Effectively and Relatively Effectively Categorical Structures

Valentina S. Harizanov

George Washington University

Washington, DC

harizanv@gwu.edu

http://home.gwu.edu/~harizanv/

Let A be a *computable* structure.

- A is *computably categorical* (*computably stable*) if for all computable $B \cong A$, at least one (every) isomorphism from A onto B is computable.

Let α be a computable ordinal.

- A is $\mathbf{\Delta}^0_\alpha$ *categorical* if for all computable $B \cong A$, there is a $\mathbf{\Delta}^0_\alpha$ isomorphism from A onto B.

- A is *relatively* $\mathbf{\Delta}^0_\alpha$ *categorical* if for all $B \cong A$, there is an isomorphism from A onto B, which is $\mathbf{\Delta}^0_\alpha$ relative to the atomic diagram of B.

- $(Q, <)$ is relatively computably categorical.
  It is not computably stable.

- $(\omega, <)$ is not computably categorical.
  It is relatively Δ^0_2 categorical.

- (R. Miller)
  No computable tree $(T, \prec)$ of infinite height is computably categorical.

- (Lempp-McCoy-R. Miller-Solomon)
  For every $n \geq 1$, there is a computable tree of finite height,
  which is Δ^0_{n+1} categorical but not Δ^0_n categorical.

- (Goncharov-Dzgoev, Remmel)
  A computable linear order is computably categorical if and only if it has only finitely many successors.

- (LaRoche, Goncharov-Dzgoev, Remmel)
  A computable Boolean algebra is computably categorical if and only if it has finitely many atoms.

- Every computably categorical computable linear order is relatively computably categorical.

- Every computably categorical computable Boolean algebra is relatively computably categorical.

- (Lempp-McCoy-R. Miller-Solomon)
  Every computably categorical computable tree of finite height is relatively computably categorical.

- Abelian p-group G
  $g \in (G - \{0\}) \Rightarrow (\exists n \geq 1)[order(g) = p^n]$

- (Goncharov, Smith)
  An abelian p-group is computably categorical iff it can be written in one of the following forms:
  (1) $(Z(p^\infty))^l \oplus G$ for $l \in \omega \cup \{\infty\}$ and G finite, or
  (2) $(Z(p^\infty))^n \oplus G \oplus (Z_{p^k})^\infty$, where $n, k \in \omega$ and G is finite.

- Every computably categorical computable abelian p-group is relatively computably categorical.

- (Calvert-Cenzer-Harizanov-Morozov)
  Every computable computably categorical equivalence structure is relatively computably categorical. These are exactly structures with:
  (1) finitely many finite equivalence classes, and
  (2) finitely many infinite classes, bounded character, at most one finite k with infinitely many equivalence classes of size k.

- (McCoy)
  Many Δ^0_2 categorical computable linear orders and Boolean algebras are relatively Δ^0_2 categorical.

- (Calvert-Cenzer-Harizanov-Morozov)
  Many Δ^0_2 categorical computable equivalence structures are relatively Δ^0_2 categorical.

- There are syntactical conditions that imply $\mathbf{\Delta}^0_\alpha$ categoricity:
  The existence of simple Scott families.

- A *Scott family for a countable structure* A is a set Φ of formulae, with a fixed finite tuple of parameters $\overline{c}$ in A, such that:
  (1) Each tuple in A satisfies some $\psi \in \Phi$,
  (2) If $\overline{a}$, $\overline{b}$ are tuples in A satisfying the *same* formula $\psi \in \Phi$, then there is an automorphism of A taking $\overline{a}$ to $\overline{b}$.

- A *formally c.e. Scott family* is a c.e. Scott family consisting in finitary existential formulae.
  A *formally* $\mathbf{\Sigma}^0_\alpha$ *Scott family* is a $\mathbf{\Sigma}^0_\alpha$ Scott family consisting in computable $\mathbf{\Sigma}_\alpha$ formulae.

- (Ash-Knight-Manasse-Slaman, Chisholm)
  A computable structure A is *relatively* $\mathbf{\Delta}^0_\alpha$ categorical *iff*
  A has a formally $\mathbf{\Sigma}^0_\alpha$ Scott family *iff*
  A has a c.e. Scott family consisting in computable $\mathbf{\Sigma}_\alpha$ formulae.

- (Goncharov)
  (i) Assume that A is 2-decidable. If A is computably categorical, then it is relatively computably categorical.

  (ii) There is a computably categorical rigid structure that is *not relatively* computably categorical.

- (Ash)
  Let $\alpha > 1$ be a computable ordinal.
  Under some additional decidability on A, if A is $\mathbf{\Delta}^0_\alpha$ categorical, then it is relatively $\mathbf{\Delta}^0_\alpha$ categorical.

- (Kudinov)
  There is a 1-decidable structure that is computably categorical, but *not relatively* computably categorical.

- (Cholak-Goncharov-Khoussainov-Shore)
  There is a computably categorical structure A such that for every $a \in A$, the expanded structure (A, a) is not computably categorical.

- (Cholak-Shore-Solomon)
  There is a computably categorical rigid structure A with no Scott family of finitary formulae.

- (Goncharov-Harizanov-Knight-McCoy-R. Miller-Solomon)

  Let $\alpha \geq 2$ be a computable successor ordinal.
  There is a computable structure that is $\mathbf{\Delta}^0_\alpha$ categorical but
  not relatively $\mathbf{\Delta}^0_\alpha$ categorical.

- *Proof sketch.* (1) Relativize the proof for $\mathbf{\Delta}^0_1$ to $\mathbf{\Delta}^0_\alpha$.

  There is a rigid $\mathbf{\Delta}^0_\alpha$ directed graph G such that:

  (i) G has exactly one $\mathbf{\Delta}^0_\alpha$ isomorphic copy, up to $\mathbf{\Delta}^0_\alpha$ isomorphism,
  (ii) G does not have $\mathbf{\Sigma}^0_\alpha$ Scott family of finitary existential formulae.

- *Proof sketch continued.* (2) Code $\mathbf{\Delta}^0_\alpha$ directed graph G in
  a *computable* structure G^*,
  using a pair of structures B_0, B_1 such that
  B_0 codes $G \models a \to b$ and
  B_1 codes $G \models \neg(a \to b)$.

- B_0 and B_1 are *computable* structures, for which
  the standard *back-and-forth relations* $\leq_\beta$ for $\beta < \alpha$ are uniformly c.e.

- B_0 and B_1 satisfy the *same* infinitary Π_β sentences for $\beta < \alpha$.

- B_0 satisfies *some* computable Π_α sentence that is not true in B_1,
  and *vice versa.*

- B_0 and B_1 are uniformly relatively $\mathbf{\Delta}^0_\alpha$ categorical.

- (3) Show that:

  (i) G^* is $\mathbf{\Delta}^0_\alpha$ categorical
  (G had exactly one $\mathbf{\Delta}^0_\alpha$ isomorphic copy, up to $\mathbf{\Delta}^0_\alpha$ isomorphism);

  (ii) G^* does not have formally $\mathbf{\Sigma}^0_\alpha$ Scott family
  (G did not have $\mathbf{\Sigma}^0_\alpha$ Scott family of finitary existential formulae).

Open Questions

- For a computable limit ordinal α, is there a computable structure that is Δ^0_α categorical but not relatively Δ^0_α categorical?

- Is every Δ^1_1 categorical structure relatively Δ^1_1 categorical?