducibility with wtt-reducibility or tt-reducibility in Corollary 5.1.2.

Theorem 5.1.1. *For every Δ^0_2 set A, there is a computable tree $T \subset 2^{<\omega}$ with no terminal nodes such that T has a unique limit branch β, and $A \leq_T \beta \leq_{tt} A$.*

By Proposition 4.4.3, we have the following corollary.

Corollary 5.1.2 (Chisholm, et al. [5])**.** *For every Δ^0_2 set A, there is a computable linear ordering $\mathcal{L}$ of order type $\omega + \omega^*$ such that $A \leq_T \omega_{\mathcal{L}} \leq_{tt} A$.*

Proof of Theorem 5.1.1. Let A be a Δ^0_2 set. By definition, $A = \lim_{s\to\infty} A_s$ for uniformly computable sets A_s. We will define a set M of natural numbers as follows. Let $M = \{m_k\}_{k\in\omega}$ where $m_0 = 0$ and m_{k+1} is the least $m > m_k$ such that $A_m \upharpoonright k = A \upharpoonright k$, for $k \geq 0$. So, for each k, m_{k+1} is the least number where $A_{m_{k+1}}$ and A agree on the first k elements and such that $m_{k+1} > m_k$, so that the sequence is strictly increasing.

Lemma 5.1.3. *$A \leq_T M \leq_{tt} A$.*

Proof. First we will show that $M \leq_{tt} A$. We want to determine if a number $m \in \omega$ is an element of M. Enumerate $m_0, m_1, \ldots$ (we do this just using A_s and A). If $m = m_i$ for some $i \in \omega$, then $m \in M$. Since $m_0 < m_1 < \cdots$ once we get to an m_k such that $m_k > m$, then we know that $m \notin M$. This procedure converges for every oracle A and any $m \in \omega$. Therefore, we have a tt-reduction.

Next, we will show that $A \leq_T M$. We want to determine if a number $a \in \omega$ is an element of A. Since $A_{m_{a+1}} \upharpoonright a = A \upharpoonright a$, we need to just use M and $A_{m_{a+1}}$ (which is computable) to determine if $a \in A$. If $a \in A_{m_{a+1}}$, then $a \in A$. Otherwise, $a \notin A$. Therefore, we have that $A \leq_T M$. □

Now, we will define a binary tree T with no terminal nodes such that β is the unique limit branch of T and $\beta = \chi_M$. In order to do this, we will compute approximations m^t_k to m_k as follows. For all $t \in \omega$, let $m^t_0 = 0$ and let m^t_{k+1} be the least $m > m^t_k$ such that $m \leq t$ and $A_m \upharpoonright k = A_t \upharpoonright k$, as long as m^t_k is defined and such an m exists.

Before we define T, we will construct a set of binary strings T_0 and then have T be an extension of T_0. Let $\tau \in 2^{<\omega}$.

$\tau \in T_0 \iff \quad \tau = 0^\wedge 0^\wedge \cdots {}^\wedge 0$ or

$\tau(n) = 1$ if and only if $n = m_i^t$ where t is maximal such that $\tau(t) = 1$, and

$0 \leq i < k$ where k is the number of 1's in τ.

Now, we will let $T \subseteq 2^{<\omega}$ such that every string $\sigma \in T$ is such that if $\tau \subseteq \sigma$ then $\tau \in T_0$. The tree T is computable, since we have an algorithm to find T_0 and therefore to find T. Also, T has no terminal nodes since if $\tau \in T$ then $\tau^\wedge 0 \in T$.

Lemma 5.1.4. *M is the unique limit branch of T.*

Proof. First, we will show that M is a branch of T. We will view M as its characteristic function χ_M, and show that χ_M is a branch of T. To do this we will show that every sequence τ which is extended by χ_M is an element of T_0 and then it follows that τ is also an element of T. Let $\tau \in 2^{<\omega}$ such that χ_M extends τ. The string $\tau \neq 0^\wedge 0^\wedge \cdots {}^\wedge 0$ since χ_M does not extend τ because $m_0 = 0$ and therefore $\chi_M \upharpoonright 0 = 1$. So τ must contain a 1. Let t be greatest such that $\tau(t) = 1$. So, $t = m_k$ for some $k \geq 0$, since $\chi_M \supset \tau$. If $t = m_0$, then $t = m_0^0$ also, so $\tau \in T_0$ and is also an element of T. Otherwise, we have that $t = m_k = m_k^t$ and therefore $A_t \upharpoonright (k-1) = A \upharpoonright (k-1)$. Therefore we have that $m_j = m_j^t$ for all $0 \leq j \leq k$. So, $\tau(n) = 1$ if and only if $n = m_j = m_j^t$ for some j. Again, by definition of T_0, $\tau \in T_0$. Therefore, M is a branch of T.

Second, we will show that M is a limit branch of T. Let σ be a finite initial segment of χ_M.

Then, $\sigma^\wedge \underbrace{0^\wedge 0^\wedge \cdots {}^\wedge 0}_{i} \in T$ for all $i \in \omega$, by definition of T_0 (and T). Also, $\sigma^\wedge \underbrace{0^\wedge 0^\wedge \cdots {}^\wedge 0}_{j} {}^\wedge 1 \in T$ for some $j \in \omega$ since $\chi_M \supset \sigma$. So, M is a limit branch of T.

Finally, we need to show that M is the unique limit branch of T. Let C be another limit branch of T. We will show that $C = M$. Let $\{C_n\}_{n\in\omega}$ be the set of all branches of T such that $\lim_{n\to\infty} C_n = C$ and $C_n \neq C$ for all n.

We will first argue that $\cup_{n\in\omega} C_n$ is infinite. Suppose, by way of contradiction, that $\cup_{n\in\omega} C_n$ is finite. Therefore, there are only a finite number of different C_n's, and so there are infinitely many of them that are the same. Let B be a collection of these infinitely many sets that are all the same. So, $B = \lim_{n\to\infty} C_n = C$. So, $C = C_n$ for infinitely many n, which contradicts our original assumption. Therefore, $\cup_{n\in\omega} C_n$ is infinite.

Now, we will show that $C = M$ by showing that $\chi_M \upharpoonright k$ is an initial segment of C for all $k \in \omega$. Fix $k \in \omega$. Choose N large enough such that $m_i^t = m_i$ for all $i \leq k$ and all $t \geq N$. Choose τ such that $\tau \subseteq C_p$ for some p such that $C_p \upharpoonright (m_k + 1) = C \upharpoonright (m_k + 1)$, and such that t' is the largest number such that $\tau(t') = 1$ where $t' \geq N$. We can find such a τ since $\cup_{n\in\omega} C_n$ is infinite and $\lim_{n\to\infty} C_n = C$. Then, $\tau \in T$ since $C_p \in [T]$, by assumption. So we have that $\tau \in T_0$, by definition of T, and since $\tau \subseteq C_p$ and $C_p \in [T]$. Therefore, it must be the case that $\tau(n) = 1$ if and only if $n = m_i^{t'}$, since $\tau \neq 0^\wedge 0^\wedge \cdots {}^\wedge 0$, by assumption. By choice of n, we have that $\{m_0^{t'}, \ldots, m_k^{t'}\} = \{m_0, \ldots, m_k\}$. So, $\{m_0, \ldots, m_k\}$ is an initial segment of τ, and is therefore an initial segment of C_p, and therefore also of C. So, $C = M$ and M is the unique limit branch of T. □

□

5.2 Extension: Strengthening Limit Computable to Computably Enumerable

We extend our first application to interval trees in Theorem 5.2.2. We can actually have the set A from Corollary 5.1.2 be c.e., and we also get that $\omega_{\mathcal{L}}$ is c.e. Since we trivially have that $A \equiv_T \omega_{\mathcal{L}}$, we have that every c.e. T-degree contains a c.e. set that is the ω-part of a computable linear ordering of order type $\omega + \omega^*$. Before proving this theorem, recall the following definition.

Definition 5.2.1 (Dekker [8]). *A set A is* regressive *if there is a one-to-one enumeration $\{a_i\}_{i\in\omega}$ of A and there is a partial computable function ψ such that $\psi(a_0) = a_0$ and $\psi(a_{i+1}) = a_i$, for $i \geq 0$. If the enumeration is also in order of magnitude, then A is* retraceable.

Note that the enumeration of A does not need to be computable.

Theorem 5.2.2 (Chisholm et al. [5]). *For every c.e. set A, there is a computable linear ordering $\mathcal{L}$ of order type $\omega + \omega^*$ such that $A \leq_T \omega_{\mathcal{L}} \leq_{tt} A$ and $\omega_{\mathcal{L}}$ is c.e.*

Proof. Let A be a noncomputable, c.e. set, since the theorem trivially holds if A is computable. Since A is c.e., A is the range of some one-to-one computable function f. We will let B be the Dekker deficiency set of A for the enumeration f, so $B = \{s : (\exists t > s)[f(t) < f(s)]\}$. We will first show that $A \leq_T B$ (this proof may be found in [31], Theorem V.2.5). It is clear that $\overline{B}$ is infinite. Let $p(x)$ be the xth greatest element of $\overline{B}$. To see if an element x is in A, first enumerate the elements $f(0), f(1), \ldots, f(p(x))$. Since $x \leq f(p(x))$, if x is one of the elements in the enumeration, then $x \in A$. Otherwise, $x \notin A$. So, $A \leq_T B$.

Next, we will show that $B \leq_{tt} A$. Let $A_s = \{f(0), \ldots, f(s)\}$. To see if an element s is in B, find the least t such that $A_t \restriction f(s) = A \restriction f(s)$. If $t > s$, then it must be the case that $f(t) < f(s)$. So, $s \in B$. Otherwise, we have that $s \notin B$. This is a tt-reduction since this algorithm halts for all oracles. Therefore, we have shown that $A \leq_T B \leq_{tt} A$.

We need to show that $B = \omega_{\mathcal{L}}$ for some computable linear ordering $\mathcal{L} \cong \omega + \omega^*$. Note that B is a Σ_1 set, and is therefore c.e. Next, we will show that $\overline{B}$ is retraceable, and therefore regressive (this proof may be found in [27], Theorem II.6.16). Let $\{b_k\}_{k \in \omega}$ be a one-to-one enumeration of $\overline{B}$ such that $b_{k+1} > b_k$ for all k. We must define a partial computable function ψ such that $\psi(b_0) = b_0$ and $\psi(b_{k+1}) = b_k$ for all k. Given $b_k \in \overline{B}$, we need to effectively find b_{k-1} (where $b_{-1} = b_0$). Let $c < b_k$. Then,

$$c \in \overline{B} \iff (\forall t > c)[f(t) > f(c)].$$

Since for all $t > b_k$ we have that $f(t) > f(b_k)$, by assumption, we only need to check values of f for elements less than b_k. So,

$$c \in \overline{B} \iff (\forall t)[c < t \leq b_k \Rightarrow f(t) > f(c)].$$

Let $\psi(b_k)$ be the biggest $c < b_k$ such that the above line holds, if a c exists. If not, then $b_k = b_0$ and let $\psi(b_0) = b_0$. Therefore, we have shown that $\overline{B}$ is retraceable, and therefore regressive. We can now use the following theorem.

By Theorem 3.2 of [17], if B is c.e. and $\overline{B}$ is regressive, then B is semirecursive. By Theorem 4.1 of [17], B is semirecursive if and only if B is the initial segment of some computable linear ordering on ω. Let $\mathcal{L}_0$ be this computable linear ordering on ω.

Now, we will show that $\overline{B}$ is immune by showing that $\overline{B}$ is hyperimmune (this proof may be found in [31], Theorem V.2.5). If $\overline{B}$ is not hyperimmune, then some computable function, say g, majorizes p. Therefore, x is an element of A if and only if $x \in \{f(0), f(1), \ldots, f(g(x))\}$, which means that A is computable, which contradicts our original assumption. Therefore, $\overline{B}$ is hyperimmune, which implies that $\overline{B}$ is immune. So, the linear ordering $\mathcal{L}_0$ restricted to the elements in $\overline{B}$ is isomorphic to ω^*.

We are going to now construct a computable linear ordering $\mathcal{L}$ such that $B = \omega_{\mathcal{L}}$ by rearranging the elements of B in the linear ordering $\mathcal{L}_0$. When doing this we want to make sure that the elements of B have order type ω, that the elements of B are below the elements of $\overline{B}$, and that $\overline{B}$ still has order type ω^*. Since B is c.e., we have a c.e. enumeration of B, namely $\{B_s\}_{s\in\omega}$ such that $B_s \subseteq B_{s+1}$ and $B = \cup_{s\in\omega} B_s$.

Construction:

Suppose that $\mathcal{L}$ is defined on all numbers less than n for $n \in \omega$. Let $I(B_n) = \{i < n : (\exists k < n)[k \in B_n \wedge i \leq_{\mathcal{L}} k]\}$, the set of all elements of ω less than n and to the left of an element k of B_n, also less than n.

If $i \in I(B_n)$, then put n to the right of i, so $i <_{\mathcal{L}} n$.

If $i < n$ but $i \notin I(B_n)$, then keep the same ordering of i and n as in $\mathcal{L}_0$. So, $i <_{\mathcal{L}} n$ if and only if $i <_{\mathcal{L}_0} n$.

This ends the construction.

Now we need to show that this construction works. First we will show that $\leq_{\mathcal{L}}$ defines a linear ordering on ω. It is easy to see that $\leq_{\mathcal{L}}$ is both reflexive and symmetric. We need to show that $\leq_{\mathcal{L}}$ is also transitive. Fix $n \in \omega$ and assume that $\leq_{\mathcal{L}}$ is transitive on the set

$\{i : i \leq n\}$. We will show that $\leq_{\mathcal{L}}$ is transitive on the set $\{i : i \leq n+1\}$. We will show that if $a \leq_{\mathcal{L}} b$ and $b \leq_{\mathcal{L}} c$ then $a \leq_{\mathcal{L}} c$ where $a, b, c \in \{i : i \leq n+1\}$. It is clear that $\leq_{\mathcal{L}}$ is transitive when any two or three of these elements are $n+1$.

First, assume that $a = n+1$ and $b, c < n+1$. Since $b < n+1$ and $b \not<_{\mathcal{L}} n+1$ we have that $b \notin I(B_{n+1})$. It must be the case that $n+1 <_{\mathcal{L}_0} b$. Suppose, by way of contradiction, that $c \in I(B_{n+1})$, so $c <_{\mathcal{L}} n+1$. Then, we can find a $l < n+1$ such that $l \in B_{n+1}$ and $c \leq_{\mathcal{L}} l$, by definition of $I(B_{n+1})$. Since $b <_{\mathcal{L}} c$ and $c \leq_{\mathcal{L}} l$ and $\leq_{\mathcal{L}}$ is transitive on $\{i : i \leq n\}$, we have that $b \leq_{\mathcal{L}} l$, so $b \in I(B_{n+1})$, a contradiction. Therefore, c and $n+1$ are ordered the same in $\mathcal{L}$ as they are in $\mathcal{L}_0$.

Suppose that $c <_{\mathcal{L}_0} n+1$. Since $\leq_{\mathcal{L}_0}$ is transitive and $n+1 <_{\mathcal{L}_0} b$, we have that $c <_{\mathcal{L}_0} b$. Since $b <_{\mathcal{L}} c$, it must be the case that $b \in I(B_c) \subseteq I(B_{n+1})$, a contradiction. So, $n+1 <_{\mathcal{L}_0} c$ and therefore $n+1 <_{\mathcal{L}} c$. We can use a similar strategy to show that transitivity holds when either $b = n+1$ or $c = n+1$. Therefore, we have shown that $\leq_{\mathcal{L}}$ is transitive and that it is a linear ordering.

Next, we will show that all of the elements of B are to the left of the elements in $\overline{B}$ with respect to the ordering $\mathcal{L}$. Let $n \in \omega$. Assume that $i, j < n$ for $j \in B$ and that if $i <_{\mathcal{L}} j$ then $i \in B$. In other words, we are assuming that if $i \in I(B_n)$, then $i \in B$. So, $I(B_n) \subseteq B$. We will show that if $a, b < n+1$ for $b \in B$ and $a <_{\mathcal{L}} b$, then $a \in B$.

If $a, b < n$, then we are done by assumption.

If $a = n$ and $b < n$, we need to show that $n \in B$. Since we are assuming that $n <_{\mathcal{L}} b$, it must be the case that $b \notin I(B_n)$. Since $b \in B$, pick s large enough such that $b \in B_s$ and $n < s$. Since $b < s$ and $n <_{\mathcal{L}} b$, we have that $n \in I(B_s)$. So, $n \in B$. We can then conclude that $I(B_k) \subset B$ for all $k \in \omega$, so all of the elements of B are to the left of the elements in

$\overline{B}$. A similar argument holds when $b = n$ and $a < n$.

Next, we will show that $B = \omega$. Let $b \in B$ and we will show that b has only finitely many predecessors in $\mathcal{L}$. By construction, we place b to the right (with respect to the linear ordering $\mathcal{L}$) of all of the elements in $I(B_b)$. Since $I(B_b)$ is finite, and we never place any more elements to the left of b, b has only finitely many predecessors. So, $B = \omega$. It is easy to see that $\overline{B} = \omega^*$ since if $\bar{b} \in \overline{B}$, then $\bar{b} \notin I(B_n)$ for any $n \in \omega$. So, the ordering stays the same in $\mathcal{L}$ as in $\mathcal{L}_0$ and thus $\overline{B} = \omega^*$.

Therefore, the above construction works and gives us a computable linear ordering $\mathcal{L} \cong \omega + \omega^*$ such that $B = \omega_{\mathcal{L}}$. □

5.3 Cannot Replace Turing Reducibility with Weak Truth-Table Reducibility

Theorem 5.1.1 does not hold if we replace T-reducibility with wtt-reducibility. We can use interval trees to show that there is a c.e. set that is not wtt-reducible to $\omega_{\mathcal{L}}$, where $\mathcal{L}$ is a computable linear ordering of order type $\omega + \omega^*$. This is a negative conclusion to trying to extend Theorem 3.1.9 for tt-degrees and for wtt-degrees. We will be able to prove a much stronger result in Section 5.4 by finding a c.e. set that is not wtt-reducible to any initial segment of any computable scattered linear ordering.

Theorem 5.3.1. *There exists a c.e. set D such that whenever $T \subset 2^{<\omega}$ is a computable tree with no terminal nodes having a unique limit branch, then D is not wtt-reducible to any branch of T.*

By Proposition 4.4.3, we have the following corollary.

Corollary 5.3.2 (Chisholm et al. [5]). *There exists a c.e. set D such that $D \nleq_{wtt} \omega_{\mathcal{L}}$, where $\mathcal{L}$ is a computable linear ordering of order type $\omega + \omega^*$.*

To see why we cannot extend Theorem 3.1.9 for the strong degree spectrum, let $\deg(D) = \mathbf{d}$, where D is as in Corollary 5.3.2. Then, $\mathbf{d} \notin DgSp_{\mathcal{L}}^{wtt}(\omega_{\mathcal{L}})$. So, $DgSp_{\mathcal{L}}^{wtt}(\omega_{\mathcal{L}})$ does not contain all Δ_2^0 wtt-degrees, which implies that $DgSp_{\mathcal{L}}^{tt}(\omega_{\mathcal{L}})$ does not contain all Δ_2^0 tt-degrees.

To prove Theorem 5.3.1, we first need the following notion of a transversal.

Definition 5.3.3. *A sequence $A =< \sigma_0, \sigma_1, \ldots >\subset 2^{<\omega}$ is a* transversal *of a tree $T \subset 2^{<\omega}$ if every $f \in [T]$ extends some σ_k.*

*Let $p \in \omega$. A transversal A of T is an (i,p)-*converging transversal *of T if $|\sigma_k| \geq \varphi_i(< p, k >)$ for every $\sigma_k \in A$.*

Proof of Theorem 5.3.1. Let $\{T_j\}_{j\in\omega}$ be a fixed enumeration of all partial computable subtrees of $2^{<\omega}$. Our goal is to construct a c.e. set D where $D = \cup D_{eij}$, where $D_{eij} \subset \{< e, i, j, k >\}_{k\in\omega}$ are uniformly c.e.

Construction:

Given the sth approximation D_{eij}^s to D_{eij}, construct D_{eij}^{s+1} as follows.

First, we want to find a transversal A_{eij} of T_j. If we have already found a transversal A_{eij} of T_j by stage s, then we do not need to look for it. So, move on to enumerating elements into D_{eij}. Else, we have not found a transversal of T_j yet. Let $A =< \sigma_0, \sigma_1 \ldots, \sigma_M >\in (2^{\leq s})^{<\omega}$ with $M \leq s$. (A is a string of at most length s. Each σ_k is a finite binary string.) A is an $(i, < e, i, j >)$-converging transversal of T_j if $|\sigma_k| \geq \varphi_{i,s}^{\sigma_k}(< e, i, j, k >)$ for every $\sigma_k \in A$ and every $f \in [T_j^s]$ extends some σ_k. In particular, the branches $f \in 2^L \cap T_j$ for $L \leq s$.

If we can find such an A, let A_{eij} be the least such (with respect to the code of $< \sigma_0, \sigma_1, \ldots, \sigma_M >$). Enumerate $< e, i, j, k >$ into D_{eij}^{s+1} if and only if $\varphi_{e,s}^{\sigma_k}(< e, i, j, k >) = 0$.

Otherwise, A_{eij} remains undefined, so we do not enumerate any elements into D_{eij}^{s+1}. Therefore, $D_{eij}^{s+1} = D_{eij}^{s} = \emptyset$.

This ends the construction.

Lemma 5.3.4. *Assume that T_j is total, and that T_j has a unique limit branch β. Fix $i \in \omega$. Then, for every $p \in \omega$, T_j has a finite (i, p)-converging transversal.*

Proof. We will define the transversal $A = < \sigma_0, \sigma_1, \ldots >$ of T_j by setting $\sigma_0 = (\beta \upharpoonright \varphi_i(< p, 0 >))$. Let $B = \{\beta_1, \beta_2, \ldots\}$ where $\beta_k \in [T_j]$ and σ_0 is not a subset of β_k for $k > 0$. Since T_j has only one limit branch, B is a finite set. Similarly, let $\sigma_k = (\beta_k \upharpoonright \varphi_i(< p, k >))$ for $1 \leq k \leq |B|$. So, A is a transversal of T_j since every $f \in T_j$ extends some element of A, by how it was constructed. Since $|\sigma_k| = \varphi_i(< p, k >)$ for every $\sigma_k \in A$, A is also a finite (i, p)-converging transversal of T_j. □

Now, fix j such that T_j is total and has a unique limit branch. Therefore we can find a branch $\beta \in [T_j]$. Assume that $D \leq_{wtt} \beta$ via φ_e with use bounded by φ_i. So, $D(x) = \varphi_e^{(\beta \upharpoonright \varphi_i(x))}(x)$ for all $x \in \omega$. Note that φ_i is total since for all $x \in \omega$ we need to compute $\varphi_i(x)$.

By Lemma 5.3.4, T_j has a finite $(i, < e, i, j >)$-converging transversal, which we will call A_{eij}. Some $\sigma_k \in A_{eij}$ must lie on β, by definition of transversal. Let $n =< e, i, j, k >$. Then,

$$\varphi_e^{\beta}(n) = \varphi_e^{(\beta \upharpoonright \varphi_i(n))}(n) = \varphi_e^{\sigma_k}(n).$$

Therefore, $\varphi_e^\beta(n) = 1 \Leftrightarrow \varphi_e^{\sigma_k}(n) = 1$. Then we have

$$n \in D \Leftrightarrow n \in D_{eij} \Leftrightarrow \varphi_{e,s}^{\sigma_k}(n) = 0 \Leftrightarrow \varphi_e^{\sigma_k}(n) = 0,$$

which is a contradiction to $D \leq_{wtt} \beta$. Therefore, $D \nleq_{wtt} \beta$ via φ_e with use bounded by φ_i. □

5.4 Extension to Computable Scattered Linear Orderings

We can significantly strengthen Theorem 5.3.1 by showing that there exists a c.e. set D that is not only not wtt-reducible to the ω-part of $\mathcal{L}$, where $\mathcal{L}$ is a computable copy of $\omega + \omega^*$, but also that D is not wtt-reducible to any initial segment of any computable scattered linear ordering. In order to prove this strengthened theorem, we need the following definition.

Definition 5.4.1 (Cenzer et al. [4])**.** *A real $f \in 2^\omega$ is* ranked *if $f \in P$ for some countable Π_1^0 class P.*

The following proposition shows that we can view initial segments of computable scattered linear orderings as ranked sets.

Proposition 5.4.2. *Let $\mathcal{L}$ be a computable scattered linear ordering. Then, every initial segment of $\mathcal{L}$ is ranked.*

Proof. Let $\mathcal{L}$ be a computable scattered linear ordering. Since $\mathcal{L}$ is scattered, $\mathcal{L}$ has only countably many initial segments by Proposition 4.4.1. Let P be the Π_1^0 class of initial segments of $\mathcal{L}$. Then, P is countable. Therefore, any initial segment of $\mathcal{L}$, which is an element of P, is ranked. □

Theorem 5.4.3. *There exists a c.e. set D such that whenever $T \subset 2^{<\omega}$ is a computable tree with countably many infinite branches, then D is not wtt-reducible to any branch of T. Therefore, there exists a c.e. set D which is not wtt-reducible to any ranked set. So, the wtt-cone above D is disjoint from every countable Π_1^0 class.*

We get the following corollary about initial segments of computable scattered linear orderings using Proposition 5.4.2.

Corollary 5.4.4 (Chisholm et al. [5])**.** *There exists a c.e. set D which is not wtt-reducible to any initial segment of any computable scattered linear ordering.*

Proof. According to Theorem 5.4.3 there exists a c.e. set D which is not wtt-reducible to any ranked set. Let $\mathcal{L}$ be a computable scattered linear ordering. By Proposition 5.4.2, every initial segment of $\mathcal{L}$ is ranked. It follows that D is not wtt-reducible to any initial segment of any computable scattered linear ordering. □

Proof of Theorem 5.4.3. We will prove this theorem using exactly the same technique as when we proved Theorem 5.3.1. Recall that we let $\{T_j\}_{j\in\omega}$ be a fixed enumeration of all partial computable subtrees of $2^{<\omega}$. Define $D = \cup D_{eij}$ where $D_{eij} \subset \{< e, i, j, k >\}_{k\in\omega}$ are uniformly c.e. We need to show, as in Lemma 5.3.4, that every total T_j has a finite (i, p)-converging transversal. In the proof of Lemma 5.4.5 we do not have a subtree of $2^{<\omega}$ with a unique limit branch, as we did in the proof of Lemma 5.3.4. Instead, we have a subtree of $2^{<\omega}$ with countably many branches. Therefore, we need to prove the following lemma.

Lemma 5.4.5. *Let T_j be total and suppose that T_j has countably many branches. Fix $i \in \omega$ such that φ_i is total. Then, for all $p \in \omega$, T_j has a finite (i,p)-converging transversal.*

Proof. Fix $p \in \omega$. Assume that φ_i is an increasing function. We can assume this, since in the proof of Theorem 5.4.3 we will assume that $D \leq_{wtt} \beta$ via φ_e and φ_i where β is a branch of T_j, and for some $e \in \omega$. The function φ_i bounds the use of the oracle β and can therefore be assumed to be increasing.

Next, we will define a computable function g by recursion. We define this function g, using φ_i, in order to pick out a subtree of T_j where each node has length $g(n)$ for some $n \in \omega$ (this subtree also contains the empty node). Therefore, the length of these nodes can be expressed in terms of φ_i. A collection of these nodes will eventually be the transversal we are looking for, and because of how we defined g, it will be easy to see that $|\sigma_k| \geq \varphi_i(< p, k >)$ for all σ_k in the transversal (which is what we need to have an (i,p)-converging transversal).

Here is the definition of g.

- Let $g(0) = \varphi_i(< p, 0 >)$.
- Let $g(l+1) = \varphi_i(< p, w_l >)$ where w_l is the number of nodes $\sigma \in T_j$ such that either $|\sigma| = g(l)$ or σ is a terminal node of T_j with $|\sigma| < g(l)$.

Now we call $\sigma \in T_j$ a *g-node* if $|\sigma| = 0$ or $|\sigma| = g(n)$ for some n. This collection of g-nodes forms a subtree of T_j. Let $\sigma, \sigma' \in T_j$ both be g-nodes such that $\sigma \subset \sigma'$ and such that there does not exist $\beta \in T_j$ such that $\sigma \subset \beta \subset \sigma'$ and β is a g-node. Then, σ is called a *g-predecessor* of σ' and σ' is called a *g-successor* of σ. A g-node with no g-successor is called a *terminal g-node*. Otherwise, it is called a *nonterminal g-node*.

Next, we define the notion of *splitting rank*. We show that the empty node, <>, has some finite splitting rank, say R. Then, we construct the transversal of T_j by finite extension such that $A^0 \subset A^1 \subset \cdots \subset A^{R+1}$ and such that A^{R+1} is the transversal we are looking for. We define these A^l's by also constructing sets B^l such that A^{l+1} contains the g-successors of each element in B^l with the highest splitting rank, if it exists. Constructing A^{l+1} in this way gives us an effective procedure for finding the transversal, and also makes it easy to check that A^{R+1} is in fact the (i,p)-converging transversal we are looking for.

Let $n \in \omega$ and $\sigma \in T_j$. We will define the relation $\mathrm{SR}(\sigma) \geq n$, where SR stands for Splitting Rank, recursively as follows.

- $\mathrm{SR}(\sigma) \geq 0$ if and only if σ is a g-node.
- $\mathrm{SR}(\sigma) \geq (n+1)$ if and only if $\mathrm{SR}(\tau) \geq n$ for at least two distinct $\tau \in T_j$ that are g-successors of σ.

Now, we define what the splitting rank of a node in T_j is actually equal to.

- $\mathrm{SR}(\sigma) = \infty$ if and only if $\mathrm{SR}(\sigma) \geq m$ for all $m \in \omega$.
- $\mathrm{SR}(\sigma) = n$ if and only if n is the greatest element of ω such that $\mathrm{SR}(\sigma) \geq n$.

According to the definition of splitting rank, it is easy to see that if σ is a g-predecessor of σ', then $\mathrm{SR}(\sigma) \geq \mathrm{SR}(\sigma')$. In fact, we see by the following lemma that we can get even more precise than this. (The proof of this lemma is obvious, using the definition of splitting rank.)

Lemma 5.4.6. *Let σ be a nonterminal g-node of T_j. Let σ' be the g-successor of σ with greatest splitting rank.*

1. *If $SR(\sigma') = \infty$, then $SR(\sigma) = \infty$.*

2. *If $SR(\sigma') = n$ for some $n \in \omega$ and if σ' is the only g-successor of σ with splitting rank n, then $SR(\sigma) = n$.*

3. *If $SR(\sigma') = n$ for some $n \in \omega$ and if σ' is not the only g-successor of σ with splitting rank n, then $SR(\sigma) = n + 1$.*

We have the following very important corollary to the above lemma. According to this corollary, it must be the case that no $\sigma \in T_j$ has splitting rank ∞. If this is the case then T_j must contain a perfect subtree, however, this contradicts our assumption that T_j has only countably many branches.

Corollary 5.4.7. *There is no $\sigma \in T_j$ such that $SR(\sigma) = \infty$.*

Proof. Let $\sigma \in T_j$ be such that $\mathrm{SR}(\sigma) = \infty$. Then, σ must have at least two distinct g-successors both with splitting rank ∞, and so on. This set of g-nodes generates a perfect subtree of T_j which must have uncountably many branches, a contradiction. Therefore, $\mathrm{SR}(\sigma) < \infty$. □

By Corollary 5.4.7, we have that $\mathrm{SR}(<>) = R$ for some $R \in \omega$. We will use this observation frequently. Now, we will start constructing sets A^l and B^l for $0 \leq l \leq R + 1$ with $A^0 \subset A^1 \subset \cdots \subset A^R \subset A^{R+1}$ forming better and better approximations of the (i, p)-converging transversal of T_j that we are looking for.

Construction:

Let $A^0 = \emptyset$, and $B^0 = \{<>\}$.

Given $A^l = \{\sigma_0, \dots, \sigma_m\}$ and $B^l = \{\tau_1, \dots, \tau_n\}$, we define A^{l+1} and B^{l+1} as follows. For each $\tau \in B^l$, let ρ_τ be the g-successor of τ having maximum splitting rank. If τ has no g-successor, then ρ_τ does not exist. Let σ_{m+j}, for $1 \leq j \leq n$, be the jth member of the set $\{\rho_\tau : \tau \in B^l\}$ (where the set is ordered with respect to the numbering of the indices on the τ's). Then, let A^{l+1} contain A^l and the set of all nodes σ_{m+j}. Let $\tau \in B^{l+1}$ if and only if $|\tau| = g(l)$ and τ does not extend any element of A^{l+1}. (In the special case where $g(0) = 0$, we let $B^1 = \emptyset$.)

This ends the construction.

Lemma 5.4.8. *For $0 \leq l \leq R$, we have that:*

1. *Every $\rho \in B^{l+1} \cup (A^{l+1} - A^l)$ is a g-successor of exactly one $\tau \in B^l$.*
2. *Let $\rho \in B^{l+1}$ extend $\tau \in B^l$. Then,*
   (a) *$SR(\rho) < SR(\tau)$, and*
   (b) *$SR(\rho) \leq R - (l+1)$.*
3. *$B^{l+1} \cup A^{l+1}$ is an antichain.*
4. *$|\sigma_k| \geq \varphi_i(< p, k >)$ for all $\sigma_k \in A^{l+1}$.*

Proof. We will prove this lemma by induction on l. First, we will show that this lemma holds for the case when $l = 0$. Then, we will assume that this lemma holds for some $l - 1 \geq 0$, and we will show that the lemma holds for l. Let $l = 0$.

(1) We must show that every $\rho \in B^1 \cup (A^1 - A^0)$ is a g-successor of exactly one $\tau \in B^0$. If $\rho \in B^1$, then since $|\rho| = g(0)$ and g is increasing, it is easy to see that ρ is a g-successor

of $<>$, and by definition, $<>\in B^0$. If $\rho \in (A^1 - A^0)$, then by definition, ρ is the g-successor of $<>$ with greatest splitting rank. Therefore, every $\rho \in B^1 \cup (A^1 - A^0)$ is a g-successor of exactly $<>\in B^0$

(2a) Let $\rho \in B^1$ extend $<>\in B^0$ (we know that we can find this ρ by (1)). We must show that $\mathrm{SR}(\rho) < \mathrm{SR}(<>)$. Let $\sigma_0 \in A^1$ be the g-successor of $<>$ with greatest splitting rank. $\sigma_0 \notin B^1$, by definition of B^1, so $\sigma_0 \neq \rho$. Therefore, we have that $\mathrm{SR}(\rho) \leq \mathrm{SR}(\sigma_0)$. If $\mathrm{SR}(\rho) = \mathrm{SR}(\sigma_0)$, then by Lemma 5.4.6(2), $\mathrm{SR}(<>) = \mathrm{SR}(\rho) + 1$, so $\mathrm{SR}(\rho) < \mathrm{SR}(<>)$. If $\mathrm{SR}(\rho) < \mathrm{SR}(\sigma_0)$, then by Lemma 5.4.6(2), $\mathrm{SR}(<>) = \mathrm{SR}(\sigma_0)$, so $\mathrm{SR}(\rho) < \mathrm{SR}(<>)$.

(2b) Using the same ρ and τ as in the proof of (2a) above, we must now show that $\mathrm{SR}(\rho) \leq R - 1$. In (2a) we showed that $\mathrm{SR}(\rho) < \mathrm{SR}(<>)$. Since $\mathrm{SR}(<>) = R$, by assumption, we have that $\mathrm{SR}(\rho) < R$, so $\mathrm{SR}(\rho) \leq R - 1$.

(3) We must show that $B^1 \cup A^1$ is an antichain. If $g(0) = 0$, then $B^1 = \emptyset$, by definition of B^1. Either $A^1 = \{\sigma_0\}$, where σ_0 is the g-successor of $<>$ with greatest splitting rank, or $A^1 = \emptyset$ if no g-successor of $<>$ exists. Therefore, A^l is an antichain, and so $B^l \cup A^l$ is an antichain. If $g(0) \neq 0$, then B^1 contains only elements of length $g(0)$ which do not extend any elements of A^1. If $A^1 \neq \emptyset$, then its only element, σ_0, has length $g(0)$, since it is the g-successor of $<>$ and g is increasing. Therefore, $B^l \cup A^l$ is an antichain.

(4) We must show that $|\sigma_k| \geq \varphi_i(< p, k >)$ for all $\sigma_k \in A^1$. According to our argument in (3), either $A^1 = \{\sigma_0\}$ or $A^1 = \emptyset$. Suppose $A^1 = \{\sigma_0\}$. If $g(0) = 0$, then $|\sigma_0| = g(1) = \varphi_i(< p, w_1 >)$, where w_1 is the number of nodes σ such that $|\sigma| = g(0)$. Therefore, $w_1 = 1$ (the empty node, $<>$). So, $|\sigma_0| = \varphi_i(< p, 1 >) \geq \varphi_i(< p, 0 >)$, since φ_i is increasing. Therefore, $|\sigma_0| \geq \varphi_i(< p, 0 >)$. If $g(0) \neq 0$, then $|\sigma_0| = g(0) = \varphi_i(< p, 0 >)$, by definition of g. On the other hand, if $A^1 = \emptyset$, then (4) trivially holds.

Now, assume this lemma holds for $l - 1 \geq 0$, and we will show that the lemma holds for l.

(1) By (3), $B^l \cup A^l$ is an antichain. Therefore, no element of $B^{l+1} \cup (A^{l+1} - A^l)$ is a g-successor to more than one element of B^l. Each $\sigma_{m+j} \in (A^{l+1} - A^l)$ extends $\tau_j \in B^l$ by the definition of A^l. Now, let $\rho \in B^{l+1}$ and let τ be the g-predecessor of ρ. ρ does not extend any element of A^l, by definition of B^{l+1}. So, τ does not extend any element of A^l. So, $\tau \in B^l$.

(2a) Let $B^l = \{\tau_1, \ldots, \tau_n\}$, $\tau \in B^l$, and $A^l = \{\sigma_0, \ldots, \sigma_m\}$. Fix j such that $\tau_j = \tau$. Let $\rho \in B^{l+1}$ extend $\tau \in B^l$ (we know this ρ can be found by (1)). By construction, $\sigma_{m+j} \in A^{l+1}$ has the greatest splitting rank attained by any g-successor of τ_j. Since $\sigma_{m+j} \notin B^{l+1}$, by definition of B^{l+1}, we have that $\rho \neq \sigma_{m+j}$. Therefore, we have that $\mathrm{SR}(\rho) \leq \mathrm{SR}(\sigma_{m+j})$. If $\mathrm{SR}(\rho) = \mathrm{SR}(\sigma_{m+j})$, then by Lemma 5.4.6(3), $\mathrm{SR}(\tau) = \mathrm{SR}(\rho) + 1$, so $\mathrm{SR}(\rho) < \mathrm{SR}(\tau)$. If $\mathrm{SR}(\rho) < \mathrm{SR}(\sigma_{m+j})$, then by Lemma 5.4.6(2), $\mathrm{SR}(\tau) = \mathrm{SR}(\sigma_{m+j})$, so $\mathrm{SR}(\rho) < \mathrm{SR}(\tau)$.

(2b) Assume that $\rho \in B^l$ extends $\tau \in B^{l-1}$ and that $\mathrm{SR}(\rho) \leq R - l$. Let $\rho_0 \in B^{l+1}$ extend $\tau_0 \in B^l$. We must now show that $\mathrm{SR}(\rho_0) \leq R - (l+1)$. By (1), $\tau_0 \in B^l$ is a g-successor of exactly one $\tau' \in B^{l-1}$. By assumption, $\mathrm{SR}(\tau_0) \leq R - l$. By part (2a), we have that $\mathrm{SR}(\rho_0) < \mathrm{SR}(\tau_0)$. Therefore, $\mathrm{SR}(\rho_0) < R - l$ and so $\mathrm{SR}(\rho_0) \leq R - l - 1 = R - (l+1)$.

(3) Assume that $B^l \cup A^l$ is an antichain. We must show that $B^{l+1} \cup A^{l+1}$ is an antichain. By construction, no two elements of $B^{l+1} \cup (A^{l+1} - A^l)$ are comparable. Assume that $\rho \in B^{l+1} \cup (A^{l+1} - A^l)$ is comparable with $\sigma \in A^l$. By (1), ρ is a g-successor of exactly one $\tau \in B^l$. So, σ and τ are also comparable, which is a contradiction since $B^l \cup A^l$ is an antichain. Therefore, no two elements in $B^{l+1} \cup A^{l+1}$ are comparable, so $B^{l+1} \cup A^{l+1}$ is an antichain.

(4) Assume that $|\sigma_k| \geq \varphi_i(< p, k >)$ for all $\sigma_k \in A^l$. We must show that $|\sigma_k| \geq$

$\varphi_i(< p, k >)$ for all $\sigma_k \in A^{l+1}$. So, we really only need to show that $|\sigma_k| \geq \varphi_i(< p, k >)$ for all $\sigma_k \in (A^{l+1} - A^l)$. Let $\sigma_k \in (A^{l+1} - A^l)$. Then, σ_k is the g-successor of some $\tau \in B^l$ with greatest splitting rank, and such $|\tau| = g(l-1)$. $|\sigma_k| = g(l)$, since g is increasing. So, $|\sigma_k| = g(l) = \varphi_i(< p, w_{l-1} >)$. If we can show that $w_{l-1} \geq k$, in other words that $w_{l-1} \geq |A^{l+1}|$, then we are done. By (1), each element of $A^{l+1} - A^l$ is a g-successor of exactly one $\tau \in B^l$. So, $|A^{l+1} - A^l| \leq |B^l|$. Therefore, $|A^{l+1}| \leq |B^l \cup A^l|$. By (2), $B^l \cup A^l$ is an antichain with members at most of length $g(l-1)$. Therefore, $|B^l \cup A^l| \leq w_{l-1}$ and so $|A^{l+1}| \leq w_{l-1}$. □

Lemma 5.4.9. *$B^{R+1} = \emptyset$.*

Proof. Assume that $\tau \in B^{R+1}$. By Lemma 5.4.8(1), τ is a g-successor of exactly one $\tau_k \in B^R$. Let $\sigma_{m+k} \in A^{R+1}$ be the g-successor of τ_k with highest splitting rank. $\sigma_{m+k} \neq \tau$ since $\tau \in B^{R+1}$ and τ cannot extend σ_{m+j}, by definition of B^{R+1}. So, $\mathrm{SR}(\tau_k) \geq 1$ since there are at least two g-successors of τ_k. By Lemma 5.4.8(2), $\mathrm{SR}(\tau_k) \leq R - (R - 1 + 1) = 0$. So, $\mathrm{SR}(\tau_k) = 0$, which contradicts our original findings. Therefore, $B^{R+1} = \emptyset$. □

Corollary 5.4.10. *A^{R+1} is a finite (i, p)-converging transversal of T_j.*

Proof. It is easy to see that for all $l \geq 0$, that $B^{l+1} \cup A^{l+1}$ is a transversal of T_j, since every node of length $g(l)$ in T_j is comparable with some element of $B^{l+1} \cup A^{l+1}$. Since $B^{R+1} = \emptyset$ by the previous lemma, A^{R+1} is a transversal of T_j, and by construction it is finite. By Lemma 5.4.8(4), $|\sigma_k| \geq \varphi_i(< p, k >)$ for all $\sigma_k \in A^{R+1}$. Therefore, A^{R+1} is a finite (i, p)-converging transversal of T_j. □

We have found a finite (i, p)-converging transversal of T_j, so this completes the proof of Lemma 5.4.5. □

We can prove Theorem 5.4.3 by using exactly the same construction as in Theorem 5.3.1 and obtaining the same conclusion. Therefore, we have finished proving the theorem. □

The following is a corollary to the proof of Theorem 5.4.3.

Corollary 5.4.11. *There exists a c.e. set D such that if D is wtt-reducible to an element of a Π^0_1 class $P \subseteq 2^{<\omega}$, then P has a perfect Π^0_1 subclass Q.*

Proof. Let $T_j \subseteq 2^{<\omega}$ such that $[T_j] = P$. If $\mathrm{SR}(<>)$ is finite, then we have the proof of Theorem 5.4.3 going through as before, and the hypothesis of this corollary does not hold. Therefore, let $\sigma \in T_j$ such that $\mathrm{SR}(\sigma) = \infty$. By Lemma 5.4.6, $<>$ must have two distinct g-successors with splitting rank ∞, and so on. Let I be the set of all of the nodes of T_j which have infinite splitting rank. In other words, $I = \{\sigma : (\forall m)[\mathrm{SR}(\sigma) \geq m]\}$. Checking if $\mathrm{SR}(\sigma) \geq m$ is computable in g, and since g is defined in terms of h_i and h_i is computable, we have that checking if $\mathrm{SR}(\sigma) \geq m$ is a computable condition. (Recall that h_i is an increasing total computable function used in the proof of Lemma 5.4.5, and consequently in the proof of Theorem 5.4.3.) Therefore, I is a Π^0_1 subset of nodes of T_j.

If we let Q be the set of branches through the tree generated by I, so $Q = \{\sigma : (\exists\tau)(\forall m)[\mathrm{SR}(\tau) \geq m \;\wedge\; \sigma \subseteq \tau]\}$, then Q is a Σ^0_2 set of nodes. Therefore, $[Q]$ will not be the Π^0_1 subclass we are looking for.

Therefore, let I_0 be the set of g-nodes σ such that every g-node τ extended by σ is in I, so $I_0 = \{\sigma : \mathrm{SR}(\sigma) \geq 0 \;\wedge\; (\forall\tau)[(\mathrm{SR}(\tau) \geq 0 \;\wedge \tau \subseteq \sigma) \Rightarrow (\forall m)(\mathrm{SR}(\tau) \geq m)]\}$. I_0 is also a Π^0_1 subset of nodes of T_j. We can see that $<> \in I_0$, and also all of the g-successors of $<>$

which are members of I are in I_0. In this manner, we can see that if $\sigma \in I_0$, then all of the g-successors of σ which are members of I are also members of I_0. Similarly, we get that I_0 is closed downwards.

Now, we will let $\sigma \in U_j$ if and only if σ has a g-successor in I_0. So, $U_j = \{\alpha : (\forall\beta)[(\beta \supseteq \alpha \wedge \mathrm{SR}(\beta) \geq 0) \Rightarrow \beta \in I_0]\}$. So, U_j is the tree generated by I_0. U_j is also a Π^0_1 subset of nodes of T_j. Notice that $U_j \neq \emptyset$ since $<>\in U_j$. Let $Q = [U_j]$. Then, Q is a Π^0_1 subclass of $[T_j]$. Q is also perfect since every g-node σ in U_j has at least two distinct g-successors of infinite splitting rank. □

We can now translate Corollary 5.4.11 to linear orderings.

Corollary 5.4.12 (Chisholm et al. [5])**.** *There exists a c.e. set D such that if D is wtt-reducible to an initial segment of a computable linear ordering $\mathcal{L}$, then $\mathcal{L}$ is not scattered. In fact, $|\mathcal{L}|$ has a subset S such that $(S, <_{\mathcal{L}}) \cong (\mathbb{Q}, <)$ and $S \leq_T \emptyset'$.*

Proof. Let D be as in Corollary 5.4.11, and suppose that $D \leq_{wtt} A$, where A is an initial segment of some computable linear ordering $\mathcal{L}$. Let P be the class of all initial segments of $\mathcal{L}$. We can view these initial segments as a collection of infinite branches through the full binary tree, as in the proof of Proposition 4.4.1. This set of infinite branches forms a Π^0_1 class, so P is a Π^0_1 class. Since $A \in P$, P has a perfect Π^0_1 subclass Q by Corollary 5.4.11. Let $T \subseteq 2^{<\omega}$ be a perfect tree such that $[T] = Q$. Since T is Π^0_1 we have that $T \leq_T \emptyset'$. Since T is perfect, T does not have countably many branches, so $\mathcal{L}$ does not have countably many initial segments. Therefore, by Proposition 4.4.1, $\mathcal{L}$ is not scattered. So, $|\mathcal{L}|$ has a subset S such that $(S, <_{\mathcal{L}}) \cong (\mathbb{Q}, <)$. In the proof of Proposition 4.4.1, it is shown that $S \leq_T T$, so $S \leq_T \emptyset'$. □

In the other direction, we have the following proposition.

Proposition 5.4.13 (Chisholm et al. [5]). *Let $\mathcal{L}$ be a computable linear ordering which has a computable subset S such that $(S, <_{\mathcal{L}}) \cong (\mathbb{Q}, <)$. Then, $\mathcal{L}$ has initial segments of every tt-degree.*

Proof. We will show that for every set $D \subseteq \omega$, there is an initial segment I of $\mathcal{L}$ such that $I \equiv_{tt} D$. Fix $D \subseteq \omega$. Let the universe of $\mathcal{L}$ be ω. Let $l, r \in \omega$ be such that $l <_{\mathcal{L}} r$. Let (l, r) denote the set of all $x \in \omega$ such that $l <_{\mathcal{L}} x <_{\mathcal{L}} r$. We will build I in stages, using the following idea. At each stage n we will choose elements $l_n, x_n, r_n \in S$ such that $n \notin (l_n, r_n)$. We will then decide if $x_n \in I$ depending on whether $n \in D$ or not.

Construction:

Stage $s = 0$: Let $l_0, x_0, r_0 \in S$ with $l_0 <_{\mathcal{L}} x_0 <_{\mathcal{L}} r_0$ be the least triple of S (with respect to the coding $< l_0, x_0, r_0 >$) such that $0 \notin (l_0, r_0)$. Then, we let $z \in I$ if and only if $z \leq_{\mathcal{L}} l_0$.

Stage $s = n+1$: We have $l_n <_{\mathcal{L}} x_n <_{\mathcal{L}} r_n$ where $l_n, x_n, r_n \in S$ is the least triple of S such that $n \notin (l_n, r_n)$.

Case 1: $n \in D$. Let $l_{n+1}, x_{n+1}, r_{n+1} \in S$ with $l_{n+1} <_{\mathcal{L}} x_{n+1} <_{\mathcal{L}} r_{n+1}$ be the least triple of $S \cap (l_n, r_n)$ such that $n+1 \notin (l_{n+1}, r_{n+1})$ and $x_n <_{\mathcal{L}} l_{n+1}$.

Case 2: $n \notin D$. Let $l_{n+1}, x_{n+1}, r_{n+1} \in S$ with $l_{n+1} <_{\mathcal{L}} x_{n+1} <_{\mathcal{L}} r_{n+1}$ be the least triple of $S \cap (l_n, r_n)$ such that $n+1 \notin (l_{n+1}, r_{n+1})$ and $r_{n+1} <_{\mathcal{L}} x_n$.

In either case, we let $z \in I$ if and only if $z \leq_{\mathcal{L}} l_{n+1}$. Notice that deciding if $n+1 \in I$ is decided at stage $n+1$. Also, Case 1 and Case 2 give us that $x_n \in I$ if and only if $n \in D$.

This ends the construction.

First, we will show that $I \leq_{tt} D$. We want to find out if an element $n+1 \in \omega$ is an element of I. In order to do this, we go through the previous construction until we get to stage $n+1$. If $n+1 \leq_{\mathcal{L}} l_{n+1}$, then $n+1 \in I$. Otherwise, $n+1 \notin I$. In order to run this construction, we need to use D, S and $<_{\mathcal{L}}$. Since S and $\mathcal{L}$ are both computable, and since this construction would halt for any oracle D, we have that $I \leq_{tt} D$.

Now, all we must show is that $D \leq_{tt} I$. We want to find out if an element $k+1 \in \omega$ is an element of D. By our previous observation, $k+1 \in D$ if and only if $x_{k+1} \in I$. Therefore, we need to find the element x_{k+1}. In order to do this, define a total function φ as follows, using I. Let $\varphi^I(0) = I(x_0)$. Given $\varphi^I(0), \ldots, \varphi^I(k)$, in order to find $\varphi^I(k+1)$, go through the previous construction with φ^I in place of D. So, instead of asking whether $k \in D$, we will ask whether $\varphi^I(k) = 1$, in stage $k+1$, and so on. In stage $k+1$, we will define x_{k+1}, using S and $<_{\mathcal{L}}$, which are both computable. So, $\varphi^I(k+1) = I(x_{k+1})$. If $I(x_{k+1}) = 1$, then $k+1 \in D$. Otherwise, $k+1 \notin D$. Since this algorithm will halt for any oracle I, we have that this is a tt-reducibility, and that $D \leq_{tt} I$. $\square$

5.5 Extension: Strengthening Computably Enumerable to Also Be Low

This section is an expanded version of parts of Section 5 from [5]. We would like to extend Theorem 5.4.3 by showing that there exists a low c.e. set D which is not wtt-reducible to any ranked set, however, we cannot make D low by adding permitting to the proof, as we will see in Proposition 5.5.3. In order to use permitting, we let A be any noncomputable c.e. set. Then, we construct $D \leq_T A$. Since we can let A be low, D can be made low.

According to Proposition 5.5.3, we cannot do this since there is a noncomputable c.e. set A such that every D computable in A is also wtt-reducible to a ranked set. In the proof of the proposition, we will let our noncomputable c.e. set A be of strongly contiguous degree, which Downey introduced in [9], Observation $(2.1)'$.

Definition 5.5.1. *A c.e. T-degree* $\mathbf{d}$ *is* strongly contiguous *if all sets of T-degree* $\mathbf{d}$ *have the same wtt-degree.*

Lemma 5.5.2 (Downey [9])**.** *There exists a noncomputable c.e. strongly contiguous T-degree.*

Proposition 5.5.3 (Chisholm et al. [5])**.** *There exists a noncomputable c.e. set A such that every set $D \leq_T A$ is wtt-reducible to a ranked set. In fact, $D \leq_{wtt} \omega_{\mathcal{L}}$, where $\mathcal{L}$ is a computable linear ordering of order type $\omega + \omega^*$.*

Proof. Let A be a noncomputable c.e. set of strongly contiguous degree. In Theorem 5.2.2 and in Example 2.4 in [15] it was shown that every c.e. T-degree contains a c.e. set that is the ω-part of a computable linear ordering $\mathcal{L}$ of order type $\omega + \omega^*$. Let B be this c.e. set such that $B = \omega_{\mathcal{L}}$ and $B \equiv_T A$. Let $D \leq_T A$, so $D \leq_T B = \omega_{\mathcal{L}}$. By Proposition 5.4.2, B is ranked. Therefore, D is wtt-reducible to a ranked set. □

In order to make D low, we will use the same proof techniques as in Theorems 5.3.1 and 5.4.3 by constructing a low $\mathbf{d}$-computable set D for every so-called *uniformly array noncomputable* degree $\mathbf{d}$ (see Definition 5.5.7 below). Then we will show that D is not wtt-reducible to any infinite branch of a tree with countably many infinite branches. Therefore, D is not wtt-reducible to any ranked set. We will now recall some further notions needed for the proof.

Definition 5.5.4. *1. Let f be a total function. We call f ω-c.e. if there are computable functions h and p such that $f(n) = \lim_{s\to\infty} h(n,s)$ for all $n \in \omega$ and $|\{s : h(n,s) \neq h(n,s+1)\}| \leq p(n)$.*

2. (Downey, Greenberg, and Weber [10]) *A c.e. degree* $\mathbf{d}$ *is* totally ω-c.e. *if for all* $\mathbf{d}$-*computable functions g, g is ω-c.e.*

In other words, let f be a total function that can be approximated by a computable function h. We call f ω-c.e. if the number of mind changes that $h(n,s)$ makes (for its approximation of $f(n)$) is computably bounded by $p(n)$, for a computable function p. If all of the functions computable in some degree are ω-c.e., then the degree is called totally ω-c.e.

It is a well known fact that there exists a uniformly Δ^0_2 enumeration $\{f_e\}_{e\in\omega}$ of all of the functions which are wtt-reducible to $\emptyset'$. The following lemmas are other known facts relating ω-c.e. functions and totally ω-c.e. degrees with functions that are wtt-reducible to $\emptyset'$.

Lemma 5.5.5 (Downey, Jockusch, and Stob [11])**.** *Let f be a total function. Then, $f \leq_{wtt} \emptyset'$ if and only if f is ω-c.e.*

Using the definition of totally ω-c.e., along with the previous lemma, we get the following proposition.

Proposition 5.5.6 (Downey, Greenberg, and Weber [10])**.** *A c.e. degree* $\mathbf{d}$ *is totally ω-c.e. if and only if for all* $\mathbf{d}$-*computable functions g, $g \leq_{wtt} \emptyset'$.*

The following definition is due to Downey, Jockusch, and Stob from [11].

Definition 5.5.7. *1. A degree* $\mathbf{d}$ *is* array noncomputable (ANC) *if for each $f \leq_{wtt} \emptyset'$ there is a* $\mathbf{d}$-*computable function g such that f does not dominate g, so $g(n) > f(n)$*

for infinitely many n.

If a degree is not ANC, then it is called array computable*.*

2. *A degree* $\mathbf{d}$ *is* uniformly ANC *if there is a fixed* $\mathbf{d}$*-computable function* g *that is not dominated by any* $f \leq_{wtt} \emptyset'$*. So,* $g(n) > f(n)$ *for infinitely many* n*, and for all* $f \leq_{wtt} \emptyset'$*.*

We want to understand the structure of these degrees, and we will ultimately show that there exists a low c.e. uniformly ANC degree. First, we will show that there are c.e. ANC degrees which are not uniformly ANC. It is obvious that every uniformly ANC degree is ANC. The following lemma says that the c.e. array computable degrees are properly contained in the totally ω-c.e. degrees.

Lemma 5.5.8 (Downey, Greenberg, and Weber [10])**.** *There are c.e. degrees that are totally ω-c.e. and are not array computable.*

Downey and Greenberg in [5] found a connection between totally ω-c.e. degrees and uniformly ANC degrees which extends the previous lemma. In this proposition, it is easy to see that every c.e. array computable degree is totally ω-c.e.

Proposition 5.5.9 (Downey and Greenberg [5])**.** *Let* $\mathbf{d}$ *be a c.e. degree. Then,* $\mathbf{d}$ *is totally ω-c.e. if and only if* $\mathbf{d}$ *is not uniformly ANC.*

Proof. Let us first assume that $\mathbf{d}$ is totally ω-c.e and let g be a $\mathbf{d}$-computable function. We will show that $\mathbf{d}$ is not uniformly ANC by showing that g is dominated by some function which is wtt-reducble to $\emptyset'$. Since g is totally ω-c.e., by Proposition 5.5.6, $g \leq_{wtt} \emptyset'$. Since g

dominates itself, $\mathbf{d}$ is not uniformly ANC. (Notice that we do not use the fact that $\mathbf{d}$ is c.e. in this direction of the proof.)

Now, we will let $\mathbf{d}$ be a c.e. degree which is not uniformly ANC. Let g be a $\mathbf{d}$-computable function. We will show that $\mathbf{d}$ is totally ω-c.e. by showing that $g \leq_{wtt} \emptyset'$, and then conclude by Proposition 5.5.6 that $\mathbf{d}$ is totally ω-c.e. Let $D \in \mathbf{d}$ be a c.e. set. Since $g \leq_T D$, we have that $g = \varphi_e^D$ for some $e \in \omega$. Let $h(n)$ be the least s such that $\varphi_{e,s}^D(n) \downarrow$. The function h is computable in D (and therefore is computable in $\mathbf{d}$), so h is dominated by some function, say f, such that $f \leq_{wtt} \emptyset'$. Therefore, $f(n) \geq h(n)$ for infinitely many n. Then, we also have that $g(n) = \varphi_{e,f(n)}^D(n)$. Therefore, $g \leq_{wtt} f \leq_{wtt} \emptyset'$, which implies that $g \leq_{wtt} \emptyset'$. □

Using Lemma 5.5.8, along with Proposition 5.5.9, we have the following corollary.

Corollary 5.5.10 (Chisholm et al. [5])**.** *There are c.e. ANC degrees that are not uniformly ANC.*

We will now introduce a class of degrees. We introduce this class since the proof of Proposition 5.5.14, which says that if a degree is not in this class then it is uniformly ANC, is similar to the proof that there exists a low c.e. uniformly ANC degree.

Definition 5.5.11 (Jockusch and Posner [20])**.** *A degree* $\mathbf{d} \in GL_2$ *if and only if* $\mathbf{d}'' = (\mathbf{d} \cup \mathbf{0}')'$.

We will use the following equivalence when we talk about the class of degrees GL_2.

Lemma 5.5.12 (Jockusch and Posner [20])**.** *For any degree* $\mathbf{d}$*,* $\mathbf{d} \in GL_2$ *if and only if there exists a function* f *such that* $\deg(f) \leq (\mathbf{d} \cup \mathbf{0}')$ *and such that for all* $\mathbf{d}$*-computable functions* g*,* $f(n) \geq g(n)$ *for infinitely many* n*.*

According to this lemma, a degree $\mathbf{d}$ is not in GL_2, so $\mathbf{d} \in \overline{GL_2}$, if and only if for each function f such that $\deg(f) \leq (\mathbf{d} \cup \mathbf{0}')$, there is a $\mathbf{d}$-computable function g such that $g(n) > f(n)$ for infinitely many n. With this observation, we immediately obtain the next lemma.

Lemma 5.5.13 (Downey, Jockusch, and Stob [11])**.** *If* $\mathbf{d} \in \overline{GL_2}$*, then* $\mathbf{d}$ *is ANC.*

Recall that a degree $\mathbf{d} \leq \mathbf{0}'$ is *low*$_2$ if $\mathbf{d}'' = \mathbf{0}''$. The following proposition holds for all degrees that are not *low*$_2$ since *low*$_2$ and GL_2 degrees coincide on degrees $\mathbf{d} \leq \mathbf{0}'$.

Proposition 5.5.14 (Chisholm et al. [5])**.** *Let* $\mathbf{d} \in \overline{GL_2}$*. Then,* $\mathbf{d}$ *is uniformly ANC. In fact, every degree* $\mathbf{d} \leq \mathbf{0}'$ *that is not low*$_2$ *is uniformly ANC.*

Proof. Let $\mathbf{d} \in \overline{GL_2}$. By Lemma 5.5.13, $\mathbf{d}$ is ANC, so no $\mathbf{0}'$-computable function dominates every $\mathbf{d}$-computable function. Let $\{f_e\}_{e\in\omega}$ be a Δ^0_2 enumeration of all functions that are wtt-reducible to $\emptyset'$. Let $F(n) = \max_{e\leq n} f_e(n)$. $F \leq_{wtt} \emptyset'$, so $F \leq_T \emptyset'$. Therefore, F is $\mathbf{0}'$-computable. So, there exists some $\mathbf{d}$-computable function g such that $g(n) > F(n)$ for infinitely many n. Fix e such that $f_e \leq_{wtt} \emptyset'$. It is easy to see that $g(n) > f_e(n)$ for infinitely many n. So, g dominates all $\mathbf{0}'$-computable functions. Therefore, $\mathbf{d}$ is uniformly ANC. □

Proposition 5.5.15 (Chisholm et al. [5])**.** *There exists a low c.e. uniformly ANC degree.*

Proof. Similarly to the proof of the previous proposition, let $\{f_e\}_{e\in\omega}$ be a Δ^0_2 enumeration of all functions that are wtt-reducible to $\emptyset'$. Let $F(n) = \max_{e\leq n} f_e(n)$. Enumerate a co-infinite c.e. set A in stages s, $A = \cup_{s\geq 0} A_s$ and A_s finite, using the finite injury priority method with the standard lowness requirement

$$N_e : (\exists^\infty s)[\varphi_e^{A_s}(e) \downarrow \Longrightarrow \varphi_e^A(e) \downarrow],$$

and also satisfying

$$(\exists^\infty n)[a_n > F(n)],$$

where $\overline{A} = \{a_0 < a_1 < \cdots\}$. □

Theorem 5.5.16 (Chisholm et al. [5]). *1. For every uniformly ANC degree* **d***, there exists a* **d***-computable set D such that D is not wtt-reducible to any ranked set. In fact, every Π^0_1 class that contains an element P such that $D \leq_{wtt} P$ has a perfect Π^0_1 subclass.*

2. Furthermore, when **d** *is a c.e. degree, D is also c.e.*

We immediately obtain the next corollary by Proposition 5.5.15 and Proposition 5.4.2.

Corollary 5.5.17. *There exists a low c.e. set D such that D is not wtt-reducible to any initial segment of any computable scattered linear ordering.*

Proof of Theorem 5.5.16. Let **d** be a uniformly ANC degree. We will construct a **d**-computable set D where $D = \cup D_{eij}$ and where $D_{eij} \subset \{< e, i, j, r, k >\}_{r,k \in \omega}$. We will make D **d**-computable by having $D \leq_T p$ where p is the **d**-computable function that is not dominated by any function $f \leq_{wtt} \emptyset'$ (since **d** is uniformly ANC). In order to make $D \leq_T p$, we will only let an element into D when p allows us, as we will see in the conditions below.

We will use a similar technique as when we proved Theorems 5.3.1 and 5.4.3. Recall that $\{T_j\}_{j \in \omega}$ is a fixed enumeration of all partial computable subtrees of $2^{<\omega}$. Similarly to Theorem 5.4.3, we will be finding transversals of T_j, however, in this proof we want to have infinitely many transversals $\{A^r_{eij}\}_{r \in \omega}$ to construct D^{s+1}_{eij}.

Construction:

Given the sth approximation D^s_{eij} to D_{eij}, construct D^{s+1}_{eij} as follows.

First, we want to define the transversals $\{A^r_{eij}\}_{r\in\omega}$ of T_j. If we have already found a transversal A^r_{eij} of T_j by stage s, then we do not need to look for it. So, move on to enumerating elements into D^{s+1}_{eij}. Then, we may go on looking for a transversal A^{r+1}_{eij} of T_j. If we have not found a transversal A^r_{eij} by stage s, let $A =< \sigma_0, \sigma_1, \ldots, \sigma_M >\in (2^{\leq s})^{<\omega}$ with $M \leq s$. A is an $(i, < e, i, j, r >)$-converging transversal of T_j if $|\sigma_k| \geq \varphi^{\sigma_k}_{i,s}(< e, i, j, r, k >)$ for every $\sigma_k \in A$ and every $f \in [T^s_j]$ extends some σ_i.

If we can find such an A, let A^r_{eij} be the least such (with respect to the code of $< \sigma_0, \sigma_1, \ldots, \sigma_M >$). Then, we can go on to enumerating elements into D^{s+1}_{eij}.

If we can not find a transversal A^r_{eij}, we can move on to finding this transversal in stage $s+1$. If by a certain stage t we have that A^r_{eij} is still not defined by stage t and $t > p(r)$, then we know that $< e, i, j, r, k >\notin D^{t+1}_{eij}$ and in fact $< e, i, j, r, k >\notin D$, as we will see in the conditions below. In this case, we stop looking for A^r_{eij} and say that it is undefined by stage $p(r)$.

Given that $< e, i, j, r, k >\notin D^s_{eij}$, then $< e, i, j, r, k >\in D^{s+1}_{eij}$ if and only if

1. $s \leq p(r)$,

2. A^r_{eij} has been defined by stage s, and

3. $\varphi^{\sigma_k}_e(< e, i, j, r, k >) = 0$, where $A^r_{eij} =< \sigma_0, \sigma_1, \ldots, \sigma_k, \ldots >$.

This ends the construction.

By these above conditions it is easy to see that $D \leq_T p$, so therefore D is **d**-computable.

Recall that the proof of (1) is very similar to the proof of Theorems 5.3.1 and 5.4.3, and therefore we must show that $< e, i, j, r, k >\in D$ if and only if $\varphi^{\sigma_k}_{e,s}(< e, i, j, r, k >) = 0$, for all $k < |A^r_{eij}|$, which is what the proofs of these theorems hinge on.

Lemma 5.5.18. *Assume that A^r_{eij} exists for every r. Then, there is some r such that: $<e,i,j,r,k>\in D$ if and only if $\varphi^{\sigma_k}_{e,s}(<e,i,j,r,k>)=0$, for all $k<|A^r_{eij}|$.*

Proof. We will prove this theorem by way of contradiction, so assume otherwise. We will define the following sequence using φ_e. For all $r\in\omega$ and all $k<|A^r_{eij}|$, define

$$t_{r,k}=\begin{cases}0, & \text{if } \varphi^{\sigma_k}_{e,s}(<e,i,j,r,k>)\uparrow,\\ \text{least } s \text{ such that } \varphi^{\sigma_k}_{e,s}(<e,i,j,r,k>)\downarrow, & \text{otherwise.}\end{cases}$$

Let $f(r)=\sup_k t_{r,k}$. $f(r)$ is the least s' such that, for every $k'<|A^r_{eij}|$ such that $\varphi^{\sigma_{k'}}_{e}(<e,i,j,r,k'>)\downarrow$, we have that $\varphi^{\sigma_{k'}}_{e,s'}(<e,i,j,r,k'>)\downarrow$ (so $s'-1$ is the largest number we need in the evaluation of the computations that converge). We will define a computable approximation $\{f_s\}_{s\in\omega}$ to f. First, we will define the following sequence, which can be found computably. For all $r,s\in\omega$, and for all $k<|A^r_{eij}|$, define

$$t_{r,k,s}=\begin{cases}0, & \text{if } \varphi^{\sigma_k}_{e,s}(<e,i,j,r,k>)\uparrow,\\ \text{least } u\le s \text{ such that } \varphi^{\sigma_k}_{e,u}(<e,i,j,r,k>)\downarrow, & \text{otherwise.}\end{cases}$$

Now, let $f_s(r)=\sup_k t_{r,k,s}$. It is easy to see that $\{f_s\}_{s\in\omega}$ is a computable approximation to f. Also, $f_{s+1}(r)\neq f_s(r)$ for at most $|A^r_{eij}|$ values of s, since for every $r,s\in\omega$, $f_s(r)$ is the supremum of $|A^r_{eij}|$ elements of the sequence $\{t_{r,k,s}\}$ (since $k<|A^r_{eij}|$). Therefore, as we evaluate $f_0(r),f_1(r),\ldots$, since each is the supremum of only $|A^r_{eij}|$ elements of the sequences $\{t_{r,k,0}\},\{t_{r,k,1}\},\ldots$, and since the increasing value of s only means that we can use more numbers in the computation of $\varphi^{\sigma_k}_{e,s}(<e,i,j,r,k>)$, $f_{s+1}(r)\neq f_s(r)$ for at most $|A^r_{eij}|$ values of s.

Since there are computable functions $\{f_s\}_{s\in\omega}$ such that $f(r) = \lim_{s\to\infty} f_s(r)$ for all $r \in \omega$ and $|\{s : f_s(r) \neq f_{s+1}(r)\}| \leq |A^r_{eij}|$, and we can computably find $|A^r_{eij}|$ for every $r \in \omega$ by the construction and since all A^r_{eij} exist by assumption, f is ω-c.e. By Lemma 5.5.5, $f \leq_{wtt} \emptyset'$. Since p is not dominated by any function wtt-reducible to $\emptyset'$, we have that $p(r) > f(r)$ for infinitely many r.

According to the three conditions in the construction, $p(r)$ is the maximum stage where we may define A^r_{eij} for every r. For a particular r, if A^r_{eij} is defined at a stage $t \geq f(r)$, then $\varphi^{\sigma_k}_{e,t}(< e,i,j,r,k >)$ has a large enough t to converge (if it does). If $t < f(r)$, then at stage $f(r)$, $\varphi^{\sigma_k}_{e,f(r)}(< e,i,j,r,k >)$ will have enough numbers to use in its computation to either converge or diverge. Therefore, we have that $\varphi^{\sigma_k}_{e,s}(< e,i,j,r,k >) = 0$, for all $k < |A^r_{eij}|$, by the construction. □

Now we will finish the proof of (1). Fix $j \in \omega$ such that $T_j \subset 2^{<\omega}$ and T_j has only countably many branches. By Lemma 5.4.5, for every $r \in \omega$ we have an $(i, < e,i,j,r >)$-converging transversal. Therefore, Lemma 5.5.18 gives us that $< e,i,j,r,k > \in D$ if and only if $\varphi^{\sigma_k}_{e,s}(< e,i,j,r,k >) = 0$, for all $k < |A^r_{eij}|$. We can now finish the proof using exactly the same proof as in Theorem 5.3.1 and obtaining the same conclusion.

Now we will prove (2). Let **d** be a c.e. degree. We must show that D also has c.e. degree. Elements can only enter D when allowed to do so by the function p, that is **d**-computable and is not dominated by any function $f \leq_{wtt} \emptyset'$. In particular, in order for an element $< e,i,j,r,k >$ to enter D^{s+1}_{eij}, the stage s must be such that $s \leq p(r)$. This condition is computable in **d**, however, for D to be c.e., we need all three conditions, including this one in particular, to all be computable. (Note that the second condition is computable, checking that the transversal A^r_{eij} has been defined by stage s, since you need to just go through the construction up to stage s and check if the transversal A^r_{eij} has been defined yet. Constructing the transversals is a computable procedure, since T_j is partial computable. The third condition is also computable, checking if $\varphi_e^{\sigma_k}(< e, i, j, r, k >) = 0$, where $A^r_{eij} = < \sigma_0, \sigma_1, \ldots, \sigma_k, \ldots >$. We only need to check if $\varphi_{e,s}^{\sigma_k}(< e, i, j, r, k >) = 0$ where s is the stage.) Our goal is to change the condition checking that $s \leq p(r)$ into a computable condition.

Let $X \in \mathbf{d}$ and $e_0 \in \omega$ be such that the function $g = \varphi_{e_0}^X$ is not dominated by any function $f \leq_{wtt} \emptyset'$. We know this function g exists since $\mathbf{d}$ is ANC. Define $p_s(r) = \max_{t \leq s} \varphi_{e_0,t}^{X_t}(r)$, which are computable functions for all $s \in \omega$. Notice that $p_s(r) \leq p_{s+1}(r)$ for all $r, s \in \omega$. Define $p(r) = \lim_{s \to \infty} p_s(r)$. Therefore, $p(r)$ is the limit of uniformly computable functions. We now have that $g(r) \leq \lim_{s \to \infty} p_s(r) = p(r)$, for all r. Since $g(r)$ is not dominated by any function $f \leq_{wtt} \emptyset'$, and $g(r) \leq p(r)$, $p(r)$ is not dominated by any function $f \leq_{wtt} \emptyset'$. It is also obvious that $p(r)$ is $\mathbf{d}$-computable since it is defined using X. We can now check the new condition that $s \leq p_s(r)$, which is a computable condition. Since all three conditions are computable, the D we construct is c.e.

To finish the proof, notice that Lemma 5.5.18 still holds, and therefore we can use exactly the same proof as in Theorem 5.3.1 and obtain the same conclusion. □

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